MATH SCAND. 62 (1988), 59–74

SERIES OF INDEPENDENT VECTOR VALUED RANDOM VARIABLES AND ABSOLUTE CONTINUITY OF SEMINORMS

T. BYCZKOWSKI and M. RYZNAR

Let E be a separable metric vector space and let $q: E \to [0, \infty]$ be a measurable seminorm. Suppose, further, that (X_i) is a sequence of E-valued independent and symmetric random vectors such that $\sum_i X_i$ converges a.s. in E. The purpose of this paper is to investigate conditions under which series of this kind converge a.s. with respect to q. We also extend here results of [4] concerning absolute continuity of seminorms, under various assumptions.

Section 1 is preliminary. In Section 2 the basic result is contained in Theorem 2.2, which is a version of Itô-Nisio Theorem concerning convergence of random series with E-valued independent and symmetric components. Let us note that although some partial results of this type have been known for some time (see [4], [17]) our theorem seems to be new even in the case of $E = \mathbf{R}^\infty$ and q being the supremum seminorm. In the sequel of this section we study, for a measurable seminorm q, the relation between the q-separability of random vectors and the condition that all q-balls centered at 0 are of positive mass. Given a seminorm q we also prove a creterion of concentration of p-stable measure on the subspace $\{q < \infty\}$, in terms of its Lévy measure, for $0 < p < 1$. In the end of this section we present some examples clarifying problems discussed there.

Section 3 is devoted to investigation of absolute continuity of distributions of seminorms of random series, under some assumptions on independent components. In particular, absolute continuity of distribution of Gaussian seminorms follows at once from our result. We improve also earlier obtained results concerning absolute continuity of stable seminorms.

Let us note that Section 2 is essentially self-contained. Section 3 relies more on results of the paper [4] and some acquaintance with [4] may be necessary.

Received October 31, 1985; in revised form November 20, 1986.

1. Preliminaries.

Throughout the whole paper E will denote a complete separable metric vector space over the real numbers, unless stated otherwise. We assume that the reader is familiar with standard concepts of weak convergence, properties of convolutions of probability measures, etc.

A Borel measurable function $q: E \to [0, \infty]$ is called a measurable seminorm if it is subadditive and homogeneous, i.e. if $q(x+y) \leqq q(x)+q(y)$ and $q(ax) = |a|q(x)$, for all $x,y \in E$ and $a \in \mathbb{R}$.

We first state some versions of the Lévy Inequality, needed in the sequel (see [11]).

LEMMA 1.1. *Let $X_1, \ldots, X_n$ be E-valued independent and symmetric random vectors and let q be a measurable seminorm. Then for every $\varepsilon > 0$ we have*

$$\text{(a)} \qquad P\left\{\max_{1 \leqq j \leqq 2} q(X_j) > \varepsilon\right\} \leqq 2P\left\{q\left(\sum_{i=1}^{2} X_i\right) > \varepsilon\right\},$$

$$\text{(b)} \qquad P\left\{\max_{1 \leqq j \leqq n} q\left(\sum_{i=1}^{j} X_i\right) > \varepsilon\right\} \leqq 2P\left\{q\left(\sum_{i=1}^{n} X_i\right) > \varepsilon\right\}.$$

Now, we collect some basic facts concerning convolution semigroups on E. Note that when E is a Banach space or locally convex space, then every infinitely divisible probability measure is embeddable into a unique continuous convolution semigroup, as a consequence of the Lévy-Khinchine Theorem. Thus, the readers interested only in these spaces may use the standard form of the Lévy-Khinchine Theorem [1] or [5], instead of facts stated in the remainder of this section.

A convolution semigroup on E is a family $(\mu_t)_{t>0}$ of probability measures such that $\mu_t * \mu_s = \mu_{t+s}$, for $t,s > 0$. It is called continuous if μ_t converges weakly to δ_0 (the point mass at 0) as $t \to 0+$; it is called symmetric if all μ_t's are symmetric.

Next, let m be a finite Borel measure on E. For every $t > 0$ we define

$$\exp tm = e^{-tm(E)} \sum_{k=0}^{\infty} (t^k/k!)m^{*k},$$

where $m^{*0} = \delta_0$. It is easy to see that the above series converges in the total variation norm and defines a continuous convolution semigroup.

We also need a version of the Lévy-Khinchine formula for continuous convolution semigroups. Suppose that $(\mu_t)_{t>0}$ is a symmetric continuous convolution semigroup on E. Then there exists a nonnegative measure ν such that for every open neighbourhood U of 0 $\nu|_{U^c}$ is finite and $(1/t)\mu_t|_{U^c}$

converges weakly to $\nu|_{U^c}$, as $t \to 0+$, whenever $\nu(\partial U) = 0$. Moreover, $\mu_t = \chi_t * \gamma_t$, where $(\chi_t)_{t>0}$ and $(\gamma_t)_{t>0}$ are symmetric and continuous convolution semigroups, $\gamma_t = \lim \exp t\nu|_{F_n}$, for every increasing sequence of symmetric Borel subsets such that $\nu|_{F_n}$ is finite and $\cap_n F_n^c = \{0\}$ and

$$\lim_{t \to 0+} (1/t)\chi_t(U^c) = 0,$$

for every open neighbourhood of 0. The measure ν, appearing in the above theorem will be called the Lévy measure of $(\mu_t)_{t>0}$. The semigroup γ_t will be denoted in the sequel by $\exp t\nu$. All these facts are taken from [4].

Now, let $(\mu_t)_{t>0}$ be a symmetric continuous convolution semigroup on E. $(\mu_t)_{t>0}$ is called symmetric p-stable, $0 < p < 2$, if $\mu_t = \exp t\nu$ and

$$\nu(sA) = (1/s^p)\nu(A), \tag{1.1}$$

for all Borel sets A and all $s > 0$. (1.1) is equivalent to the following property:

$$\mu_t(A) = \mu_{ts}(s^{1/p}A),$$

for all $t,s > 0$ and all Borel sets A. Again, this last property shows that in the case of locally convex spaces this definition is consistent with the usual definition of stable measures.

Finally, we need one simple consequence of Lemma 1.1. We begin with a definition.

Let q be a measurable seminorm. A convolution semigroup $(\mu_t)_{t>0}$ is called q-continuous if

$$\lim_{t \to 0+} \mu_t\{q > \varepsilon\} = 0, \quad \text{for all } \varepsilon > 0.$$

It is easy to see that if $(\mu_t)_{t>0}$ is p-stable symmetric semigroup then it is q-continuous for every measurable seminorm q satisfying $q < \infty$ μ_1-a.s. The next lemma is taken from [3]. q stands here for a measurable seminorm.

LEMMA 1.2. *Let $(\mu_t)_{t>0}$ be a q-continuous convolution semigroup on E. Then for all $\varepsilon > 0$*

$$\limsup_{t \to 0+} (1/t)\mu_t\{q > \varepsilon\} < \infty. \tag{1.2}$$

2. Convergence and boundedness of random series.

Let (X_i) be a sequence of E-valued independent and symmetric random vectors and let q denote a measurable seminorm. We always assume that $S = \sum_{i=1}^{\infty} X_i$ converges a.s. in E and that $q(S) < \infty$ a.s. Let us denote

$$S_n = \sum_{i=1}^{n} X_i; \quad R_n = \sum_{i=n+1}^{\infty} X_i.$$

We begin with a simplified version of Lemma 5.1 from the paper [4]. Its proof is included here for the sake of completeness.

LEMMA 2.1. *Under the assumptions as above* $\limsup q(R_n)$ *is equal* a.s. *to a finite constant.*

PROOF. That $\limsup q(R_n)$ is a.s. constant is a well-known consequence of the Kolmogorev $0-1$ Law. Denote this constant by c. We show that $q(S) < \infty$ a.s. implies that $c < \infty$. Applying the inequality (b) from Lemma 1.1 we obtain that

$$P\left\{\max_{k \leqq n \leqq m} q(R_n) > \alpha\right\} \leqq 2P\{q(R_k) > \alpha\},$$

for every $\alpha > 0$. When $m \to \infty$ we get

$$P\left\{\sup_{n \geqq k} q(R_n) > \alpha\right\} \leqq 2P\{q(R_k) > \alpha\}.$$

This, in turn, gives

$$\liminf P\{q(R_n) > \alpha\} \geqq 1/2P\{\limsup q(R_n) > \alpha\}. \tag{2.1}$$

Next, applying the part (a) of Lemma 1.1 with S_n as X_1 and R_n as X_2 we get

$$P\{q(R_n) > \alpha\} \leqq P\{\max(q(S_n), q(R_n)) > \alpha\} \leqq 2P\{q(S) > \alpha\},$$

for $n = 1, 2, \ldots$. When $\alpha < c = \limsup q(R_n)$, then by (2.1) and the above inequality we obtain $2P\{q(S) > \alpha\} \geqq 1/2$. This clearly ends the proof.

Now, we are ready to state and prove the basic result of this section.

THEOREM 2.2. *Let* (X_i) *be a sequence of independent symmetric random vectors with values in* E *and let* q *be a measurable seminorm. Suppose that* $S = \sum_{i=1}^{\infty} X_i$ *converges* a.s. *in* E *and that* $q(S) < \infty$ a.s. *Then* $q(R_n)$ *converges* a.s. *to a constant* $c < \infty$. *Moreover,* $q(R_n) \geqq c$ a.s., *for* $n = 0, 1, \ldots$. *The series* $\sum_{i=1}^{\infty} X_i$ *converges* a.s. *with respect to* q *(that is* $c = 0$*) whenever* S *satisfies the following condition*:

$$P\{q(S) < \varepsilon\} > 0, \quad \textit{for every } \varepsilon > 0. \tag{*}$$

PROOF. 1. We first prove our theorem under the assumption $Eq(S) < \infty$. Note that then we have $Eq(R_n) < \infty$, for $n = 1, 2, \ldots$. This follows by Lemma 1.1 (a) and by integration by parts formula. Next, let

$\mathscr{B}_n = \sigma\{(X_i)_{i \geqq n+1}\}$. By the assumptions of separability of E and measurability of q it follows that we can compute conditional expectations in the same way as for real random variables:

$$E\{q(R_n)|\mathscr{B}_{n+1}\} = E_{n+1}q(X_{n+1}+R_{n+1})$$

where E_{n+1} denotes the integration with respect to the distribution of X_{n+1}. Next, by the symmetry of X_{n+1} we obtain that

$$\begin{aligned} E_{n+1}q(X_{n+1}+R_{n+1}) &= E_{n+1}[\tfrac{1}{2}q(X_{n+1}+R_{n+1})+\tfrac{1}{2}q(-X_{n+1}+R_{n+1})] \\ &\geqq q(R_{n+1}), \end{aligned}$$

This shows that $\{q(R_n), \mathscr{B}_n\}$ is a reversed positive submartingale, thus it converges a.s. and in L^1. By the Kolmogorov $0-1$ Law, $\lim q(R_n)$ is constant a.s. Denote this constant by c.

If we now denote by $\mathscr{F}_n = \sigma\{(X_i)_{i \leqq n}\}$ then, by the same arguments as before, we obtain

$$E\{q(S)|\mathscr{F}_n\} = E^{(n)}q(S_n+R_n) \geqq Eq(R_n),$$

where $E^{(n)}$ denotes the integration with respect to $(X_{n+1}, X_{n+2}, \ldots)$. Since $\lim Eq(R_n) = c$ and $\lim E\{q(S)|\mathscr{F}_n\} = q(S)$, a.s., we get

$$q\left(\sum_{i=1}^{\infty} X_i\right) \geqq c \quad \text{a.s.}$$

Since it is clear that the above inequality remains valid if we replace $i = 1$ by $i = n$, for $n = 2, 3, \ldots$, this proves our theorem under the assumption $Eq(S) < \infty$.

2. To remove the assumption of integrability of $q(s)$ we consider the case when $X_i = r_i x_i$, (r_i) is a Rademacher sequence and (x_i) is a sequence of elements of E. As before, we assume that $S = \sum_{i=1}^{\infty} r_i x_i$ converges a.s. in E and that $q(S) < \infty$ a.s. Then we have

$$Eq(S) < \infty.$$

Indeed, an obvious inequality

$$q(S) \leqq \sup q(S_n) + \limsup q(R_n)$$

and the fact that $\limsup q(R_n)$ is equal a.s. to a constant, show that it is enough to prove that $E \sup q(S_n) < \infty$ and that this constant is finite. The first fact is well-known for Rademacher series (see [8]), the second one is contained in Lemma 2.1.

The remaining part of the proof is standard; it is included here for the sake of completeness. Let (X_i) be an arbitrary sequence of E-valued random vectors, defined on a probability space $(\Omega_1, \mathscr{A}_1, P_1)$ with the properties as in our theorem. Let (r_i) be a Rademacher sequence defined on another space $(\Omega_2, \mathscr{A}_2, P_2)$. Then $(r_i X_i)$, defined on the product space, has the same distribution as (X_i) hence it satisfies all the assumptions of our theorem. Let $c = \limsup q(R_n)$ a.s. By the Fubini Theorem, the first part of 2. and part 1. of the proof we obtain that $q(\sum_{i=n+1}^{\infty} r_i X_i(\omega))$ converges P_2 – a.s. for P_1 – almost all ω to the constant c and that $q(\sum_{i=n+1}^{\infty} r_i X_i(\omega)) \geqq c$ P_2 – a.s. for P_1 – almost all ω, for $n = 0, 1, \ldots$. Applying once more Fubini's theorem and the fact that the distributions of (X_i) and $(r_i X_i)$ are identical, we obtain the conclusion.

The next theorem is a version of Itô-Nisio Theorem (see [10] and [8]). Let us recall that a random vector Y is called q-separable if for every $\varepsilon > 0$ there exists a sequence $K(x_i, \varepsilon)$ of q-balls with centers x_i and of radius ε such that $Y \in \cup K(x_i, \varepsilon)$ a.s. Equivalently, Y is q-separable if there exists a Borel vector subspace F which is q-separable and such that $Y \in F$ a.s.

THEOREM 2.3. *Let (X_i), E, q be as in Theorem 2.2. Suppose that $\sum_{i=1}^{\infty} X_i$ is q-separable. Then $\sum_{i=1}^{\infty} X_i$ converges* a.s. *with respect to q.*

We first prove a lemma.

LEMMA 2.4. *Let X be a q-separable random vector with values in E. Then $Y = X^s$, the symmetrization of X, has the property* $(*)$.

PROOF. Let $\varepsilon > 0$. Denote by $K(\varepsilon)$ the closed ball of radius ε centered at 0. Let

$$E_\varepsilon = \{x ; P\{X \in K(\varepsilon)+x\} \leqq P\{q(Y) \leqq \varepsilon\}\}.$$

We claim that $P\{X \in E_\varepsilon\} > 0$. Indeed, if not, then

$$P\{X \in K(\varepsilon)+x\} > P\{q(Y) \leqq \varepsilon\} \quad \text{a.s.}$$

Integrating the both sides with respect to the distribution of X we obtain a contradiction:

$$P\{q(Y) \leqq \varepsilon\} = \int P\{X \in K(\varepsilon)+x\} P_X(dx) > P\{q(Y) \leqq \varepsilon\}.$$

Now, let $\{x_i\}$ be a countable subset of E_ε such that

$$P\left\{X \in \bigcup_{i=1}^{\infty} (K(\varepsilon)+x_i) \cap E_\varepsilon\right\} = P\{X \in E_\varepsilon\}.$$

Then we have

$$0 < P\{X \in E_\varepsilon\} \leqq P\left\{X \in \bigcup_{i=1}^{\infty} (K(\varepsilon)+x_i)\right\}.$$

Let N be a positive integer such that

$$P\left\{X \in \bigcup_{i=1}^{N} (K(\varepsilon)+x_i)\right\} > 0.$$

Since $x_i \in E_\varepsilon$, we obtain

$$0 < P\left\{X \in \bigcup_{i=1}^{N} (K(\varepsilon)+x_i)\right\} \leqq \sum_{i=1}^{N} P\{X \in K(\varepsilon)+x_i\} \leqq NP\{q(Y) \leqq \varepsilon\},$$

which completes the proof of the lemma.

PROOF OF THEOREM 2.3. Let (X_n') be an independent copy of the sequence (X_n) and let

$$Y = \sum_{n=1}^{\infty} X_n' + \sum_{n=1}^{\infty} X_n.$$

Then Y is the symmetrization of $\sum_{n=1}^{\infty} X_n$, thus by Lemma 2.4, Y satisfies $(*)$. Applying Theorem 2.2 to the sequence Z_n, where $Z_1 = \sum_{n=1}^{\infty} X_n'$, $Z_{n+1} = X_n$, we obtain that $q(\sum_{n=k}^{\infty} X_n)$ converges to 0 a.s. This ends the proof of Theorem 2.3.

COROLLARY 2.5. *Let $(T, \mathscr{F}, \nu)$ be a finite measure space and let $(X_i(t))$ be a sequence of independent symmetric measurable stochastic processes. Suppose that for ν – almost all t the series $\sum_i X_i(t)$ converges* a.s. *and that*

$$\operatorname{ess\,sup} \left|\sum_i X_i(t)\right| < \infty \quad \text{a.s.}$$

Then $\sum_i X_i$ converges a.s. *in $L^\infty(\nu)$ provided $Y = \sum_i X_i$ satisfies $(*)$ with respect to $q = q_\infty = \operatorname{ess\,sup}$ or when Y is q_∞ – separable.*

PROOF. Let $E = L_0(\nu)$ be the space of all measurable functions on T with convergence in measure. Denote by $\tilde{Y}$ and $\tilde{X}_i$ the E-valued random vectors determined by $Y(t)$ and $X_i(t)$, respectively. Using results of Chung and Doob [6] we infer that $\tilde{Y}$ and $\tilde{X}_i$ have separable ranges in E. Restricting our attention to a separable subspace of E containing the ranges of $\tilde{X}_i$ and applying Theorem 2.2 and Theorem 2.3 we obtain the conclusion.

REMARK 2.6. (i) One may ask what is the relation between the q-separability of a given symmetric random vector Y and the property $(*)$.

By Theorem 2.2 and Theorem 2.3 it follows directly that for a given measurable seminorm q and Y of the form $Y = \sum_i X_i$, where X_i are q-separable, $\sum_i X_i$ is a.s. convergent in E and $q(Y) < \infty$ a.s., we have that Y is q-separable if and only if Y^s satisfies $(*)$ and this is exactly the case when $\sum_i X_i$ converges in q a.s.

(ii) By the proof of Lemma 2.4 we obtain that if $(\mu_t)_{t>0}$ is a symmetric convolution semigroup on E and q is a measurable seminorm such that μ_1 is q-separable, then $\mu_t\{q < \varepsilon\} > 0$, for every $\varepsilon > 0$ and every $t > 0$.

(iii) If E is locally convex and Y is symmetric Gaussian, then applying (i) we obtain that Y is q-separable if and only if Y satisfies $(*)$. Indeed, in this case Y can be represented as the distribution of a series of the form $\sum_i \xi_i e_i$, where (ξ_i) is a sequence of independent standard normal random variables, $e_i \in E$ and the above series converges a.s. in E. The same fact is also true when Y is p-stable, $0 < p < 2$, with the purely atomic spectral measure. However, as we will see later, for every p, $0 < p < 2$, it is possible to construct p-stable Y satisfying $(*)$, which is not q-separable.

Now, we investigate when for a given lower semicontinuous seminorm q the concentration of a stable measure on $\{q < \infty\}$ can be expressed in terms of the corresponding Lévy measure.

THEOREM 2.7. *Let q_n be a nondecreasing sequence of continuous seminorms on E and let $\mu_t = \exp tv$ be a symmetric p-stable semigroup with the corresponding Lévy messure v. Define $q = \sup q_n$. If $0 < p < 1$, then $q < \infty$ μ_1-a.s. if and only if $v\{q > 1\} < \infty$.*

PROOF. In Section 3 we show that $q < \infty$ μ_1-a.s. always implies that $v\{q > 1\} < \infty$, for every measurable seminorm q, regardless of p. For the sake of simplicity we prove this fact here only for lower semicontinuous seminorms q. To do this, we use the definition of v. Namely, since $\{q > 1\}$ is open, for every open neighbourhood U of 0 such that $v(\partial U) = 0$ we get

$$\begin{aligned} v(\{q > 1\} \cap U^c) &\leqq \liminf_{t\to 0+} (1/t)\mu_t|_{U^c}\{q > 1\} \\ &\leqq \liminf_{t\to 0+} (1/t)\mu_t\{q > 1\} < \infty, \end{aligned}$$

where the last inequality follows by Lemma 1.2. Thus, $v\{q > 1\} < \infty$.

Now, let $c_n = v\{q_n > 1\}$. Since q_n are continuous, by the definition of Lévy measure we obtain

$$\lim_{t\to 0+} (1/t)\mu_t\{q_n > 1\} = c_n.$$

On the other hand, when $n \to \infty$, we obtain

$$\nu\{q_n > 1\} = c_n \to c_0 = \nu\{q > 1\}.$$

Assume that $c_0 < \infty$. Let X_i be independent copies of a random vector X with the distribution μ_1. Then

$$P\{q_n(X) > t\} \leqq P\left\{(1/k^{1/p}) \sum_{i=1}^{k} q_n(X_i) > t\right\}. \tag{2.2}$$

Now, $q_n(X)$ belongs to the normal domain of attraction of non-negative real p-stable random variable Θ_n with the Lévy measure m_n satisfying

$$m_n\{u; u > 1\} = c_n = \lim_{t \to \infty} t^p P\{\Theta_n > t\}$$

(see, e.g. [7, chapter XVII]).

When $k \to \infty$, (2.2) implies that $P\{q_n(X) > t\} \leqq P\{\Theta_n > t\}$. On the other hand, as $n \to \infty$, then $c_n \to c_0$, so $\Theta_n \to \Theta_0$ in distribution, where Θ_0 is a nonnegative real p-stable random variable with the Lévy measure m_0 determined by the condition

$$m_0\{u; u > 1\} = c_0 = \nu\{q > 1\}.$$

Thus, as $n \to \infty$, we obtain

$$P\{q(X) > t\} \leqq P\{\Theta_0 > t\} < 1.$$

By the $0-1$ Law for stable measures we conclude that $q < \infty$ μ_1-a.s.

The following example indicates that the weaker condition $\nu\{q = \infty\} = 0$ is not sufficient to ensure that $\mu\{q < \infty\} = 1$, for $\mu = \exp \nu$.

EXAMPLE 2.8. Suppose that $0 < p < 2$ and that (Z_i) is a sequence of independent symmetric random variables with the following properties:

$$E|Z_i|^p < \infty, \quad E \sup |Z_i|^p = \infty \quad \text{but} \quad \sup |Z_i| < \infty.$$

Let σ be the distribution of the sequence (Z_i) on $\mathbf{R}^\infty$. Define $d\nu = d\sigma \times dt/t^{1+p}$, that is, for every nonnegative and product measurable function f we put

$$\int f(x) d\nu(x) = \iint f(tx)/t^{1+p} d\sigma(x) dt.$$

It is easy to check that ν satisfies (1.1), so it is a Lévy measure of a symmetric p-stable semigroup on $\mathbf{R}^\infty$. Let $q(x) = \sup|x_i|$.

Now, since $q(Z) = \sup|Z_i| < \infty$ a.s., we get

$$0 = \int_0^\infty \sigma\{q = \infty\}/t^{1+p}dt = \int 1_{\{q=\infty\}}(x)d\nu(x) = \nu\{q = \infty\}.$$

On the other hand, as we have seen in the first part of the proof of Theorem 2.7, the condition $\mu_1\{q < \infty\} = 1$ implies that $\nu\{q > 1\} < \infty$. Using the explicit form of ν, we see that this last condition is equivalent to the following one: $\int q^p d\sigma < \infty$. However, because of the following equality $\int q^p d\sigma = E\sup|Z_i|^p = \infty$, we obtain that $\mu_1\{q < \infty\} = 0$.

The next example shows that for $p \geqq 1$ the condition $\nu\{q > 1\} < \infty$ no longer guarantees the concentration of p-stable measure $\mu_1 = \exp\nu$ on the subspace $\{q < \infty\}$.

EXAMPLE 2.9. Let (Θ_i) be a sequence of independent, identically distributed standard symmetric p-stable random variables, $1 \leqq p < 2$, and let β_i be a sequence of real numbers such that $\sum_i|\beta_i|^p < \infty$ but

$$\sum_i |\beta_i|^p \ln(1 + 1/|\beta_i|) = \infty.$$

Let μ be the distribution of $(\beta_i\Theta_i)$ on R^∞. It is easy to check that $d\nu = d\sigma \times dt/t^{1+p}$ is the Lévy measure of μ, where

$$\sigma = 1/2 \sum_i |\beta_i|^p(\delta_{e_i} + \delta_{-e_i}),$$

with e_i being the ith coordinate unit vector. Now, if we put

$$q(x) = \left(\sum_i |x_i|^p\right)^{1/p},$$

then

$$q((\beta_i\Theta_i)) = \left(\sum_i |\beta_i|^p|\Theta_i|^p\right)^{1/p},$$

hence $q(x) = \infty$ μ—a.s. However, $\nu\{q > 1\} = \sum_i|\beta_i|^p < \infty$.

Since p-stable random vextors with $p < 1$ always satisfy (*) (see [12]), application of Theorem 2.2 and Theorem 2.7 gives:

COROLLARY 2.10. *Let (X_i) be a sequence of independent E-valued symmetric p-stable random vectors, $0 < p < 1$, such that $X = \sum_i X_i$ converges* a.s. *in* E *and let q be a measurable seminorm. If $q(X) < \infty$* a.s. *then $\sum_i X_i$ converges* a.s. *in q. When, additionally, E is locally convex and q lower semicontinuous then $q(X) < \infty$* a.s. *whenever $\nu\{q > 1\} < \infty$, where ν is the Lévy measure of X.*

The above corollary generalizes the fact that in Banach spaces the boundedness of $\sum_i X_i$ implies its convergence, for p-stable X_i's, $0 < p < 1$. That result was proved in [14], by domination technique. For a similar result, see [13].

The following example indicates that the condition (*) does not imply q-separability.

EXAMPLE 2.11. Let (ξ_i) be a sequence of independent standard normal random variables. Now, if (a_n) is a sequence of positive numbers such that $\sup|\xi_i|/a_i < \infty$ a.s., but $\sum_i \exp(-a_n^2) = \infty$, then $q(x) = \sup|x_i|$ has the property: $q(x) < \infty$ a.s. and $q(x) \geqq 2^{1/2}$ a.s. with respect to the distribution on R^∞ induced by $(\xi_i/a_i) = Y$ (see [9]). By Remark 2.6, (i), it follows that this distribution is not q-separable. Now, let $0 < p < 2$ and let Θ be a positive $p/2$-stable random variable with the Laplace transform $\exp(-t^{p/2})$, which is independent of Y. Then $Z = \sqrt{\Theta}\, Y$ is a p-stable random vector (see e.g., [7, chapter VI]). It is easy to see that $q(Z) < \infty$ a.s. and that Z satisfies (*). However, Z is not q-separable.

3. Absolute continuity of seminorms.

In this section we extend the main result of [4] for series of the form $\sum_i \xi_i x_i$, where $x_i \in E$ and ξ_i are independent symmetric real random variables with absolutely continuous distributions. As before, we assume that $\sum_i \xi_i x_i$ converges a.s. in E and that q is a measurable seminorm on E such that $q(\sum_i \xi_i x_i) < \infty$ a.s. Let μ be the distribution of $\sum_i \xi_i x_i$. Then we have the following result:

THEOREM 3.1. *Under the assumptions as above, $F(t) = \mu\{q < t\}$ is absolutely continuous on (c, ∞), where $c = \inf\{t; F(t) > 0\}$. When μ is q-separable or q is strictly convex, then F is absolutely continuous on* $(0, \infty)$.

The proof is an adaptation of that of Theorem 5.4 in [4] and is based on the following lemma (see Corollary 4.2 in [4]):

LEMMA 3.2. *Suppose that q is a measurable seminorm on E and that y, $x_1, \ldots, x_n$ are elements of E such that $q(x_i) < \infty$, $q(y) < \infty$. Let*

$$m = \inf_r \left\{ q\left(\sum_{j=1}^n r_j x_j + y \right); r = (r_1, \ldots, r_n) \in \mathsf{R}^n \right\}. \tag{3.1}$$

Then

$$\int_{\mathsf{R}^n} \mathbf{1}_N \left(q\left(\sum_{j=1}^n r_j x_j + y \right) \right) dr_1 \ldots dr_n = 0 \tag{3.2}$$

for each N of linear Lebesque measure 0 that does not contain the point m.

PROOF OF THEOREM 3.1. Let χ_n be the distribution of $\sum_{j=n+1}^{\infty}\xi_j x_j$. Suppose that N is a subset of linear Lebesque measure 0 such that $\inf N > \varepsilon > c$. Then

$$P\left\{q\left(\sum_{j=1}^{\infty}\xi_j x_j\right)\in N\right\} = \int P\left\{q\left(\sum_{j=1}^{n}\xi_j x_j + y\right)\in N\right\}d\chi_n(y)$$

$$\leqq \int_{\{q\leqq\varepsilon\}}\int_{R^n} \mathbf{1}_N\left(q\left(\sum_{j=1}^{n} r_j x_j + y\right)\right)\prod_{j=1}^{n} f_j(r_j)dr_1\ldots dr_n d\chi_n(y) + \chi_n\{q>\varepsilon\},$$

where f_i is the density of ξ_i. Using Theorem 2.2 we obtain that $q(\sum_{i=n+1}^{\infty}\xi_i x_i)$ converges a.s. to some constant, which is $\leqq c$, as $n\to\infty$. Since $\varepsilon > c$, we obtain that

$$\chi_n\{q>\varepsilon\} = P\left\{q\left(\sum_{i=n+1}^{\infty}\xi_i x_i\right) > \varepsilon\right\}$$

tends to 0, as $n\to\infty$. Moreover, since the constant m determined by (3.1) has the property $m \leqq q(y)$, the first part of our conclusion follows by Lemma 3.2. When μ is q-separable, then by Theorem 2.3 we obtain that $\chi_n\{q > \varepsilon\}\to 0$ for every $\varepsilon > 0$. Finally, when q is strictly convex, then the infimum m in (3.1) is attained exactly at one point, as a consequence of strict convexity, hence (3.2) holds for all N of Lebesque measure 0. This clearly ends the proof.

COROLLARY 3.3. *Let E be a locally convex separable vector space and let* μ *be a Gaussian measure on E. Suppose that q is a measurable seminorm on E with the property* $q<\infty$ a.s. *Then* $F(t)=\mu\{q<t\}$ *is absolutely continuous on* (c,∞), *where* $c=\inf\{t; F(t)>0\}$. *When* μ *is q-separable or q is strictly convex then F is absolutely continuous on* $(0,\infty)$.

PROOF. Our corollary follows immediately by Theorem 3.1 and the well-known fact that μ can be represented as the distribution of a.s. convergent series of the form $\sum_i \xi_i x_i$, where ξ_i is a sequence of independent standard real normal random variables and x_i are appropriate elements of E.

The absolute continuity of $F(t)=\mu\{q<t\}$ was proved by Tsirel'son [18], for Gaussian μ. This result can also be obtained using logarithmic convexity of Gaussian measures [2]. Our method is more elementary and much more simple.

Now, we improve somewhat the main result of [4], dropping the assumption that q is lower semicontinuous. To do so, we need some preparatory lemmas. The first one is taken from [4].

LEMMA 3.4. *Let* $\mu_t=\exp t\xi * v_t$, *where* ξ *is symmetric and finite and* v_t *is* *a continuous and q-continuous symmetric semigroup. Then for every $\eta > 0$ and all open subsets A of E we have*

$$\liminf_{t \to 0+} (1/t)\mu_t|_{\{q > \eta\}}(A) \geqq \xi|_{\{q > \eta\}}(A). \tag{3.3}$$

LEMMA 3.5. *Let q be a measurable seminorm on E. Suppose that $\mu_t = \exp t\nu$ is a symmetric continuous semigroup. If it is q-continuous then $\nu\{q > \eta\} < \infty$, for every $\eta > 0$.*

PROOF. Using formula (1.2) and some standard compactness arguments (see [4] for details) we obtain that $(1/t)\mu_t|_{\{q > \eta\}}$ is weakly conditonally compact, as $t \to 0+$. Let $\nu^{(\eta)}$ be an accumulation point of this family, as $t \to 0+$. By (1.2), $\nu^{(\eta)}(E) < \infty$. Now, let U be an open neighbourhood of 0. Then

$$\mu_t = \exp t\nu|_{U^c} * \chi_t^U,$$

for some symmetric continuous and q-continuous semigroup χ_t^U. Using (3.3) we obtain

$$\nu^{(\eta)}(E) \geqq \nu|_{U^c}\{q > \eta\} \uparrow \nu\{q > \eta\}.$$

THEOREM 3.6. *Let $(\mu_t)_{t>0}$ be a symmetric p-stable semigroup on E, $0 < p < 2$, and let q be a measurable seminorm such that $q < \infty$ μ_1-a.s. Then $F(t) = \mu_1\{q < t\}$ is absolutely continuous on (c, ∞), where*

$$c = \inf\{t; F(t) > 0\}.$$

If either $0 < p < 1$ or μ_1 is q-separable or q is a strictly convex norm, then F is absolutely continuous on $(0, \infty)$.

PROOF. We first prove that F is absolutely continuous on (c, ∞), with c determined as above. To show this, let U_n be a decreasing sequence of symmetric open neighbourhoods of 0 such that $\cap_n U_n = \{0\}$. Further, let us decompose ν into two parts: $\nu = \nu^{(1)} + \nu^{(2)}$, where $\nu^{(1)} = \nu|_{\{q=0\}}$ and $\nu^{(2)} = \nu|_{\{q>0\}}$. Then $\mu_t = \mu_t^{(1)} * \mu_t^{(2)}$, with $\mu_t^{(i)} = \exp t\nu^{(i)}$, $i = 1, 2$, being symmetric p-stable semigroups. Note that if $\nu^{(2)} = 0$ then, by virtue of Lemma 5.2 in [4], $q =$ const. a.s., in which case the conclusion is trivially true. Hence, we may assume that $\nu^{(2)} \neq 0$. Now, put

$$\chi^{(n)} = \exp(\nu^{(1)}|_{U_n} + \nu^{(2)}|_{\{q > 1/n\}}).$$

Then

$$\mu_1 = \exp(\nu^{(1)}|_{U_n^c}) * \exp(\nu^{(2)}|_{\{q > 1/n\}}) * \chi^{(n)}, \tag{3.4}$$

for $n = 1, 2, \ldots$. Moreover, μ_1 can be represented as the distribution of a.s.

convergent in E series $\sum_{i=1}^{\infty} X_i$ of independent symmetric random vectors X_i with distributions

$$\exp(\nu^{(1)}|_{(U_{i-1} \setminus U_i)} + \nu^{(2)}|_{\{1/i < q \leq 1/i-1\}}),$$

where $U_0 = E$. $\chi^{(n)}$ is then clearly the distribution of $\sum_{i=n+1}^{\infty} X_i$. Applying Theorem 2.2 we obtain that

$$\lim \chi^{(n)}\{q > \alpha\} = 0, \quad \text{for every } \alpha > c. \tag{3.5}$$

On the other hand, the fact that $\nu^{(1)}|_{U_n^c}\{q > \varepsilon\} = 0$, for every $\varepsilon > 0$, and the trivial induction yield

$$q = 0 \quad \exp(\nu^{(1)}|_{U_n^c}) - \text{a.s.} \tag{3.6}$$

The formulas (3.4) and (3.6) thus show that on the sets of the form $\{q \in N\}$, with Borel N, we get

$$\mu_1\{q \in N\} = \exp(\nu^{(2)}|_{\{q > 1/n\}}) * \chi^{(n)}\{q \in N\}.$$

Since, by Lemma 3.5, $\nu^{(2)}\{q > 1/n\}$ is finite, the above formula can be written as

$$\mu_1\{q \in N\} = e^{-\beta_n} \sum_{k=1}^{\infty} (1/k!)(\nu^{(2)}|_{\{q > 1/n\}})^{*k} * \chi^{(n)}\{q \in N\} + e^{-\beta_n}\chi^{(n)}\{q \in N\},$$

for $n = 1, 2, \ldots$, where $\beta_n = \nu^{(2)}\{q > 1/n\} \to \infty$, as $n \to \infty$. The remaining part of the proof is the same as in [4]. Namely, it is enough to show that each component of the above series vanishes when $N \subseteqq (c, \infty)$ is of Lebesque measure 0. To show this, suppose that $N \subseteqq (\alpha, \infty)$, where $\alpha > c$. By (3.5) we have only to show that

$$\int_{\{q \leq \alpha\}} (\nu^{(2)}|_{\{q > 1/n\}})^{*k}\{x : q(x+y) \in N\}\chi^{(n)}(dy) = 0,$$

for $k = 1, 2, \ldots$ and $n = 1, 2, \ldots$. By the property (1.1) and measurability of q the expression under the integral sign is equal to

$$\int_{S_q^k} \int_{1/n} \ldots \int_{1/n} \mathbf{1}_N\left(q\left(\sum_{j=1}^{k} r_j s_j + y\right)\right) \frac{dr_1 \ldots dr_k}{(r_1 \ldots r_k)^{1+p}} \sigma(ds_1) \ldots \sigma(ds_k) \tag{3.7}$$

where $S_q = \{q = 1\}$ and σ is a finite nonzero Borel measure on S_q. Since we integrate the above expression over the set $\{y; q(y) \leqq \alpha\}$, the same arguments as in the end of the first part of the proof of Theorem 3.1 show that (3.7) vanishes, which completes this part of the proof.

When μ_1 is q-separable or $0 < p < 1$ then our conclusion will follow by the first part of the theorem if we show that $c = 0$. In these both cases, however, we have $\mu_1\{q < \varepsilon\} > 0$; in the first one, by Remark 2.6 (ii); in the second, by the property of p-stable random vectors, $0 < p < 1$, mentioned before the formulation of Corollary 2.10. Finally, when q is a norm then $\nu^{(1)} = 0$, so $\nu^{(2)}$ is nonzero. Hence, if q is a strictly convex norm, we can apply the same observation as in the end of the proof of Theorem 3.1.

REMARK 3.7. Theorem 3.6 was also proved in [15], [16], and in [13] under the assumption that E is locally convex, by means of a representation of p-stable measures as mixtures of Gaussian measure and application of Tsirel'son's theorem for Gaussian measures.

ACKNOWLEDGEMENT. The authors are grateful to the referee for suggesting the use of the martingale method in the proof of Theorem 2.2 which considerably simplified the original version of that proof, based previously on one inequality from [4].

REFERENCES

1. A. de Acosta, A. Araujo, and E. Giné, *On Poisson measures, Gaussian measures and the Central Limit Theorem in Banach spaces*, in *Probability on Banach Spaces*, ed. J. Kuelbs (Advances in Probability and Related Topics 4), pp. 1–68. Marcel Dekker, Inc., New York - Basel, 1978.
2. C. Borell, *Gaussian Radon measures on locally convex spaces*, Math. Scand. 38 (1976), 265–284.
3. T. Byczkowski and T. Żak, *Asymptotic properties of semigroups of measures on vector spaces*, Ann. Probab. 9 (1981), 211–220.
4. T. Byczkowski and K. Samotij, *Absolute continuity of stable seminorms*, Ann. Probab. 14 (1986), 299–312.
5. E. Detweiller, *Grenzwertsatze für Wahrscheinlichkeitsmase auf Badrikianschen Raumen*, Z. Wahrsch. Verw. Gebiete 34 (197), 285–311.
6. K. L. Chung, and J. L. Dob, *Fields, optionality and measurability*, Amer. J. Math. 87 (1965), 397–424.
7. W. Feller, *An Introduction to Probability Theoru and Its Applications* II, 2nd ed. John Wiley and Sons, New York - London - Sydney, 1971.
8. J. Hoffmann-Jørgensen, *Sums of independent Banach space valued random variables*, Studia Math. 52 (1974), 159–186.
9. J. Hoffmann-Jørgensen, L. A. Shepp, and R. M. Dudley, *On the lower tail of Gaussian seminorms*, Ann. Probab. 7 (1979), 319–342.
10. K. Itô and M. Nisio, *On the convergence of sums of independent random variables*, Osaka J. Math. 5 (1968), 35–48.
11. J.-P. Kahane, *Some Random Series of Functions*, D. C. Heath and Co., Lexington, Massachusetts, 1968.
12. M. Ryznar, *Asymptotic behaviour of stable measures near the origin*, Ann. Probab. 14 (1986), 287–298.
13. M. Ryznar, *Density of stable seminorms*, Bull. Polish Acad. Sci. Math. 33 (1985), 431–440.
14. R. Sztencel, *On boundedness and convergence of some Banach space valued random series*, Probab. Math. Statist. 2 (1981), 83–88.
15. R. Sztencel, *On the lower tail of stable seminorms*, Bull. Polish Acad. Sci. Math. 32 (1984), 715–719.

16. R. Sztencel, *Absolute continuity of the lower tail of stable seminorms*, Bull. Polish Acad. Sci. Math. 34 (1986), 231–234.
17. M. Talagrand, *Sur l'intégrabilité des vecteurs gaussiens*, Z. Wahrsch. Verw. Gebiete 68 (1984), 1–8.
18. V. S. Tsirel'son, *The density of the distribution of the maximum of a Gaussian process*, Theory Probab. Appl. 20 (1975), 847–855.

INSTITUTE OF MATHEMATICS
WROCLAW TECHNICAL UNIVERSITY
WYB. WYSPIANSKIEGO 27
50-370 WROCLAW
POLAND