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RESIDUES, CURRENTS, AND THEIR RELATION TO IDEALS OF HOLOMORPHIC FUNCTIONS

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CONTENTS

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1. Introduction.

If a function f is holomorphic in a domain $D \subset \mathbf{C}$ and has a simple zero at $a_1 \in D$, then a necessary and sufficient condition on a holomorphic function h to be of the form $h = gf$ (with g holomorphic) near a_1 is that $h(a_1) = 0$ or, equivalently, that the residue of h/f at a_1 be equal to zero. More generally, we shall see that, by generalizing the notion of residue somewhat, one may characterize those holomorphic functions h which belong to the ideal (of holomorphic functions on D) generated by any f, holomorphic in $D \subset \mathbf{C}$, by the vanishing of the residue of h/f. Let us sketch briefly how this can be done. Assume first that

$$V = \{z \in D\,; f(z) = 0\} = \{a_1, a_2, \ldots\}$$

and that g is holomorphic on $D \setminus V$. For $w \in D$ we denote by $\operatorname{res}_w(g)$ the ordinary residue of g at w. That is,

$$\operatorname{res}_w(g) = \frac{1}{2\pi i} \int_{\partial D_w} g(z)dz,$$

∂D_w being the oriented (and reasonably smooth) boundary of a neighborhood D_w of w such that $(D_w \setminus \{w\}) \cap V = \phi$. We know that g may be expanded in a convergent Laurent series around w:

$$g(z) = \sum_{k=-\infty}^{+\infty} \alpha_k^w (z-w)^k,$$

and we have

$$\operatorname{res}_w(g) = \alpha_{-1}^w.$$

One way of extracting more information from the residue is to let it act on test functions $\psi \in C_0^\infty(D)$, and one is then led to consider expressions of the form

$$\sum_{a_j \in V} \frac{1}{2\pi i} \int_{\partial D_{a_j}} g(z)\psi(z)dz. \tag{1}$$

A problem with (1) is that it depends on the paths ∂D_{a_j} and there are two ways in which one can go about in order to recover the independence of the choice of D_{a_j}.

The first way is to make a restriction on the test functions and to consider only such ψ which are holomorphic near V. (This will be our viewpoint in Chapter 3.)

The expression

$$\operatorname{Res}(g)(\psi) = \frac{1}{2\pi i}\sum \int_{\partial D_{a_j}} g(z)\psi(z)dz$$

is then well defined as long as the D_{a_j} are contained in the neighborhood of V in which ψ is holomorphic. Writing $\mathscr{O}_V^{1,0}(D)$ for the sheaf of germs at V of holomorphic (1,0)-forms $\psi = \psi(z)dz$, we thus get a mapping

$$\operatorname{Res}(g): \mathscr{O}_V^{1,0}(D) \to \mathbf{C}.$$

If, in particular, $\psi(z) \equiv 1$ near w and $\psi(z) \equiv 0$ near $V \setminus \{w\}$, one has

$$\operatorname{Res}(g)(\psi) = \operatorname{res}_w(g).$$

But in general $\psi(z)$ is of the form

$$\psi(z) = \sum_{k=0}^{\infty} \beta_k^w (z-w)^k$$

near w and we get

$$\operatorname{Res}(g)(\psi) = \sum_{k=0}^{\infty} \beta_k^w \alpha_{-1-k}^w.$$

The second way of dealing with (1) is to make a restriction on g by demanding that it be meromorphic on D, say $g = g_1/g_2$ with g_1 and g_2 holomorphic. One then considers the expression

$$\frac{1}{2\pi i}\int_{\{|g_2| = \varepsilon\}} g(z)\psi(z)dz, \tag{2}$$

which for ε small enough is of the form (1). Using a partition of unity we can write $\psi = \sum \psi_j$ with $\psi_j(z) \equiv 0$ near $V \setminus \{a_j\}$. One may then show that

$$\lim_{\varepsilon \to 0} \int_{\{|g_2| = \varepsilon\}} g(z)\psi_j(z)dz = \lim_{\varepsilon \to 0} \int_{\{|z-a_j| = \varepsilon\}} g(z)\psi_j(z)dz,$$

see e.g. Herrera and Lieberman [19, Proposition 6.6.], and the latter limit is easily computed explicitly by expanding ψ_j in a Taylor series around a_j. This yields an expression involving derivatives at a_j of ψ_j up to the order $m_j - 1$, where m_j is the order of the pole a_j of g. In other words, the limit of (2) as ε tends to zero (which we will denote by $\bar{\partial}g(\psi)$) defines the action of a (0,1)-current on D. This residue current $\bar{\partial}g$ will be of order $m_j - 1$ at a_j, $j = 1, 2, \ldots$. (We will pursue this in Chapter 4.)

Returning to our original function f, we ask whether or not a given holomorphic function h is in the ideal I_f, generated by f. The answer is of course simple: h belongs to I_f precisely if it is zero at a_j to at least the same order as is f. The point is that this condition can be reformulated as

(i) $\qquad \operatorname{Res}(h/f) = 0$

or

(ii) $\qquad \bar{\partial}(h/f) = h\bar{\partial}(1/f) = 0.$

In this thesis we give a meaning to the conditions (i) and (ii) for several complex variables and we show that they characterize certain ideals of holomorphic functions. One of our main tools will be the weighted integral formulas of Berndtsson and Andersson [3]. This method will enable us to obtain similar results in analytic rings with certain growth conditions. The residue currents which we use are those given in Coleff and Herrera [7] but we will also need more general currents, the existence and adequate properties of which are proved in Chapter 4.

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2. Results.

In Chapter 3 we start by defining a rather general residue homomorphism Res, which to any $\bar{\partial}$-closed (p,q)-current (modulo exact ones) assigns an element in $\operatorname{Hom}(H_{c,V}^{n-p,n-q-1}(D), \mathbf{C})$, where $H_{c,V}^{*}(D)$ means Dolbeault cohomology of germs at V of test forms on D. We observe that in the one-dimensional case Res becomes the residue defined in the introduction (modulo a multiplicative constant). For slightly different points of view we refer to Gordon [11] and the survey in Dolbeault [8].

The second section of Chapter 3 deals with a more specific situation. Given a holomorphic mapping $\mathbf{C}^n \supset D \xrightarrow{f} \mathbf{C}^p$ we define a $(0,p-1)$-form Ω_f on the complement of the common zero set V of the components f_j by

$$\Omega_f = (|f_1|^2 + \ldots + |f_p|^2)^{-p} \sum (-1)^{j+1} \bar{f}_j d\bar{f}_1 \wedge \ldots \underset{j}{\wedge} \ldots \wedge d\bar{f}_p.$$

We then show that, in a sense made precise in Proposition 3.2.1, integration of the $(n, n-p)$-form $\psi/f_1 \ldots f_p$ over a certain tube is equivalent to integration of $\psi \wedge \Omega_{f}$, over a much larger set. See Harvey [13] and Griffiths and Harris [12].

Chapter 4 begins with a collection of definitions and notation which will be used in the sequel. The main result is the existence of limits of the form

$$\lim_{\delta \to 0} \int_{D^{\delta}_{IJ}(\varepsilon, f)} \lambda\psi,$$

where D^{δ}_{IJ} is the oriented tubular domain

$$\{z \in D\,;\, |f_i(z)| = \varepsilon_i(\delta),\ i \in I,\ |f_j(z)| > \varepsilon_j(\delta),\ j \in J\},$$

$f = (f_1, \ldots, f_p)$ is a holomorphic map, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_p)$ is an admissible path (i.e. a mapping $[0, 1] \to \mathbf{R}^p_>$ such that ε_j tends to zero much quicker than ε_{j+1} as $\delta \to 0$; see Definition 4.1.1), λ is a semimeromorphic function (Definition 4.1.2) whose poles are contained in $\bigcup_{k \in I \cup J}\{f_k = 0\}$ and ψ is a test form of bidegree $(n, n-|I|)$. In fact, the above limit is independent of the path ε and defines a $(0, |I|)$-current supported on $\bigcap_{k \in I}\{f_k = 0\}$ and denoted by R^{λ}_{IJ}.

The proof is in two major steps: In Section 4.2 we assume that we have normal crossings, i.e. the functions f_j behave locally as monomials (Theorem 4.2.1), and in Section 4.3 we use a result on the resolution of singularities given in Hironaka [20] in order to reduce the general case to the one already treated (Theorem 4.3.1). The two special cases

(i) $\qquad I = \{1, \ldots, p\}, \quad J = \emptyset,$

and

(ii) $\qquad I = \{1, \ldots, p-1\}, \quad J = \{p\},$

have been proved in Coleff and Herrera [7] and the general idea of our proof resembles the one used in their paper.

In the last section of Chapter 4 we restrict our attention to the case when f is a complete intersection (Definition 4.4.1). We show that if λ is smooth outside $\bigcup_{k \in I \cup J}\{f_k = 0\}$ then for $i, j \in \{1, \ldots, p\} \setminus I \cup J$,

$$R^{\lambda}_{I, J \cup \{j\}} = R^{\lambda}_{I,J}$$

and

$$R^{\lambda}_{I \cup \{i\}, J} = 0.$$

(Propositions 4.4.2 and 4.4.3). Once again the corresponding statements in the special cases (i) and (ii) are contained in Coleff and Herrera [7].

The first section of Chapter 5 presents those results from Berndtsson and Andersson [3] and Berndtsson [2] on weighted integral representation formulas which will be needed later. To clarify how these formulas work and how they are connected to well-known ones, we include some examples.

A related application of these formulas to the ideal of all holomorphic functions, vanishing on a variety and belonging to certain Lipschitz classes, is made in Bonneau, Cumenge, and Zériahi [5]. More general facts about integral formulas can be found in Øvrelid [24] and Henkin [17].

Section 5.2 features a result (Proposition 5.2.2 and, in greater generality, Proposition 5.2.3), which shows how the $(0,|I|)$-current acting on $(n,n-|I|)$-forms as integration over D_{IJ}^{δ} may be realized as a limit of smooth forms. In the final part of the chapter we combine some of the results we have obtained so far into a formula representing a holomorphic function as certain currents acting on some test forms (Theorem 5.3.2). More precisely, we choose particular weights in the integral formulas of Section 5.1 and after a limiting process we arrive at the residue currents which were treated in Chapter 4. We also prove a related result for entire mappings (Proposition 5.3.5).

In Chapter 6 we introduce the ideal I_f generated by functions $f_1,\ldots,f_p$, which are holomorphic in a neighborhood of a strictly pseudoconvex domain D and which form a complete intersection. We show that, by choosing the weights of our integral formulas in a particular way, the results of Section 5.3 may be sharpened so as to have representation formulas of the following type for a holomorphic function h:

$$(1) \qquad h(w) = \sum g_k(w)f_k(w) + h\bar\partial\frac{1}{f_1}\wedge\ldots\wedge\bar\partial\frac{1}{f_p}(\psi),$$

where g_k are given as currents acting on certain test forms (which are holomorphically parametrized by w),

$$\bar\partial\frac{1}{f_1}\wedge\ldots\wedge\bar\partial\frac{1}{f_p}$$

denotes the residue current $R_{I\phi}^{\lambda}$ with $I=\{1,\ldots,p\}$ and $\lambda=(f_1\ldots f_p)^{-1}$ and ψ is a test form of bidegree $(n,n-p)$. This is Theorem 6.1.1. As a corollary we get that h belongs to I_f precisely if the residue current vanishes when multiplied by h. Similar division problems have been studied by e.g. Skoda [27], Berenstein and Taylor [1] and Berndtsson [2]. We consider entire functions in Section 6.2 and define, for certain convex functions φ, the rings A_φ, consisting of entire functions which are bounded by $Ce^{C\varphi}$ for some constant C. Assuming that the f_k are polynomials and introducing the corresponding ideals $I_{\varphi,f} \subset A_\varphi$, we find a representation similar to (1) where the functions g_k belong to A_φ if h does. A similar division problem is solved with L^2-estimates in Hörmander [22, Theorem 7.6.11].

In Section 6.3 we give an alternative proof of the fact that, locally, $h \in I_f$ is equivalent to

$$h\bar{\partial}\frac{1}{f_1} \wedge \ldots \wedge \bar{\partial}\frac{1}{f_p} = 0.$$

It follows from this new proof that we also have: $h \in I_f$ is equivalent to $\mathrm{Res}[h\Omega_f] = 0$. In other words, we have generalized to several complex variables the argument we gave in the introduction.

The last chapter of the thesis consists of some, hopefully clarifying, examples and remarks. We describe for instance the relation between residue currents and ordinary currents of integration and we show (Example 7.1.3) that the condition of complete intersections is necessary.

3. Some generalities on residues.

3.1. The cohomological residue.

Let D be a domain in $\mathbf{C}^n$ and let V be a closed subset of D. We use the following notation:

$\mathscr{D}^{p,q}(D) = \{$smooth and compactly supported forms on D of bidegree $(p,q)\}$,
$\mathscr{D}^{p,q}(D \setminus V) = \{\psi \in \mathscr{D}^{p,q}(D);\ \mathrm{supp}\,\psi \subset D \setminus V\}$,
$\mathscr{D}^{p,q}_V(D) = \mathscr{D}^{p,q}(D)/\mathscr{D}^{p,q}(D \setminus V) = \{$germs at V of test forms on $D\}$.

We clearly have an exact sequence

$$0 \to \mathscr{D}^{p,q}(D \setminus V) \xrightarrow{i} \mathscr{D}^{p,q}(D) \xrightarrow{\pi} \mathscr{D}^{p,q}_V(D) \to 0, \tag{1}$$

where i and π denote the natural injection and projection respectively.

Define $\bar{\partial}: \mathscr{D}^{p,q}_V(D) \to \mathscr{D}^{p,q+1}_V(D)$ by

$$\bar{\partial}\pi(\psi) = \pi(\bar{\partial}\psi), \quad \text{for} \quad \psi \in \mathscr{D}^{p,q}(D).$$

(Note that $\pi(\psi) = \pi(\psi')$ implies $\psi = \psi' + \varphi$, where $\varphi \in \mathscr{D}^{p,q}(D \setminus V)$, and hence $\pi(\bar{\partial}\psi) = \pi(\bar{\partial}\psi')$ so the mapping is well defined and $\bar{\partial}^2 = 0$.)

We thus have the short exact sequence of complexes

$$0 \to \mathscr{D}^{p,\cdot}(D \setminus V) \xrightarrow{i} \mathscr{D}^{p,\cdot}(D) \xrightarrow{\pi} \mathscr{D}^{p,\cdot}_V(D) \to 0. \tag{2}$$

Writing

$$H_c^{p,q}(D) = \{\psi \in \mathcal{D}^{p,q}(D); \bar\partial\psi = 0\}/\bar\partial\mathcal{D}^{p,q-1}(D), \quad q > 0,$$
$$H_c^{p,0}(D) = \{\psi \in \mathcal{D}^{p,0}(D); \bar\partial\psi = 0\}$$

and similarly for $H_c^{p,q}(D \setminus V)$ and $H_{c,V}^{p,q}(D)$, we obtain Dolbeault cohomology groups for compact supports.

By a standard diagram chase, (2) gives rise to a long exact sequence in cohomology:

$$\cdots \to H_c^{p,q}(D \setminus V) \xrightarrow{i^*} H_c^{p,q}(D) \xrightarrow{\pi^*} H_{c,V}^{p,q}(D) \xrightarrow{\delta^*} H_c^{p,q+1}(D \setminus V) \to \cdots .$$

Next, we write

$$\hat{\mathcal{D}}^{p,q}(D \setminus V) = \{\text{currents of bidegree } (p,q) \text{ on } D \setminus V\},$$

that is $\hat{\mathcal{D}}^{p,q}(D \setminus V)$ is the dual space to $\mathcal{D}^{n-p,n-q}(D \setminus V)$ equipped with its usual topology.

For $T \in \hat{\mathcal{D}}^{p,q}(D \setminus V)$ we define the $(p, q+1)$-current $\bar\partial T$ by

$$\bar\partial T(\psi) = (-1)^{p+q} T(\bar\partial\psi), \quad \psi \in \mathcal{D}^{n-p,n-q-1}(D \setminus V).$$

Also, any smooth (p,q)-form ω on $D \setminus V$ is identified in the usual manner with the (p,q)-current defined by

$$\psi \mapsto \int_{D \setminus V} \omega \wedge \psi.$$

We then put

$$\hat{H}^{p,q}(D \setminus V) = \{T \in \hat{\mathcal{D}}^{p,q}(D \setminus V); \bar\partial T = 0\}/\bar\partial\hat{\mathcal{D}}^{p,q-1}(D \setminus V), \quad q > 0,$$
$$\hat{H}^{p,0}(D \setminus V) = \{T \in \hat{\mathcal{D}}^{p,0}(D \setminus V); \bar\partial T = 0\}.$$

(We have in fact an isomorphism $H^{p,q}(D \setminus V) \cong \hat{H}^{p,q}(D \setminus V)$, where $H^{p,q}(D \setminus V)$ denotes the ordinary Dolbeault group, see Griffiths and Harris [12, p. 382].)

For each $[T] \in \hat{H}^{p,q}(D \setminus V)$ one obtains a natural homomorphism

$$[T]^* : H_c^{n-p,n-q}(D \setminus V) \to \mathbf{C}$$

by defining

$$[\psi] \mapsto T(\psi).$$

(Note that $(T + \bar\partial T')(\psi + \bar\partial\psi') = T(\psi)$ if T and ψ are $\bar\partial$-closed.) In other words, $\hat{H}^{p,q}(D \setminus V)$ is a subgroup of $\mathrm{Hom}(H_c^{n-p,n-q}(D \setminus V), \mathbf{C})$.

We will denote by Res the homomorphism

$$\mathrm{Hom}(H_c^{p,q+1}(D \setminus V), \mathbf{C}) \to \mathrm{Hom}(H_{c,V}^{p,q}(D), \mathbf{C})$$

which is induced by δ^* and we will call it the *cohomological residue homomorphism*. In particular, it assigns to any $\bar\partial$-closed (p,q)-current on $D\setminus V$ an element in $\mathrm{Hom}(H^{n-p,n-q-1}_{c,V}(D),\mathbf{C})$.

Let us now interpret this residue mapping in a more concrete manner. We start by considering the homomorphism δ^*.

Pick $[\omega]\in H^{n-p,n-q-1}_{c,V}$ and assume that $\omega=\pi(\psi)$ for $\psi\in\mathscr{D}^{n-p,n-q-1}(D)$. The fact that ω is $\bar\partial$-closed implies that ψ is $\bar\partial$-closed near V. Hence $\bar\partial\psi\in\mathscr{D}^{n-p,n-q}(D\setminus V)$ and since $\bar\partial^2=0$ we get a class $[\bar\partial\psi]\in H^{n-p,n-q}_c(D\setminus V)$. This then defines δ^* by

$$\delta^*([\omega])=[\bar\partial\psi].$$

(The definition makes sense because $\pi(\psi)=\pi(\psi')$ implies

$\psi-\psi'\in\mathscr{D}^{n-p,n-q}(D\setminus V)$ and therefore $[\bar\partial(\psi-\psi')]=0$ in $H^{n-p,n-q}_c(D\setminus V)$.)

It follows that if $[T]\in\hat H^{p,q}(D\setminus V)$, $q<n$, we have

$$\mathrm{Res}[T]([\omega])=[T](\delta^*([\omega]))=[T]([\bar\partial\psi])=T(\bar\partial\psi).$$

In other words, for any $\bar\partial$-closed current T, of bidegree (p,q) on $D\setminus V$, the action of $\mathrm{Res}[T]$ on the class of a $\bar\partial$-closed germ ω at V of bidegree $(n-p,n-q-1)$ is given as follows:

(i) Extend ω to a test form ψ on D which is $\bar\partial$-closed near V.
(ii) Let T act on $\bar\partial\psi$.

A $\bar\partial$-closed $(p,q+1)$-current on D with support on V can be considered as an element in $\mathrm{Hom}(H^{n-p,n-q-1}_{c,V}(D),\mathbf{C})$ and we shall see further on that, for certain $\bar\partial$-closed $T\in\hat{\mathscr{D}}^{p,q}(D\setminus V)$, $\mathrm{Res}[T]$ may be identified with such a current.

Finally notice that, if $n=1$, $V=\{a_1,a_2,\ldots\}$ and $\psi=\psi(z)dz$ is $\bar\partial$-closed near V, then $\operatorname{supp}\bar\partial\psi$ is contained in $\Omega=D\setminus\bigcup_{a_j\in V}D_{a_j}$ for small neighborhoods D_{a_j} of a_j. It follows that, if g is holomorphic on $D\setminus V$ (hence a $\bar\partial$-closed $(0,0)$-current), we get

$$\mathrm{Res}[g]([\pi(\psi)])=g(\bar\partial\psi)=\int_\Omega g(z)\bar\partial\psi=\int_{\partial\Omega}g(z)\psi=-\sum\int_{\partial D_{a_j}}g(z)\psi(z)dz,$$

so (up to a multiplicative constant) we get the residue of Chapter 1.

3.2. The Dolbeault isomorphism.

Let D be a domain in $\mathbf{C}^n$ and $f=(f_1,\ldots,f_p)\colon D\to\mathbf{C}^p$ a holomorphic mapping such that, if we put

$$V = \{z \in D; f_1(z) = \ldots = f_p(z) = 0\},$$

then $\dim_{\mathbf{C}} V = n-p$, that is f is a complete intersection (Definition 4.4.1). Then take $\psi \in \mathscr{D}^{n,n-p}(D)$ such that $\bar\partial\psi = 0$ in some open neighborhood of V and choose $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_p) \in \mathbf{R}^p$ with $\varepsilon_j > 0$ and so small that if we put

$$D^\varepsilon = \{z \in D; |f_j(z)| < \varepsilon_j,\ j = 1, \ldots, p\}$$

then ψ is $\bar\partial$-closed on D^ε. (Note that we need the fact that ψ has compact support.) We also define

$$D_f^\varepsilon = \{z \in D; |f_j(z)| = \varepsilon_j,\ j = 1, \ldots, p\}$$

and orient it by requiring that

$$d(\arg f_1) \wedge \ldots \wedge d(\arg f_p) \wedge \beta^{n-p} \geqq 0$$

where β is the usual Kähler form.

We will show in Section 4.4 that ε can be chosen so that D_f^ε is a regular real-analytic manifold of dimension $2n-p$, and hence integration on D_f^ε is an elementary operation. We will be interested in integrals of the form

$$\int\limits_{D_f^\varepsilon} \frac{\psi}{f_1 \cdots f_p}$$

and, since the integrand is closed and equal to zero near ∂D, it follows that the particular choice of ε does not affect the value of the integral.

Next, we define the $(0, p-1)$-form Ω_f on $D \setminus V$ by

$$\Omega_f(z) = \left(\sum_{j=1}^{p} (-1)^{j+1} \bar f_j(z) d\bar f_1(z) \wedge \ldots \underset{j}{\wedge} \ldots \wedge d\bar f_p(z)\right) \Big/ \left(\sum_{j=1}^{p} |f_j(z)|^2\right)^p, \tag{3}$$

where $\underset{j}{\wedge}$ means that $d\bar f_j$ is omitted. It is a straightforward calculation to verify that Ω_f is $\bar\partial$-closed on $D \setminus V$.

We have the following result.

PROPOSITION 3.2.1. *Let f, D^ε and D_f^ε be given as above. Let ψ be a test form of bidegree $(n, n-p)$ such that $\bar\partial\psi = 0$ on D^ε. Then there exists a constant c_p, depending solely on p, such that*

$$\int\limits_{D_f^\varepsilon} \frac{\psi}{f_1 \cdots f_p} = c_p \int\limits_{\partial D^\varepsilon} \psi \wedge \Omega_f,$$

Ω_f being given by (3).

PROOF. Put $V_j = \{z; f_j(z) = 0\}$, $U_j = D^\varepsilon \setminus V_j$ and observe that $\{U_j\}_{j=1,\dots,p}$ is a covering of $D^\varepsilon \setminus V$. We then notice that $\psi/f_1 \dots f_p$ is a section over $U_1 \cap \dots \cap U_p$ of the sheaf $\mathscr{Z}^{n,n-p}$, consisting of germs of $\bar\partial$-closed $(n, n-p)$-forms. It may therefore be considered as a Čech $(p-1)$-cochain for the covering $\{U_j\}_{j=1,\dots,p}$. Since the covering consists of just p open sets there are no nontrivial p-cochains and we get a class in Čech cohomology

$$[\psi/f_1 \dots f_p] \in \check{H}^{p-1}(D^\varepsilon \setminus V, \mathscr{Z}^{n,n-p}).$$

By (the proof of) the Dolbeault theorem, see Griffiths and Harris [12, p. 45], we have an isomorphism

$$\Delta : \check{H}^{p-1}(D^\varepsilon \setminus V, \mathscr{Z}^{n,n-p}) \to H^{n,n-1}(D^\varepsilon \setminus V).$$

Keeping in mind that ψ vanishes on ∂D, we can use precisely the same argument as in Griffiths and Harris [12, p. 651–653] (i.e. tracing $\psi/f_1 \dots f_p$ through the long exact sequence which arises in the proof of the Dolbeault theorem) to show that

$$\int_{D_f^\varepsilon} \frac{\psi}{f_1 \dots f_p} = \int_{\partial D^\varepsilon} \eta,$$

for any $\eta \in \Delta([\psi/f_1 \dots f_p])$ with $\eta \equiv 0$ near ∂D. Continuing to follow the argument of Griffiths and Harris one introduces the functions $|f_j|^2/(|f_1|^2 + \dots + |f_p|^2)$, which may be thought of as a partition of unity for the covering $\{U_j\}_{j=1,\dots,p}$ of $D^\varepsilon \setminus V$, and, once more scrutinizing the proof of the Dolbeault theorem, one constructs a canonical representative for the class $\Delta([\psi/f_1 \dots f_p])$. This representative turns out to be

$$c_p \psi \wedge \Omega_f.$$

The proposition follows.

REMARK 3.2.2. Notice that since $\psi \equiv 0$ on ∂D and $\bar\partial\psi \equiv 0$ on D^ε we have

$$-\int_{\partial D^\varepsilon} \psi \wedge \Omega_f = \int_{\partial(D \setminus D^\varepsilon)} \psi \wedge \Omega_f = \int_{D \setminus D^\varepsilon} \bar\partial\psi \wedge \Omega_f = \int_D \bar\partial\psi \wedge \Omega_f.$$

Hence, if we put $\omega = \pi(\psi)$, where π is the projection in (1), and consider $[\omega] \in H_{c,V}^{n,n-p}(D)$, $[\Omega_f] \in \hat{H}^{0,p-1}(D \setminus V)$, we have found that

$$c_p \operatorname{Res}[\Omega_f]([\omega]) = \int_{D_f^\varepsilon} \frac{\psi}{f_1 \dots f_p}.$$

We will use this identity in Section 6.3.

4. Principal value currents, residue currents and those in between.

4.1. Definitions and notation.

We introduce here most of the notation which will be used in this and the following chapters. We also give some definitions.

B is the open unit polydisk in $\mathbf{C}^n$, i.e. the unit ball for the norm $|z| = \max_j |z_j|$. We will write $|z_j| = \varrho_j$ for $z_j \in \mathbf{C}$.

$$\mathbf{R}^m_+ = \{x \in \mathbf{R}^m; x_j \geqq 0, \forall j\},$$

$$\mathbf{R}^m_> = \{x \in \mathbf{R}^m; x_j > 0, \forall j\}.$$

If $C \subset \{1, \ldots, n\}$, we put $|C|$ for the cardinality of C and define the following projections

$$\pi_C : \mathbf{C}^n \to \mathbf{C}^{|C|}, \quad \text{which forgets all coordinates } z_c, c \notin C$$

$$\pi(C) : \mathbf{C}^n \to \mathbf{C}^{n-|C|}, \quad \text{which forgets } z_c, c \in C.$$

If $+_m : \mathbf{C}^m \to \mathbf{R}^m_+$ is given by $+_m(z) = (\varrho_1, \ldots, \varrho_m) = \varrho$, then

$$\pi_C^+ = +_{|C|} \circ \pi_C$$

and

$$\pi^+(C) = +_{n-|C|} \circ \pi(C).$$

Using this we may now introduce

$$z_C = \pi_C(z)$$

$$z(C) = \pi(C)(z)$$

$$\varrho_C = \pi_C^+(z)$$

$$\varrho(C) = \pi^+(C)(z).$$

For a multi-index $a = (a_1, \ldots, a_n) \in \mathbf{N}^n$, we put as usual

$$z^a = z_1^{a_1} \ldots z_n^{a_n} \quad \text{and} \quad \varrho^a = \varrho_1^{a_1} \ldots \varrho_n^{a_n},$$

but also, for $C = \{c_1, \ldots, c_s\} \subset \{1, \ldots, n\}$,

$$z_C^a = z_{c_1}^{a_{c_1}} \ldots z_{c_s}^{a_{c_s}}, \quad z(C)^a = z^a / z_C^a \quad \text{etc.}$$

We use an analogous notation for differentials

$$dz_C = \bigwedge_{c \in C} dz_c, \quad dz(C) = \bigwedge_{c \notin C} dz_c, \quad \text{and so on.}$$

For any $I, J \subset \{1, \ldots, n\}$, the differential $dz_I \wedge d\bar{z}_J$ will sometimes be written in the following canonical way

$$dz_I \wedge d\bar{z}_J = \pm dz_A \wedge d\bar{z}_B \wedge (dz \wedge d\bar{z})_C,$$

where A, B, and C are mutually disjoint, in fact $A = I \setminus J$, $B = J \setminus I$ and $C = I \cap J$.

If g is a holomorphic function in some domain $D \subset \mathbf{C}^n$, we write

$$V(g) = \{z \in D; g(z) = 0\}.$$

Similarly, if $f: D \mapsto \mathbf{C}^p$ is a holomorphic mapping given by $z \mapsto (f_1(z), \ldots, f_p(z))$, we write $V_j = V(f_j)$ and also

$$V = \bigcap_{j=1}^{p} V_j, \quad V_A = \bigcup_{j \in A} V_j \quad \text{for any } A \subset \{1, \ldots, p\},$$

$$V_f = \bigcup_{j=1}^{p} V_j = V(f_1 \ldots f_p).$$

DEFINITION 4.1.1. An *admissible path* in $\mathbf{R}^p$ is an analytic map

$$\varepsilon = (\varepsilon_1(\delta), \ldots, \varepsilon_p(\delta)):]0, 1] \to \mathbf{R}^p_>$$

such that

$$\lim_{\delta \to 0} \varepsilon_p(\delta) = 0 \quad \text{and, if } p > 1,$$

$$\lim_{\delta \to 0} \varepsilon_j(\delta)/\varepsilon_{j+1}(\delta)^q = 0, \quad 1 \leqq j \leqq p-1, \quad \text{for all } q \in \mathbf{N}.$$

Once again, let $f: D \to \mathbf{C}^p$ be a holomorphic mapping defined in some domain $D \subset \mathbf{C}^n$ and $\varepsilon:]0, 1] \to \mathbf{R}^p_>$ an admissible path. Then, for $I, J \subset \{1, \ldots, p\}$, consider the tubular set

$$D^\delta_{IJ} = D^\delta_{IJ}(\varepsilon, f) = \{z \in D; |f_i(z)| = \varepsilon_i(\delta), i \in I; |f_j(z)| > \varepsilon_j(\delta), j \in J\}.$$

Clearly, D^δ_{IJ} is an open subset of the analytic variety $D^\delta_{I\emptyset}$ and it is shown in Coleff and Herrera [7, Sections 1.5 and 1.6] how one can give this latter set an orientation and define a current of integration on it. We shall assume that this has been done once and for all. It is thus meaningful to talk about integrals like

$$\int_{D^\delta_{IJ}} \psi, \quad \psi \text{ being a continuous test form.}$$

DEFINITION 4.1.2. Let $D \subset \mathbf{C}^n$ be a domain and $V \subset D$ a complex analytic subvariety. A smooth function $\lambda: D \setminus V \to \mathbf{C}$ will be called *semimeromorphic* *with poles contained in V* if, for any $z_0 \in D$, one can find a neighborhood U_{z_0} of z_0 such that $\lambda(z) = a(z)/g(z)$, $z \in U_{z_0}$, with a smooth on U_{z_0}, g holomorphic on U_{z_0} and $V(g) \cap U_{z_0} \subset V \cap U_{z_0}$. *A semimeromorphic (q,r)-form on D* is a differential form of bidegree (q,r) whose coefficients are semimeromorphic functions. If all the coefficients have their poles contained in V, the form will clearly be smooth outside V.

Finally we define a family of seminorms on the space of differential forms on D with coefficients in C^N. Let

$$\psi = \sum_{I,J \subset \{1,\dots,n\}} \psi_{IJ}(z) dz_I \wedge d\bar{z}_J.$$

Then, for $K \subset\subset D$ and $\alpha, \beta \in \mathbf{N}^n$, $|\alpha| + |\beta| \leqq N$,

$$p^K_{\alpha\beta}(\psi) = \sum_{I,J} \lim_{z \in K} |(\partial/\partial z)^\alpha (\partial/\partial \bar{z})^\beta \psi_{IJ}(z)|.$$

4.2. Existence in the case of normal crossings.

This section is devoted to the proof of the following theorem, which will be generalized in Section 4.3.

Theorem 4.2.1. *Let D be a domain in $\mathbf{C}^n$ and f a holomorphic mapping $D \to \mathbf{C}^p$ defined by $z \mapsto (f_1(z), \dots, f_p(z))$ and such that, for any point $z_0 \in D$, there is a neighborhood of z_0 on which f is of the form*

$$f_j(z) = u_j(z)(z - z_0)^{\alpha_j}, \quad \textit{where} \quad \alpha_j = (a_{j1}, \dots, \alpha_{jn}) \in \mathbf{N}^n$$

and the u_j's are holomorphic and never vanishing. Suppose that I and J are disjoint subsets of $\{1, \dots, p\}$ with $|I| = s$. Let λ be a semimeromorphic function on D which is smooth outside $V_{I \cup J}$ and let $\varepsilon:]0,1] \to \mathbf{R}^p_>$ be an admissible path. Then, for any $(2n-s)$-form ψ with coefficients in $C_0^\infty(D)$, the following limit exists and does not depend on the choice of ε

$$R^{\lambda;f}_{I,J}(\psi) = R^\lambda_{I,J}(\psi) = \lim_{\delta \to 0} \int_{D^\delta_{I,J}(\varepsilon, f)} \lambda\psi.$$

For every compact $K \subset D$ one can furthermore find an integer q^K and positive real numbers $c^K_{\alpha\beta}$, α and β being multi-indices satisfying $|\alpha| + |\beta| \leqq q^K$, such that

$$|R^\lambda_{I,J}(\psi)| \leqq \sum_{\alpha,\beta} c^K_{\alpha\beta} p^K_{\alpha\beta}(\psi)$$

for all ψ with $\operatorname{supp}\psi \subset K$. *In fact, $R^\lambda_{I,J}$ is a current of bidegree* $(0,s)$.

Proof. First we notice that the statements are local. It is enough to prove them for test forms ψ with support in some small neighborhood U_{z_0} of an arbitrary point $z_0 \in D$. After translating and dilating the coordinates, if necessary, we can of course assume that $z_0 = 0$ and that $U_{z_0} = B$.

Also there is no loss of generality in assuming that λ may be represented on all of B by a quotient $a(z)/z^\lambda$ where $a \in C^\infty(B)$ and $V(z^\lambda) \subset V(z^{\alpha_1} \dots z^{\alpha_p})$.

We now need a lemma concerning the relationship between the $(p \times n)$-matrix α (having α_j as its rows) and the tubular domain $D^\delta_{I,J}$.

Lemma 4.2.2. *If* rank $\alpha < p$, *then, for small* $\delta > 0$, $D^\delta_{I,J} \cap B$ *is either empty or it can be defined by fewer than p relations,* i.e. *at least one of the (in)equalities* $|f_j(z)| = |u_j(z)z^{\alpha_j}| = \varepsilon_j(\delta)$ $(> \varepsilon_j(\delta))$ *is superfluous.*

Proof. If $I \cup J \subsetneqq \{1, \dots p\}$ there is nothing to prove. Suppose therefore $|I| + |J| = p$. Suppose also rank $\alpha < p$. Then the vectors $\alpha_1, \dots, \alpha_p$ are linearly dependent so that we can find r, $1 \leqq r < p$, such that

$$\alpha_r = \sum_{j=r+1}^{p} m_j \alpha_j, \quad \text{for some integers } m_{r+1}, \dots, m_p.$$

(This is true for $p = 1$ as well with the convention $\sum_\phi = 0$.)

Take $m_0 \geqq m_j$, $\forall j \in \{r+1, \dots, p\}$. Then we have

$$\alpha_{rk} \leqq m_0(\alpha_{r+1,k} + \dots + \alpha_{pk}), \quad \forall k \in \{1, \dots, n\}.$$

Now, put

$$M = \sup_{z \in B} |u_{r+1}(z) \dots u_p(z)|^{m_0}, \quad m = \inf_{z \in B} |u_r(z)|,$$

and notice that $M < \infty$, $m > 0$.

For $z \in B$ we then get

$$|f_r(z)| = |u_r(z) z^{\alpha_r}| \geqq m|z^{\alpha_r}| \geqq m|z^{m_0(\alpha_{r+1} + \dots + \alpha_p)}|$$

$$\geqq \frac{m}{M} |u_{r+1}(z) \dots u_p(z)|^{m_0} |z^{\alpha_{r+1} + \dots + \alpha_p}|^{m_0} = \frac{m}{M} |f_{r+1}(z) \dots f_p(z)|^{m_0}.$$

Now, assume

(1) $$|f_j(z)| \geqq \varepsilon_j(\delta), \quad \forall j \in \{r+1, \dots, p\}.$$

It follows that

$$|f_r(z)| \geqq \frac{m}{M} (\varepsilon_{r+1} \dots \varepsilon_p)^{m_0}.$$

But for small enough $\delta > 0$ we have

$$\varepsilon_r(\delta) < \frac{m}{M}(\varepsilon_{r+1}(\delta)\dots\varepsilon_p(\delta))^{m_0},$$

since ε is an admissible path. But this means that the inequality $|f_r| > \varepsilon_r$ is implied by (1) and we have two cases:

(i) $r \in I$. Since the condition $|f_r| = \varepsilon_r$ contradicts (1), D^δ_{IJ} is empty.

(ii) $r \in J$. Since the condition $|f_r| > \varepsilon_r$ is implied by (1), it is superfluous.

The lemma is proved.

Proof of Theorem 4.2.1, continued. From now on we assume that $D^\delta_{IJ} \neq \emptyset$ when $\delta > 0$ is small and that all of the relations $|f_j| = \varepsilon_j$ ($> \varepsilon_j$) are needed to define D^δ_{IJ}. There is clearly no loss of generality in making these assumptions and the above lemma ensures that we have $\operatorname{rank}\alpha = p$.

Our next step concerns the invertible functions u_j. We shall see that a local biholomorphic coordinate change reduces them all to the constant 1.

Choose $k_1, \dots, k_p \in \{1, \dots, n\}$ such that, if we put

$$\alpha' = (\alpha_{jk})_{\substack{j = 1,\dots,p \\ k = k_1,\dots,k_p}}$$

and $\Delta = \det\alpha'$, we have $\Delta \neq 0$. For simplicity we re-label the coordinates so that $k_i = i$. Next, for $(j, k) \in \{1, \dots, p\}^2$, write

$$\Delta_{j,k} = (-1)^{j+k}\det\alpha'(j,k),$$

where $\alpha'(j, k)$ is obtained from α' by excluding the jth row and the kth column.

Cramer's rule allows us to calculate $(\alpha')^{-1} = (\alpha_{jk})^{-1} = (\beta_{jk})$ by putting $\beta_{jk} = \Delta_{kj}/\Delta$. From this it follows that

(2) $$\Delta^{-1}(\alpha_{j1}\Delta_{k1} + \dots + \alpha_{jp}\Delta_{kp}) = \delta^j_k,$$

δ^j_k being the Kronecker symbol ($\delta^j_k = 1$, $j = k$; $\delta^j_k = 0$, $j \neq k$).

We want to find a local biholomorphic coordinate transformation μ such that in the new coordinates $w = \mu(z)$ we will have

$$w^{\alpha_j} = u_j(z)z^{\alpha_j}, \quad j = 1, \dots, p.$$

We therefore, for each $j \in \{1, \dots, p\}$, pick a branch of $u_j^{1/\Delta}$ on B and define

$$v_j = u_1^{\Delta_{1j}/\Delta}\dots u_p^{\Delta_{pj}/\Delta}.$$

Using the nonvanishing functions v_j, we then construct our coordinate transformation μ as follows

$$w = \mu(z) = (z_1 v_1(z), \dots, z_p v_p(z), z_{p+1}, \dots, z_n).$$

Notice that

$$\det(\partial w_i/\partial z_j(0)) = v_1(0)\dots v_p(0) \neq 0,$$

so μ is invertible in some neighborhood U of the origin. By making a change of scale ein the z coordinates we can assume that $U \supset \bar{B}$ and by modifying the w coordinates in the same way (that is, by putting $w'_j = t_j w_j$ for large enough real numbers $t_j > 0$) we obtain that $\mu(\bar{B}) \supset \bar{B}$, where B is used to denote the unit polydisk in either set of coordinates. Observe that these operations are harmless since we are dealing with local problems.

It is then easy to see that the inverse of μ will be of the form

$$z = \mu^{-1}(w) = (w_1\tilde{v}_1(w), \dots, w_p\tilde{v}_p(w), w_{p+1}, \dots, w_n),$$

where $\tilde{v}_j$ are nonvanishing, holomorphic functions on B.

By (2) we have

$$v_1(z)^{\alpha_{j,1}}\dots v_p(z)^{\alpha_{j,p}} = u_j(z), \quad j = 1, \dots, p$$

and hence

$$w^{\alpha_j} = w_1^{\alpha_{j,1}}\dots w_n^{\alpha_{j,n}} = v_1(z)^{\alpha_{j,1}}\dots v_p(z)^{\alpha_{j,p}}\cdot z^{\alpha_j} = u_j(z)z^{\alpha_j},$$

in other words, $\mu(D_{ij}^\delta \cap B)$ is set-theoretically equal to

$$\{w \in \mu(B); |w|^{\alpha_i} = \varepsilon_i(\delta),\ i \in I; |w|^{\alpha_j} > \varepsilon_j(\delta),\ j \in J\}.$$

We now observe that

$$\int_{D_{IJ}^\delta} \lambda\psi = \int_{D_{IJ}^\delta \cap B} = \int_{\mu(D_{IJ}^\delta \cap B)} (\mu^{-1})^*(\lambda\psi)$$

so we just have to show that the pull-back $(\mu^{-1})^*(\lambda\psi)$ is of the same form as $\lambda\psi$ in order to eliminate the u_j's, i.e., we will be able to assume that they are all $\equiv 1$.

Recall that $\lambda(z) = a(z)/z^\gamma$, $a \in C^\infty(B)$, $\gamma \in \mathbf{N}^n$ and that ψ is a $(2n-s)$-form with C_0^∞ coefficients. We have

$$(\mu^{-1})^*(\lambda\psi) = (\mu^{-1})^*(\lambda)\cdot(\mu^{-1})^*(\psi)$$

and

$$(\mu^{-1})^*(\lambda)(w) = \frac{a \circ \mu^{-1}(w)}{\mu^{-1}(w)^\gamma} = \frac{a \circ \mu^{-1}(w)}{\tilde{v}_1(w)^{\gamma_1}\dots\tilde{v}_p(w)^{\gamma_p}\cdot w^\gamma} = \frac{\tilde{a}(w)}{w^\gamma},$$

if we put

$$\tilde{a} = \frac{a \circ \mu^{-1}}{\tilde{v}_1^\gamma \dots \tilde{v}_p^\gamma} \in C^\infty(B),$$

whereas $(\mu^{-1})^*(\psi)$ clearly is a form of the same type as ψ. It follows that it is enough to prove Theorem 4.2.1 in the case when $f_j(z) = z^{\alpha_j}$ and $\operatorname{rank} \alpha = p$.

We are considering limits of the following form

$$\lim_{\delta \to 0} \int_{D_{IJ}^\delta} \lambda \psi,$$

where ψ is a $(2n-s)$-form and in order to prove the existence and continuity of such limits there is of course no restriction in assuming ψ to consist of just one term

$$\psi = \psi(z) dz_A \wedge d\bar{z}_B \wedge (dz \wedge d\bar{z})_C.$$

LEMMA 4.2.3. *Let f be a holomorphic mapping $D \to \mathbf{C}^p$, where D is a domain in $\mathbf{C}^n$. Suppose that f is given by $f_j(z) = z^{\alpha_j}$, $\alpha_j \in \mathbf{N}^n$, $j = 1, \ldots, p$. Let $I = \{i_1, \ldots, i_s\} \subset \{1, \ldots, p\}$ and $J \subset \{1, \ldots, p\}$, $I \cap J = \emptyset$. Let λ be a semi-meromorphic function on D which is smooth outside $V_{I \cup J}$. Then, for any smooth $(2n-s)$-form*

$$\psi = \psi(z) dz_A \wedge d\bar{z}_B \wedge (dz \wedge d\bar{z}_C$$

which is such that either $|C| > n-s$ or $|C| = n-s$ and $\det \alpha_{I, A \cup B} = 0$, *one has*

$$\int_{D_{IJ}^\delta} \lambda \psi = 0, \quad 1 \geqq \delta > 0.$$

PROOF. If we just observe that $J' \subset J$ implies that D_{IJ}^δ is an open subset of $D_{IJ'}^\delta$, it becomes clear that the proof of Lemma 2.7 in Coleff and Herrera [7, p. 71] works in our case as well.

PROOF OF THEOREM 4.2.1, CONTINUED. It follows from Lemma 4.2.3 that we may assume that we can reorder the coordinates so that $A \cup B = \{1, \ldots, s\} = S$ and that $\alpha_{I,S}$ is invertible.

Let us also show that we can assume that none of the columns of α consists of only zeros. Indeed, suppose that there is a k such that $\alpha_{j,k} = 0$, $j = 1, \ldots, p$. Since we have $\det \alpha_{I,S} \neq 0$ it is clear that $k \notin S$ and hence $k \in C$. Therefore ψ may be written as $\tilde{\psi} \wedge dz_k \wedge d\bar{z}_k$. It is also immediate that

$$D_{IJ}^\delta = \tilde{D}_{IJ}^\delta \times \mathbf{C},$$

where

$$\tilde{D}_{IJ}^\delta = \pi(k)(D_{IJ}^\delta).$$

We then have (by Fubini's theorem)

$$\int_{D^\delta_{IJ}} \lambda\psi = \int_{\tilde{D}^\delta_{IJ}} \left[\int_{\mathbf{C}} \lambda\tilde{\psi} dz_k \wedge d\bar{z}_k \right],$$

where the inner integration is with respect to z_k. Since furthermore $\lambda(z) = a(z)/z^\gamma$ and $\gamma_k = 0$ it follows that

$$\int_{\mathbf{C}} \lambda\tilde{\psi} dz_k \wedge d\bar{z}_k = \frac{1}{z(k)^\gamma} \int_{\mathbf{C}} a(z)\tilde{\psi}(z) dz_k \wedge d\bar{z}_k$$

is of the form

$$\frac{\tilde{a}(z(k))}{z(k)^\gamma} \tilde{\tilde{\psi}}(z(k)),$$

where $\tilde{a} \in C^\infty$ and $\tilde{\tilde{\psi}}$ is a test form of degree $2(n-1)-s$ which depends continuously on ψ in the seminorms $p^K_{\alpha\beta}$. That is, we have effectively eliminated the coordinate z_k. This procedure can then be continued until all zero columns of α are disposed of.

Before we proceed any further, let us indicate that we intend to do. Roughly, the idea is to parametrize the tubes D^δ_{IJ} by using polar coordinates. In this way we obtain a kind of fibration of D^δ_{IJ} with fibers $[0, 2\pi]^n$. We then integrate along the fibers and it is precisely at this point that we get rid of the singularities of λ.

To construct the parametrization, then, we recall the notation $\varrho_j = |z_j|$ and consider the system of equations occurring in the definition of D^δ_{IJ}:

$$\begin{cases} \varrho^{\alpha_{i_1}} = \varepsilon_{i_1} \\ \vdots \\ \varrho^{\alpha_{i_s}} = \varepsilon_{i_s} \end{cases},$$

or equivalently

$$\begin{cases} \varrho_S^{\alpha_{i_1}} = \varepsilon_{i_1}/\varrho(S)^{\alpha_{i_1}} \\ \vdots \\ \varrho_S^{\alpha_{i_s}} = \varepsilon_{i_s}/\varrho(S)^{\alpha_{i_s}} \end{cases} \tag{3}$$

Since $\det \alpha_{I,S} \neq 0$ we can solve for $\varrho_1, \ldots, \varrho_s$ (in terms of $\varepsilon_I = (\varepsilon_{i_1}, \ldots, \varepsilon_{i_s})$ and $\varrho(S) = (\varrho_{s+1}, \ldots, \varrho_n)$) by taking logarithms of (3) and using Cramer's rule. We get

$$\begin{cases} \varrho_1 = \varepsilon_I^{\delta_1}/\varrho(S)^{\beta_1} \\ \vdots \\ \varrho_s = \varepsilon_I^{\delta_s}/\varrho(S)^{\beta_s}, \end{cases} \tag{4}$$

for some exponents

$$\delta_j = (\delta_{j1}, \ldots, \delta_{js}) \in \mathbf{Q}^s \quad \text{and} \quad \beta_j = (\beta_{j,1}, \ldots, \beta_{j,n-s}) \in \mathbf{Q}^{n-s}.$$

Calculating these exponents explicitly, one sees also that $\det \alpha_{I,S} \neq 0$ implies that none of the δ_j's, $1 \leqq j \leqq s$, is the zero-vector.

Now put

$$W_\delta = \pi^+(S)(D_{IJ} \cap B), \quad B^+(S) = \pi^+(S)(B).$$

Clearly $W_\delta \subset B^+(S)$. It is in fact easy to see that W_δ is an open subset of $B^+(S)$. Indeed, a point $z \in B$ belongs to D_{IJ}^δ precisely if it satisfies (3) and the following system of inequalities:

$$\begin{cases} \varrho^{\alpha_{j_1}} > \varepsilon_{j_1} \\ \vdots \\ \varrho^{\alpha_{j_t}} > \varepsilon_{j_t} \end{cases},$$

or inserting the values (4) for $\varrho_1, \ldots, \varrho_s$:

$$\text{(5)} \qquad \begin{cases} \varrho(S)^{\beta'_1} > \varepsilon_{j_1} \cdot \varepsilon_I^{\delta'_1} \\ \vdots \\ \varrho(S)^{\beta'_t} > \varepsilon_{j_t} \cdot \varepsilon_I^{\delta'_t} \end{cases}$$

for some $\beta'_r \in \mathbf{Q}^{n-s}$, $\delta'_r \in \mathbf{Q}^s$, $r = 1, \ldots, t$. The fact that $z \in B$ can, in view of (4), be expressed by the following:

$$\text{(6)} \qquad \begin{cases} \varrho(S)^{\beta_1} > \varepsilon_I^{\delta_1} \\ \vdots \\ \varrho(S)^{\beta_s} > \varepsilon_I^{\delta_s}. \end{cases}$$

Thus

$$D_{IJ}^\delta \cap B = \{z \in B\,; z \text{ satisfies (4) and (5)}\}$$

and the inequalities being invariant under $\pi^+(S)$ we conclude that

$$W_\delta = \{\varrho(S) \in B^+(S)\,; \varrho(S) \text{ satisfies (5) and (6)}\};$$

hence it is an open subset.

If we now use (4) to define a mapping

$$v_\delta \colon [0, 2\pi]^n \times W_\delta \to D_{IJ}^\delta$$

by prescribing

(7) $$(\theta_1, \ldots, \theta_n; \varrho(S)) \overset{\nu_\delta}{\mapsto} (\varrho_1(\varepsilon(\delta), \varrho(S))e^{i\theta_1}, \ldots, \varrho_s(\varepsilon(\delta), \varrho(S))e^{i\theta_s}, \varrho_{s+1}e^{i\theta_{s+1}}, \ldots, \varrho_n e^{i\theta_n})$$

we find that it is smooth and one-to-one. That is, up to a sign depending on the orientations we have obtained a parametrization of D^δ_{IJ}. Notice that in order to conclude that ν_δ is injective, we need the fact that none of the columns of α vanishes so that, on D^δ_{IJ}, $\varrho_j \neq 0$, $j = 1, \ldots, n$.

It follows that we can define a push-forward $\nu_{\delta *}$ (turning a form on $[0, 2\pi]^n \times W_\delta$ into a form on D^δ_{IJ}) by

$$\nu_{\delta *} = (\nu_\delta^{-1})^*.$$

In particular

$$\nu_{\delta *}(d\theta_1 \wedge \ldots \wedge d\theta_n \wedge d\varrho_{s+1} \wedge \ldots \wedge d\varrho_n) = \nu_{\delta *}(d\theta \wedge d\varrho(S))$$

is well defined on D^δ_{IJ} and since it never vanishes it follows that the restriction to D^δ_{IJ} of any smooth $(2n-s)$-form on D equals some smooth function times this form.

We now recall that our test form ψ can be assumed to be of the form

$$\psi = \psi(z)dz_A \wedge d\bar{z}_{S \setminus A} \wedge dz(S) \wedge d\bar{z}(S),$$

and we find that

$$\psi|_{D^\delta_{IJ}} = B_A(z)\psi(z)\nu_{\delta *}(d\theta \wedge d\varrho(S)),$$

where

(8) $$B_A(z) = k_A \nu_{\delta *}(\varrho_1 \ldots \varrho_n e^{i(\theta_A - \theta_{S \setminus A})}) = k_A \prod_{r \in A} z_r \prod_{r \in A \setminus S} \bar{z}_r \cdot \varrho_{s+1} \ldots \varrho_n,$$

with $\theta_A = \sum_{a \in A} \theta_a$, $\theta_{S \setminus A} = \sum_{a \in S \setminus A} \theta_a$ and k_A being a nonzero complex constant. (Note that B_A is continuous on B and smooth on $B \cap D^\delta_{IJ}$ for each $\delta \in]0, 1]$.) Next, put

(9) $$\psi'(z) = a(z)\psi(z),$$

where $a(z)$ is the smooth function that occurs in the expression for

$\lambda = a(z)/z^{\lambda}$. It is clear that

$$p^k_{\alpha\beta}(\psi') \leqq \sum_{\substack{|\gamma| \leq |\alpha| \\ |\delta| \leq |\beta|}} c^K_{\gamma\delta} p^K_{\gamma\delta}(\psi)$$

for some constants $c^K_{\gamma\delta}$ which are independent of ψ. It follows that in order to prove the continuity part of Theorem 4.2.1 it suffices to show that

$$R^{\lambda}_{IJ}(\psi) \leqq \sum_{|\alpha| + |\beta| \leq q^K} c^K_{\alpha\beta} p^K_{\alpha\beta}(\psi')$$

for some (other) constants $c^K_{\alpha\beta}$.

In other words, we shall prove that

$$\lim_{\delta \to 0} \int_{D^{\delta}_{IJ}} z^{-\gamma}\psi'(z) B_A(z) v_{\delta *}(d\theta \wedge d\varrho(S))$$

exists and that it depends continuously on ψ' in the seminorms $p^K_{\alpha\beta}$.

We now rewrite our integrals as follows

$$\int_{D^{\delta}_{IJ}} \lambda\psi = \int_{D^{\delta}_{IJ}} z^{-\gamma}\psi'(z) B_A(z) v_{\delta *}(d\theta \wedge d\varrho(S)) = \int_{W_{\delta}} \tilde{\psi}_{\delta}(\varrho(S)) d\varrho(S),$$

where

(10) $$\tilde{\psi}_{\delta}(\varrho(S)) = \int_{[0, 2\pi]^n} v^*_{\delta}(z^{-\gamma}\psi'(z) B_A(z)) d\theta.$$

In other words, we integrate along the fibers $[0, 2\pi]^n$ and it will turn out that all singularities disappear, i.e. $\tilde{\psi}_{\delta}$ is bounded. We shall in fact see that $\tilde{\psi}_0 = \lim_{\delta \to 0} \tilde{\psi}_{\delta}$ exists pointwise and that one has an estimate

(11) $$\sup_{1 \geq \delta > 0} \sup_{\varrho \in W_{\delta}} |\tilde{\psi}_{\delta}(\varrho)| \leqq \sum_{|\alpha| + |\beta| < q^K} c^K_{\alpha\beta} p^K_{\alpha\beta}(\psi') = M.$$

But first we take a look at what happens to W_{δ} as $\delta \to 0$.

LEMMA 4.2.4. *Let D be a domain in $\mathbf{C}^n$ and let $f: D \to \mathbf{C}^p$ be a holomorphic mapping of the form $f_j(z) = z^{\alpha_j}$, $\alpha_j \in \mathbf{N}^n$. Let $A, B \subset \{1, .., p\}$ with $|A| = s$ and assume* $\det \alpha_{A,S} \neq 0$, *where α is the $(p \times n)$-matrix having α_j as its rows and $S = \{1, \ldots, s\}$. Also, let $\varepsilon = \varepsilon(\delta)$ be an admissible path and put, for* $1 \geqq \delta > 0$,

$$W_{\delta} = \pi^{+}(S)(D^{\delta}_{AB} \cap B),$$

where $\pi^+(s)$ *and* D^δ_{AB} *are defined as before. Then*

(12) $$\lim_{\delta\to 0} m(W_\delta) = 0 \text{ or } 1, \text{ independently of } \varepsilon.$$

Here m denotes Lebesque measure on R^{n-s}_+.

PROOF. Suppose that $A = \{a_1, \ldots, a_s\}$ and $B = \{b_1, \ldots, b_t\}$. We have seen before that W_δ is an open subset of $B^+(S) = \pi^+(S)(B)$ and that one can in fact find exponents $\beta_j, \beta'_k \in \mathsf{Q}^{n-s}$ and $\delta_j, \delta'_k \in \mathsf{Q}^s$, $1 \leqq j \leqq s$, $1 \leqq k \leqq t$ ($\delta_j \neq (0, \ldots, 0)$) such that

$$W_\delta = \{\varrho(S) \in B^+(S); \varrho(S)^{\beta_j} > \varepsilon_A^{\delta_j}, 1 \leqq j \leqq s \text{ and}$$
$$\varrho(S)^{\beta'_k} > \varepsilon_{b_k} \cdot \varepsilon_A^{\delta'_k}, 1 \leqq k \leqq t\},$$

(cf. (5) and (6)).

Since ε is an admissible path it follows that

$$\lim_{\delta\to 0} \varepsilon_A^{\delta_j} = \lim_{\delta\to 0} \varepsilon_{a_1}^{\delta_{j_1}} \ldots \varepsilon_{a_s}^{\delta_{j_s}} = 0 \quad \text{or} \quad +\infty \quad \text{for all } j$$

and, similarly,

$$\lim_{\delta\to 0} \varepsilon_{b_k} \varepsilon_A^{\delta'_k} = 0 \quad \text{or} \quad +\infty \quad \text{for all } k.$$

Suppose that at least one of these limits, say $\lim \varepsilon_A^{\delta_j}$, is infinite. Let us see that this implies that $\lim_{\delta\to 0} m(W_\delta) = 0$. Indeed, since

$$m\left(\bigcup_{k=s+1}^{n} \{\varrho_k = 0\}\right) = 0,$$

it is enough to have

$$\lim_{\delta\to 0} m(W_\delta \setminus \cup \{\varrho_k = 0\}) = 0.$$

But this follows from the observation that if $\varrho(S) \in B^+(S) \setminus \cup \{\varrho_k = 0\}$, then $\varrho(S)^{\beta_j} < +\infty$ and thus, for small enough δ, the inequality $\varrho(S)^{\beta_j} > \varepsilon_A^{\delta_j}$ is not satisfied and $\varrho(S) \notin W_\delta$.

Now we suppose, on the contrary, that all the limits $\lim_{\delta\to 0} \varepsilon_A^{\delta_j}$ and $\lim_{\delta\to 0} \varepsilon_{b_k} \varepsilon_A^{\delta'_k}$ equal zero. It is then clear that if $\varrho(S) \in B^+(S) \setminus \cup \{\varrho_k = 0\}$, then $\varrho(S) \in W_\delta$ for small enough δ, and since $m(B^+(S) \setminus \cup \{\varrho_k = 0\}) = 1$ we get

$$\lim_{\delta\to 0} m(W_\delta) = 1.$$

To finish the proof we merely have to observe that the limits $\lim_{\delta \to 0} \varepsilon_A^{\delta_j}$ and $\lim_{\delta \to 0} \varepsilon_{h_k} \varepsilon_A^{\delta_k}$ do not depend on our particular choice of admissible path ε.

PROOF OF THEOREM 4.2.1, CONTINUED. In Coleff and Herrera [7, Lemma 2.4] there is given a kind of partial Taylor formula which says that a smooth function ψ' may be decomposed as

$$\psi'(z) = \sum_{j=1}^{n} \left(\sum_{\sigma + \tau < \gamma_j - 1} z_j^{\sigma} \bar{z}_j^{\tau} g^j(z(j)) \right) + \Psi'(z); \quad \Psi'(z) = \sum_{\mu + \nu = \gamma - 1} z^{\mu} \bar{z}^{\nu} \Psi'_{\mu\nu}(z),$$

$$(\gamma - 1 = (\gamma_1 - 1, \ldots, \gamma_n - 1)),$$

where all functions are smooth and $\Psi'_{\mu\nu}$ depend continuously on ψ' in the seminorms $p^K_{\alpha\beta}$. We need yet another lemma.

LEMMA 4.2.5. *Let $g \in C^{\infty}(B)$, $g(z) = g(z(j))$, that is, g does not depend on z_j. Let $a, b \in \mathbf{Z}^n$ with $b_j \neq 0$. Finally, for $1 \geqq \delta > 0$, let v_δ be the mapping defined by* (7). *Then*

$$\int_{[0,2\pi]^n} \varrho^a e^{ib\theta} v_\delta^*(g(z)) d\theta = 0.$$

PROOF. By Fubini's theorem we have

$$\int_{[0,2\pi]^n} \varrho^a e^{ib\theta} v_\delta^*(g) d\theta = \pm \int_{[0,2\pi]^{n-1}} \varrho^a e^{i(b_1\theta_1 + \ldots \hat{j} \ldots + b_n\theta_n)} v_\delta^*(g) \times$$

$$\times \left[\int_0^{2\pi} e^{ib_j\theta_j} d\theta_j \right] d\theta_1 \wedge \ldots \underset{j}{\wedge} \ldots \wedge d\theta_n.$$

But

$$b_j \neq 0 \Rightarrow \int_0^{2\pi} e^{ib_j\theta_j} d\theta_j = 0$$

and the lemma is proved.

PROOF OF THEOREM 4.2.1, CONTINUED. Using the decomposition of ψ' which precedes Lemma 4.2.5 and recalling (10) we have

$$\tilde{\psi}_\delta(\varrho(S)) = \sum_{j=1}^{n}\left[\sum_{\sigma+\tau<\gamma_j-1}\ \int_{[0,2\pi]^n} v_\delta^*(z^{-\gamma}z_j^\sigma \bar{z}_j^\tau g^j(z(j))B_A(z))d\theta\right] +$$

$$+ \int_{[0,2\pi]^n} v_\delta^*(z^{-\gamma}\Psi'(z)B_A(z))d\theta = \sum I_{j\sigma\tau} + I'.$$

Each $I_{j\sigma\tau}$ can be written

$$I_{j\sigma\tau} = \int_{[0,2\pi]^n} \varrho^{-\gamma}e^{-i\gamma\theta}\varrho_1 \dots \varrho_n e^{i(\theta_A-\theta_{S\setminus A})}\varrho_j^{\sigma+\tau}e^{i(\sigma\theta_j-\tau\theta_j)}v_\delta^*(g^j)d\theta$$

$$= \int_{[0,2\pi]^n} \varrho^a \exp\{i(-\gamma\theta+\theta_A-\theta_{S\setminus A}+\sigma\theta_j-\tau\theta_j)\}v_\delta^*(g^j)d\theta.$$

If $j\in A$, we get $I_{j\sigma\tau}=0$ by Lemma 4.2.5, since $-\gamma_j+1+\sigma-\tau \neq 0$.
If $j\in S\setminus A$, we get $I_{j\sigma\tau}=0$, since $-\gamma_j-1+\sigma-\tau\neq 0$.
If $j\notin S$, we get $I_{j\sigma\tau}=0$, since $-\gamma_j+\sigma-\tau\neq 0$.
It follows that

$$\tilde{\psi}_\delta(\varrho(S)) = \int_{[0,2\pi]^n} v_\delta^*(z^{-\gamma}\Psi'(z)\,B_A(z))d\theta \tag{13}$$

$$= \sum_{\mu+\nu=\gamma-1}\ \int_{[0,2\pi]^n} k_A e^{ib_{\mu\nu}\theta}v_\delta^*(\Psi'_{\mu\nu}(z))d\theta, \quad \text{for some } b_{\mu\nu}\in \mathbf{Z}^n$$

and hence

$$|\tilde{\psi}_\delta| \leqq (2\pi)^n k_A \sum_{\mu+\nu=\gamma-1}\ \sup_{z\in B}|\Psi'_{\mu\nu}(z)|.$$

Since the right-hand side is independent of δ and ϱ but depends continuously on ψ', the estimate (11) follows.

As for the pointwise convergence of $\tilde{\psi}_\delta$ as $\delta\to 0$ we observe that by (13), it is enough to show that each

$$\lim_{\delta\to 0} v_\delta^*(\Psi'_{\mu\nu}) = \lim_{\delta\to 0}\Psi'_{\mu\nu}(v_\delta(\theta,\varrho(S))$$

exists and does not depend on the choice of admissible path ε. Since $\Psi'_{\mu\nu}$

is smooth and compactly supported in B it suffices to have either

(i) $\lim_{\delta\to 0} v_\delta$ exists (in fact equals $(0,\ldots,0,\varrho_{s+1}e^{i\theta_{s+1}},\ldots,\varrho_n e^{i\theta_n})$) for θ and $\varrho(S)$ fixed

or

(ii) if δ is small enough, then $v_\delta \notin B$.

To see this, merely recall the definition of v_δ

$$v_\delta(\theta,\varrho(S)) = \left(\frac{\varepsilon_I^{\delta_1}}{\varrho(S)^{\beta_1}}e^{i\theta_1},\ldots,\frac{\varepsilon_I^{\delta_s}}{\varrho(S)^{\beta_s}}e^{i\theta_s},\varrho_{s+1}e^{i\theta_{s+1}},\ldots,\varrho_n e^{i\theta_n}\right),$$

and notice that, for $1 \leqq j \leqq s$, $\varepsilon_I^{\delta_j} \to 0$ or ∞ as $\delta \to 0$, since ε is an admissible path and no $\delta_j = (0,\ldots,0)$. Observe also that $\lim \varepsilon_I^{\delta_j}$ (and hence $\lim \tilde{\psi}_\delta$) does not depend on the choice of admissible path.

We recall that

$$\int_{D_{IJ}^\delta} \lambda\psi = \int_{W_\delta} \tilde{\psi}_\delta$$

and by (12) we have two cases:

(A) $m(W_\delta) \to 0$:

We get

$$\left|\int_{W_\delta} \tilde{\psi}_\delta\right| \leqq \int_{W_\delta} |\tilde{\psi}_\delta| \leqq Mm(W_\delta) \quad \text{(by (11))}$$

and hence

$$R_{IJ}^\lambda(\psi) = \lim_{\delta\to 0} \int_{D_{IJ}^\delta} \lambda\psi = 0.$$

(B) $m(W_\delta) \to 1$:

Then $m(B^+(S)\setminus W_\delta) \to 0$, and since

$$\int_{W_\delta} \tilde{\psi}_\delta = \int_{B^+(S)} \tilde{\psi}_\delta - \int_{B^+(S)\setminus W_\delta} \tilde{\psi}_\delta$$

we conclude by (A) and the pointwise convergence $\tilde{\psi}_\delta \to \tilde{\psi}_0$ that

$$R_{IJ}^{\lambda}(\psi) = \int_{B^+(S)} \tilde{\psi}_0.$$

Since

$$|\tilde{\psi}_0| \leqq \sum_{|\alpha|+|\beta| \leq q^K} c_{\alpha\beta}^K p_{\alpha\beta}(\psi')$$

the existence and continuity parts of the theorem follow.

To see that $R_{IJ}^{\lambda}(\psi)$ is independent of the particular choice of admissible ε, we just observe that $\tilde{\psi}_0$ and $\lim(W_\delta)$ are independent.

Finally we wish to show that the current R_{IJ}^{λ} is of bidegree $(0, s)$, where $s = |I|$. To this end, and for future use, we give a lemma.

Lemma 4.2.6. *For every* $\delta \in]0, 1]$, *let* $W_\delta = \pi^+(S)(B \cap D_{IJ}^{\delta})$, *and let* $\tilde{\psi}_\delta : W_\delta \to \mathbf{C}$, $\psi' : B \to \mathbf{C}$, *and* $B_A : B \to \mathbf{C}$ *be the mappings defined by* (10), (9), *and* (8) *respectively. If, for some* $j \in S$, *either* ψ' *or* B_A *contains the factor* $\bar{z}_j$, i.e. *if* $\psi' B_A = \bar{z}_j \psi'' \varrho_{s+1} \cdots \varrho_n$ *for some* $\psi'' \in C_0^\infty(B)$, *then*

$$\lim_{\delta \to 0} \int_{W_\delta} \tilde{\psi}_\delta = 0.$$

Proof. In view of (11), we may assume that $\lim_{\delta \to 0} m(W_\delta) \neq 0$, m being Lebesgue measure on $B^+(S)$. Let us see that this implies that $\varrho_j = \varrho_j(\varepsilon(\delta), \varrho(S)) \to 0$ as $\delta \to 0$ for $\varrho(S)$ fixed. From (4) we have

$$\varrho_j = \frac{\varepsilon_I^{\delta_J}}{\varrho(S)^{\beta_J}},$$

and just as when we proved Lemma 4.2.4, we see that if $\varepsilon_I^{\delta_J}$ tends to infinity we have $\lim_{\delta \to 0} m(W_\delta) = 0$ contrary to our assumption. We must therefore have $\varepsilon_I^{\delta_J} \to 0$ and hence $\varrho_j \to 0$.

The decomposition which precedes Lemma 4.2.5 gives

$$\psi'' = \sum_{k=1}^{n} \left(\sum_{\sigma+\tau < \gamma_k} z_k^\sigma \bar{z}_k^\tau g^k(z(k)) \right) + \Psi'',$$

where

$$\Psi'' = \sum_{\mu+\nu=\gamma} z^\mu z^\nu \Psi''_{\mu\nu}$$

and hence

$$\tilde{\psi}_\delta = \sum_{k=1}^{n}\left(\sum_{\sigma+\tau<\gamma_k}\int_{[0,2\pi]^n} \varrho^{-\gamma}e^{-i\gamma\theta}\varrho_j e^{-i\theta_j}\varrho_k^\sigma e^{i\sigma\theta_k}\varrho_k^\tau e^{-i\tau\theta_k}\varrho_{s+1}\ldots\varrho_n v_\delta^*(g^k)d\theta\right)+$$

$$+\int_{[0,2\pi]^n} \varrho^{-\gamma}e^{-i\gamma\theta}\varrho_j^{-i\theta_j}\varrho_{s+1}\ldots\varrho_n v_\delta^*(\Psi'')d\theta = \sum I_{k\sigma\tau}+I''.$$

Now,

$$I_{k\sigma\tau} = \int_{[0,2\pi]^n} \varrho^a e^{ib\theta}v_\delta^*(g^k)d\theta, \quad \text{for some } a,b\in \mathbf{Z}^n$$

with

$$b_k = -\gamma_k+\sigma-\tau \qquad \text{if } j\neq k$$

and

$$b_k = -\gamma_k-1+\sigma-\tau \quad \text{if } j=k.$$

Since in either case $b_k<0$ and since g^k is independent of z_k, Lemma 4.2.5 implies that $I_{k\sigma\tau}=0$. Considering the remaining term I'' we see that it can be written

$$I'' = \sum I''_{\mu\nu},$$

where

$$I''_{\mu\nu} = \int_{[0,2\pi]^n} \varrho^{-\gamma}e^{-i\gamma\theta}\varrho_j e^{-i\theta_j}\varrho^\mu e^{i\mu\theta}\varrho^\nu e^{-i\nu\theta}\varrho_{s+1}\ldots\varrho_n v_\delta^*(\Psi''_{\mu\nu})d\theta$$

$$= \varrho_j\int_{[0,2\pi]^n} \varrho^{\mu+\nu-\gamma}\varrho_{s+1}\ldots\varrho_n e^{ib\theta}v_\delta^*(\Psi''_{\mu\nu})d\theta, \quad \text{for some } b\in\mathbf{Z}^n,$$

and since $\mu+\nu=\gamma$ and $\sup|\Psi''_{\mu\nu}| = M_{\mu\nu}<\infty$, we get

$$|I''| \leqq \sum|I''_{\mu\nu}| \leqq \varrho_j(2\pi)^n\sum M_{\mu\nu}.$$

It follows that $\lim_{\delta\to 0}|I''|=0$. Since furthermore

$$\left|\int_{W_\delta}\tilde{\psi}_\delta\right| \leqq \int_{B^+(S)} |\tilde{\psi}_\delta| \leqq |I''|$$

the lemma is proved.

To prove that R_{IJ}^λ is of bidegree $(0,s)$ we assume once again that ψ is of the form

$$\psi(z)dz_A \wedge d\bar{z}_B \wedge (dz \wedge d\bar{z})_C,$$

where $B = S \setminus A$.

It suffices to prove that $R^{\lambda}_{IJ}(\psi) = 0$, if $B \neq 0$. Suppose therefore that $j \in B \subset S$. Recalling (8) we find that the conditions in Lemma 4.2.6 are fulfilled and hence

$$\lim_{\delta \to 0} \int_{W_\delta} \tilde{\psi}_\delta = 0.$$

Finally we remember that

$$R^{\lambda}_{IJ}(\psi) = \lim_{\delta \to 0} \int_{W_\delta} \tilde{\psi}_\delta.$$

The proof of Theorem 4.2.1 is complete.

4.3. The general case.

In this section we prove a result analogous to Theorem 4.2.1, but without the assumption of normal crossings. The idea is to resolve the singularities of

$$V_f = \left\{ \prod_{j=1}^{p} f_j(z) = 0 \right\}$$

so as to obtain normal crossings and then apply Theorem 4.2.1. Notice that we make no assumptions about the dimension of $V = \{f_1(z) = \ldots = f_p(z) = 0\}$. This would be pointless, since even if we knew, say, that $\dim_{\mathbf{C}} V = n-p$, we still would not be able to say anything about the dimension of the desingularization of V. In other words: we cannot in general have both complete intersection and normal crossings so we settle for the latter.

Let us state the general result:

Theorem 4.3.1. *Let D be a domain in $\mathbf{C}^n$ and f a holomorphic mapping $D \to \mathbf{C}^p$ defined by $z \mapsto (f_1(z), \ldots, f_p(z))$. Suppose that I and J are disjoint subsets of $\{1, \ldots, p\}$ with $|I| = s$. Let λ be a semimeromorphic function on D which is smooth on the set $D \setminus V_{I \cup J}$ and let $\varepsilon :]0, 1] \to \mathbf{R}^p_>$ be an admissible path. Then, for any $(2n-s)$-form ψ with coefficients in $C^\infty_0(D)$, the following limit exists and does not depend on the choice of ε*

$$R_{IJ}^{\lambda,f}(\psi) = R_{IJ}^{\lambda}(\psi) = \lim_{\delta \to 0} \int_{D_{IJ}^{\delta}(\varepsilon,f)} \lambda\psi.$$

In fact, R_{IJ}^{λ} *defines a current of bidegree* $(0,s)$.

PROOF. The statements in the theorem are of a local nature and it is therefore enough to consider test forms ψ with support in some small neighborhood U of the origin. If U is small enough (which we may assume) one can find a complex manifold $\tilde{U}$ of dimension n and a proper holomorphic mapping $\pi: \tilde{U} \to U$ such that

(i) π induces an isomorphism $\tilde{U} \setminus \pi^{-1}(V_f) \to U \setminus V_f$

and

(ii) $\pi^{-1}(V_f)$ has normal crossings in $\tilde{U}$ (Hironaka [20]).

The latter property means precisely that for any point $x \in \tilde{U}$ there is a neighborhood $\mathscr{V}_x$ of x and a system of coordinates $(w_1, \ldots, w_n)$ on $\mathscr{V}_x$ centered at x such that

$$f_j \circ \pi(w) = u_j(w) w^{\alpha_j} \tag{14}$$

for some vectors $\alpha_j = (\alpha_{j1}, \ldots, \alpha_{jn}) \in \mathbf{N}^n$ and holomorphic functions u_j, invertible in some neighborhood of the origin. For simplicity we assume that this neighborhood (where the u_j's never vanish) contains $\bar{B}_x$, B_x being the unit ball around x in the coordinates w. (This can easily be brought about by a change of scale if necessary.) Since $\operatorname{supp}\psi$ is a compact subset of U and π is a proper map it follows that $\pi^{-1}(\operatorname{supp}\psi)$ is a compact subset of $\tilde{U}$. Hence it is covered by a finite number of balls $B_{x_1}, \ldots, B_{x_N}$ as above. Let $\{\eta_j\}_{j=1,\ldots,N}$ be a partition of unity subordinate to the covering $\{B_{x_j}\}_{j=1,\ldots,N}$. Now, $\eta_j(w)\lambda \circ \pi(w)$ is a semimeromorphic function on B_{x_j}, whose singular set is contained in

$$B_{x_j} \cap \pi^{-1}(V_f) = \{w \in B_{x_j}; w^{\alpha_1} \ldots w^{\alpha_p} = 0\}.$$

It is clear that we can find $\gamma_j \in \mathbf{N}^n$ such that

$$w^{\gamma_j} = 0 \Rightarrow w^{\alpha_1} \ldots w^{\alpha_p} = 0$$

and

$$w^{\gamma_j} \eta_j(w) \lambda \circ \pi(w) \in C_0^\infty(B_{x_j}).$$

Hence $\eta_j \cdot (\lambda \circ \pi)$ is of the form $a_j(w)/w^{\gamma_j}$ on B_{x_j}, where $a_j(w) \in C_0^\infty(B_{x_j})$.

We now observe that

$$D^{\delta}_{IJ}(f) \cap \operatorname{supp}\psi \subset U \setminus V_f$$

and use the fact that $\pi: \tilde{U} \setminus \pi^{-1}(V_f) \to U \setminus V_f$ is an isomorphism to conclude that

$$\int_{D^{\delta}_{IJ}} \lambda\psi = \int_{\pi^{-1}(D^{\delta}_{IJ}(f) \cap \operatorname{supp}\psi)} \pi^*(\lambda\psi) = \int_{D^{\delta}_{IJ}(f\circ\pi) \cap \pi^{-1}(\operatorname{supp}\psi)} (\lambda\circ\pi)\cdot\pi^*(\psi).$$

Then, employing the covering $\{B_{x_j}\}$ and the corresponding functions η_j, we deduce that

$$\lim_{\delta\to 0} \int_{D^{\delta}_{IJ}(f)} \lambda\psi = \lim_{\delta\to 0} \sum_{j=1}^{N} \int_{D^{\delta}_{IJ}(f\circ\pi)\cap B_{x_j}} \frac{a_j(w)}{w^{\gamma_j}}\pi^*(\psi) \tag{15}$$

and it obviously suffices to consider each term separately. But on B_{x_j} the functions $f_j \circ \pi$ are given by (14) so we can infer from Theorem 4.2.1 that the above limit exists and depends continuously on $a_j\pi^*(\psi)$ in the seminorms $p^K_{\alpha\beta}$, $K \subset B_{x_j}$. All that remains, is to observe that $a_j\pi^*(\psi)$ itself depends continuously on ψ in the seminorms $p^K_{\alpha\beta}$, $K \subset U$ and that it is of the same bidegree as ψ. The theorem follows.

Theorem 4.3.1 can be generalized in an obvious fashion by allowing λ to be a semimeromorphic form. One obtains the following result:

COROLLARY 4.3.2. *Let D be a domain in $\mathbf{C}^n$ and f a holomorphic mapping $D \to \mathbf{C}^p$. Let λ be a semimeromorphic form of bidegree (q,r) which is smooth on the set $D \setminus V_{I\cup J}$ and let $\varepsilon:]0,1] \to \mathbf{R}^p_>$ be an admissible path. Then, for any $(2n-s-q-r)$-form ψ with coefficients in $C^\infty_0(D)$, the following limit exists*

$$R^{\lambda,f}_{IJ}(\psi) = R^{\lambda}_{IJ}(\psi) = \lim_{\delta\to 0} \int_{D^{\delta}_{IJ}(\varepsilon,f)} \lambda \wedge \psi.$$

In fact, R^{λ}_{IJ} defines a current of bidegree $(q, r+s)$.

4.4. FURTHER PROPERTIES IN THE CASE OF COMPLETE INTERSECTIONS.

We prove there a few more results concerning the currents $R^{\lambda,f}_{IJ}$. None of them are however true for general holomorphic maps $f: D \to \mathbf{C}^p$ (cf. Example 7.1.3), so we have to impose a condition on f. To do this we need a definition.

DEFINITION 4.4.1. A holomorphic map $f: D \to \mathbf{C}^p$ is said to be *a complete intersection* if $V = f^{-1}(0)$ is of codimension p in D.

We shall demand that f be a complete intersection.

Our first result concerns the case, when the semimeromorphic form λ is such that one of the conditions $|f_j| > \varepsilon_j$ is unnecessary in order to avoid the polar set of λ.

PROPOSITION 4.4.2. *Let D be a domain in $\mathbf{C}^n$ and f a complete intersection $D \to \mathbf{C}^p$. Let $I, J \subset \{1, \ldots, p\}$, $I \cap J = \emptyset$ and take $j \in J$. Let λ be a semimeromorphic form of bidegree (q, r) on D which is smooth outside $V_{I \cup J'}$, where $J' = J \setminus \{j\}$. Then the two currents R^{λ}_{IJ} and $R^{\lambda}_{IJ'}$ are equal.*

PROOF. First we note that, if ψ is a test form of bidegree $(n-q, n-r-s)$, (and both currents give zero on all other test forms), then $\lambda \wedge \psi$ may be rewritten as $\hat{\lambda} \wedge \hat{\psi}$, where $\hat{\lambda}$ is a semimeromorphic function and $\hat{\psi}$ a test form of bidegree $(n, n-s)$. Since

$$R^{\lambda}_{I,J}(\psi) = R^{\hat{\lambda}}_{I,J}(\hat{\psi}),$$

we can confine ourselves to the case, when λ is a function, i.e. the given currents both have bidegree $(0, s)$. By exactly the same reasonings as in the proof of Theorem 4.3.1 we find that, for a given $(n, n-s)$-form ψ, $R^{\lambda}_{I,J}(\psi)$ can be decomposed into a finite sum of terms of the form

$$\lim_{\delta \to 0} \int_{D^{\delta}_{I,J}(f \circ \pi)} \frac{a(w)}{w^{\gamma}} \pi^*(\psi)$$

(c.f. (15)), where $\pi: \tilde{U} \to U \supset \operatorname{supp} \psi$ is a local desingularization of V_f and $a(w)$ is a smooth function with compact support near the origin in the local coordinates w on $\tilde{U}$. Furthermore, on $\operatorname{supp} a$ the holomorphic mapping $f \circ \pi: \tilde{U} \to \mathbf{C}^p$ is of the form

$$(f \circ \pi)_k(w) = u_k(w) w^{\alpha_k}, \quad k = 1, \ldots, p$$

for some invertible, holomorphic functions u_k and vectors $\alpha_k \in \mathbf{N}^n$. The corresponding decomposition for $R^{\lambda}_{IJ'}(\psi)$ is of course identical except that the integration is carried out on $D^{\delta}_{I,J'}(f \circ \pi)$ instead. (Note that since λ is smooth outside $V_{I \cup J'}$, it follows that $a(w)/w^{\gamma}$ is bounded on $D^{\delta}_{I,J'}$.) It is therefore sufficient to prove that

$$\lim_{\delta \to 0} \int_{D^{\delta}_{I,J}(f \circ \pi)} \tilde{\lambda} \pi^*(\psi) = \lim_{\delta \to 0} \int_{D^{\delta}_{I,J'}(f \circ \pi)} \tilde{\lambda} \pi^*(\psi), \tag{16}$$

where we have put

$$\tilde{\lambda}(w) = \frac{a(w)}{w^{\gamma}}.$$

Now, since the functions $(f \circ \pi)_k(w)$ are of the same form as those considered in section 4.2 we can adopt the same way of reasoning once again. First we see that we may assume that the $(p \times n)$-matrix α (obtained by using the α_k's as rows) is of maximal rank (cf. Lemma 4.2.2). Then we use this fact to construct a coordinate transformation such that the u_k's become $\equiv 1$, for $k = 1, \ldots, p$.

$\pi^*(\psi)$ is a smooth $(n, n-s)$-form on B (note that we write B for the unit polycylinder in both the coordinates w and z) so it is of the form

$$\pi^*(\psi) = \sum_{|A| = s} \psi_A(w) dw_A \wedge dw(A) \wedge d\bar{w}(A),$$

where ψ_A are C^∞-functions on B. This of course gives a decomposition of both sides of (16) and it suffices to prove the equality termwise. That is, if we reorder the coordinates so as to have $A = S$ and put $\psi_S = \psi$, what we shall prove is

$$\lim_{\delta \to 0} \int_{D^\delta_{I,J}(f \circ \pi)} \tilde{\lambda}\psi(w) dw \wedge d\bar{w}(S) = \lim_{\delta \to 0} \int_{D^\delta_{I,J'}(f \circ \pi)} \tilde{\lambda}\psi(w) dw \wedge d\bar{w}(S). \tag{17}$$

We then use Lemma 4.2.3 to conclude that both sides of (17) are equal to zero, if $\det \alpha_{I,S} = 0$, where

$$\alpha_{I,S} = (\alpha_{jk})_{\substack{j \in I \\ k \in S}}.$$

Using the notation $\varrho_j = |w_j|$, we see that the system (3) must be satisfied by a point w in order for it to belong to either of the sets $D^\delta_{I,J}(f \circ \pi)$ or $D^\delta_{I,J'}(f \circ \pi)$. Since we may assume $\det \alpha_{I,S} \neq 0$, we obtain (4).

We can of course also assume $\operatorname{supp} \tilde{\lambda} \subset B$, by changing the scale if necessary, so that, in fact, the integration in (17) is only over $D^\delta_{I,J} \cap B$ and $D^\delta_{I,J} \subset B$ respectively. Then define $\pi^+(S)$ as before and put

$$W_\delta = \pi^+(S)(D^\delta_{I,J} \cap B), \quad W'_\delta = \pi^+(S)(D^\delta_{I,J'} \cap B), \quad B^+(S) = \pi^+(S)(B).$$

W_δ will then be the open subset of $B^+(S)$ defined by the inequalities (5) and (6). If we assume that $j = j_r$, where $\{j_1, \ldots, j_t\} = J$, then W'_δ is seen to be the open subset of $B^+(S)$ defined by (5′) and (6), where (5′) is the system of inequalities obtained from (5) by neglecting the rth inequality. In other words

$$W_\delta = \{\varrho(S) \in W'_\delta\,;\, \varrho(S)^{\beta'_r} > \varepsilon_{j_r}\, \varepsilon_I^{\delta'_r}\}, \tag{18}$$

so W_δ is an open subset of W'_δ.

Next, we define a map

$$v_\delta : [0, 2\pi]^n \times W'_\delta \to D^\delta_{I,J'}$$

by (7). It will then be a parametrization of $D^\delta_{I,J'}$ and its restriction to $[0, 2\pi]^n \times W_\delta$ (which we also denote by v_δ) is a parametrization of $D^\delta_{I,J}$.

Continuing to follow the argument from Section 4.2 we define the function $\tilde{\psi}_\delta$ on W'_δ (and hence on W_δ) by (10) and rewrite (17) as

$$\lim_{\delta \to 0} \int_{W_\delta} \tilde{\psi}_\delta = \lim_{\delta \to 0} \int_{W'_\delta} \tilde{\psi}_\delta. \tag{19}$$

Recalling the estimate (11) we see that these two limits are equal if

$$\lim_{\delta \to 0} m(W_\delta) = \lim_{\delta \to 0} m(W'_\delta),$$

where m denotes Lebesgue measure on $B^+(S)$. So the proposition follows in that case. We may therefore assume that

$$\lim_{\delta \to 0} m(W_\delta) < \lim_{\delta \to 0} m(W'_\delta) \tag{20}$$

(remember that $W_\delta \subset W'_\delta$, so $m(W_\delta) \leqq m(W'_\delta)$). We shall show that in this case both sides of (19) equal zero and thereby finish the proof of the proposition.

Lemma 4.2.4 shows that both $\lim_{\delta \to 0} m(W_\delta)$ and $\lim_{\delta \to 0} m(W'_\delta)$ equal either zero or one. The inequality (20) then implies that $\lim_{\delta \to 0} m(W_\delta) = 0$ and hence

$$\lim_{\delta \to 0} \int_{W_\delta} \tilde{\psi}_\delta = 0.$$

Consequently, it remains to show that

$$\lim_{\delta \to 0} \int_{W'_\delta} \tilde{\psi}_\delta = 0.$$

To do this we start by proving that (20) implies that $\alpha_{jk} \neq 0$ for some $k \in S$ (here $j = j_r$ is the same index which occurs in the formulation of the proposition and in (18)). Suppose therefore, on the contrary, that $\alpha_{jk} = 0$ for all $k \in S$ and observe that the inequality $\varrho^{\alpha_{j_r}} > \varepsilon_{j_r}$ may then be rewritten $\varrho(S)^{\alpha_{j_r}} > \varepsilon_{j_r}$ and (18) becomes

$$W_\delta = \{\varrho(S) \in W'_\delta\,;\, \varrho(S)^{\alpha_{j_r}} > \varepsilon_{j_r}\}.$$

Letting $\delta \to 0$ we obtain

$$\lim_{\delta \to 0} m(W'_\delta \setminus W_\delta) = \lim_{\delta \to 0} m(B^+(S) \cap \{\varrho(S)^{\alpha_{j_r}} < \varepsilon_{j_r}\}) = 0$$

(recall that $\operatorname{rank} \alpha = p$ so $\alpha_{j_r} \neq (0, \ldots, 0)$). Hence

$$\lim_{\delta \to 0} m(W'_\delta) = \lim_{\delta \to 0} m(W_\delta)$$

contradicting our assumption (20).

We will use the following notation

$$P_A = \{w \in B\,;\, w_k = 0,\ k \in A\}, \quad \text{for} \quad A \subset \{1, \ldots, n\}.$$

The fact that $\det \alpha_{I,S} \neq 0$ implies of course that none of the rows of $\alpha_{I,S}$ consists of only zeros. Hence

$$\begin{aligned} P_S &\subset \{u_{i_1}(w)w^{\alpha_{i_1}} = \ldots = u_{i_s}(w)w^{\alpha_{i_s}} = 0\} \\ &= \{(f \circ \pi)_{i_1}(w) = \ldots = (f \circ \pi)_{i_s}(w) = 0\} = \bigcap_{i \in I} \pi^{-1}(V_i). \end{aligned}$$

Since furthermore $\alpha_{jk} > 0$ for some $k \in S$ we also have

$$P_S \subset \pi^{-1}(V_j).$$

It follows that

$$\pi(P_S) \subset \left\{\bigcap_{i \in I} V_i\right\} \cap V_j$$

and, finally making use of the assumption that f is a complete intersection, we see that

$$\dim_{\mathbb{C}} \left\{\bigcap_{i \in I} V_i\right\} \cap V_j < n - s.$$

We are going to end the proof of the proposition by showing that $\pi(P_S)$ being contained in an analytic variety of dimension $< n-s$ implies that

$$\lim_{\delta \to 0} \int_{W'_\delta} \tilde{\psi}_\delta = \lim_{\delta \to 0} \int_{D^\delta_{I,J'}(J \circ \pi)} \tilde{\lambda}\psi(w) dw \wedge dw(S) = 0.$$

First we return to our original test form ψ and observe that it may be written as a finite sum

$$\psi = \sum_\beta \xi_\beta \wedge \bar{\omega}_\beta,$$

where ξ_β are $(n, 0)$-forms with C_0^∞-coefficients and ω_β are $(n-s, 0)$-forms with holomorphic coefficients. Since $\pi(P_S)$ is contained in an analytic variety of complex dimension $< n-s$ it follows that the holomorphic $(n-s, 0)$-forms ω_β become zero when restricted to $\pi(P_S)$. This then implies that

$$\pi^*(\omega_\beta)|_{P_S} \equiv 0.$$

But sine $\pi^*(\omega_\beta)$ is of the form

$$\pi^*(\omega_\beta)|_B = \sum_{|M| = s} \varphi^\beta_M(w) dw(M),$$

with φ^β_M holomorphic on B, we have

$$\pi^*(\omega_\beta)|_{P_S} = \varphi^\beta_S(w) dw(S)$$

so we conclude that $\varphi^\beta_S \equiv 0$ on P_S. From this it now follows that we have

(21) $$\varphi^\beta_S(w) = \sum_{i \in S} w_i \varphi^\beta_{S,i}(w) \quad \text{for some } \varphi^\beta_{S,i}, \text{ holomorphic on } B.$$

We also have

$$\pi^*(\psi) = \sum_\beta \pi^*(\xi_\beta \wedge \bar{\omega}_\beta) = \sum_\beta \pi^*(\xi_\beta) \wedge \pi^*(\bar{\omega}_\beta) = \sum_\beta \pi^*(\xi_\beta) \wedge \overline{\pi^*(\omega_\beta)}$$

and hence we see that the term of $\pi^*(\psi)$ which we are considering, namely $\psi(w) dw \wedge d\bar{w}(S)$, can be written

$$\psi(w) dw \wedge d\bar{w}(S) = \sum_\beta \pi^*(\xi_\beta) \wedge \bar{\varphi}^\beta_S(w) d\bar{w}(S).$$

Recalling (21) we see that the function $\psi(w)$ is of the form

(22) $$\psi(w) = \sum_{i \in S} \psi_i(w),$$

where $\psi_i(w) = \bar{w}_i \hat{\psi}_i(w)$ for some smooth functions $\hat{\psi}_i$.

Applying Lemma 4.2.6 to each term of (22) and observing that

$$\lim_{\delta \to 0} \int_{W'_\delta} \tilde{\psi}_{\delta, i} = \lim_{\delta \to 0} \int_{D^\delta_{IJ'}} \tilde{\lambda}\psi_i dw \wedge dw(S),$$

where $\tilde{\psi}_{\delta,i}$ is defined as before via (9) and (10) (except that ψ is replaced by ψ_i), we conclude that

$$\lim_{\delta \to 0} \int_{D_{IJ}^{\delta}} \bar{\lambda}\psi_i dw \wedge dw(S) = 0.$$

Finally, this implies, by (22), that

$$\lim_{\delta \to 0} \int_{D_{IJ}^{\delta}} \bar{\lambda}\psi dw \wedge dw(S) = 0$$

and the proposition follows.

We now formulate another, very similar result.

PROPOSITION 4.4.3. *Let D be a domain in $\mathbf{C}^n$ and f a complete intersection $D \to \mathbf{C}^p$. Let $I, J \subset \{1, \ldots, p\}$, $I \cap J = \emptyset$, and take $i \in I$. Let λ be a semimeromorphic form on D which is smooth outside $V_{I' \cup J}$, where $I' = I \setminus \{i\}$. Then*

$$R_{IJ}^{\lambda} = 0.$$

Before we prove the proposition we need some definitions and a lemma.

DEFINITION 4.4.4. Let $f: D \to \mathbf{C}^p$ be a holomorphic mapping and define $|f|: D \to \mathbf{R}_+^p$ as before. We say that $x \in \mathbf{R}_>^p$ is *a good value for f* if there does not exist $C \subset \{1, \ldots, p\}$ such that $\pi_C(x)$ is a critical value for $\pi_C \circ |f|$.

It follows from Sard's theorem that the set of points which are not good is of Lebesque measure zero in $\mathbf{R}_>^p$, and hence in $\mathbf{R}_+^p$.

DEFINITION 4.4.5. Let $f: D \to \mathbf{C}^p$ be a holomorphic mapping. An asmissible path $\varepsilon:]0, 1] \to \mathbf{R}_>^p$ is said to be *regular for f*, if $\varepsilon(\delta)$ is a good value for f for almost all $\delta \in]0, 1]$.

It is easy to see that for any f there are admissible paths which are regular for f. Indeed, it is enough to prove that if $X \subset \mathbf{R}^p$ has Lebesgue measure zero then we can find an admissible path ε such that $\varepsilon^{-1}(X)$ has measure zero on $]0, 1]$.

To do this, choose a path ε such that ε_p is strictly increasing with δ. Solve for δ to obtain $\delta = \delta(\varepsilon_p)$ and hence $\varepsilon_j = \varepsilon_j(\varepsilon_p)$, $j = 1, \ldots, p-1$. Then, for $t = (t_1, \ldots, t_{p-1}) \in]0, 1]^{p-1}$, define

$$\varepsilon^t = (t_1\varepsilon_1, \ldots, t_{p-1}\varepsilon_{p-1}, \varepsilon_p)$$

and

$$E = \bigcup_{t \in]0,1]^{p-1}} \varepsilon^t(]0, c]),$$

where $c = \varepsilon_p(1)$. Since E has nonempty interior and we have a natural fibration of it with the admissible paths ε^t as fibers it follows from the Fubini theorem that X intersects almost all fibers in sets whose pull-back to $]0, 1]$ via ε^t and ε_p is of Lebesgue measure zero.

LEMMA 4.4.6. *Let D be a domain in $\mathbf{C}^n$ and $f: D \to \mathbf{C}^p$ a holomorphic mapping. Let $\varepsilon(\delta) \in \mathbf{R}^p_>$ be a good value for f. Let $A, B \subset \{1, \ldots, p\}$, $A \neq \emptyset$. Choose $a \in A$ and put $A' = A \setminus \{a\}$, $B' = B \cup \{a\}$. Then the following holds*

$$\pm D^\delta_{AB} = \partial D^\delta_{A'B'} - \partial D^\delta_{A'B} \cap \{|f_a| > \varepsilon_a(\delta)\}. \tag{23}$$

PROOF. Since $\varepsilon(\delta)$ is a good value we know that $D^\delta_{A'B'}$ and D^δ_{AB} are either empty or analytic manifolds of real codimension $|A'|$ and $|A|$, respectively. Let us first consider the case $D^\delta_{A'B'} = \emptyset$. Then the right-hand side of (23) is zero. Assume that $D^\delta_{AB} \neq \emptyset$ and pick $x \in D^\delta_{AB}$. There is a neighborhood U_x of x such that (after a change of coordinates)

$$U_x \cap D^\delta_{AB} = U_x \cap \{\operatorname{Re} z_i = 0, i \in A; \operatorname{Re} z_j > 0, j \in B\}.$$

Since this set is not empty, neither is

$$U_x \cap \{\operatorname{Re} z_i = 0, i \in A'; \operatorname{Re} z_j > 0, j \in B'\} = U_x \cap D^\delta_{A'B'}.$$

But we assumed $D^\delta_{A'B'} = \emptyset$. It follows that (23) holds in this case. If on the other hand $D^\delta_{A'B'} \neq \emptyset$ we observe that it has the same orientation as $D^\delta_{A'B}$ and we get

$$\partial D^\delta_{A'B'} = \partial D^\delta_{A'B} \cap \{|f_a| > \varepsilon_a\} + Y,$$

where of course set-theoretically

$$Y = D^\delta_{A'B} \cap \{|f_a| = \varepsilon_a\}.$$

Hence, taking into account orientation, we conclude that

$$Y = \pm D^\delta_{AB}.$$

The lemma is proved.

PROOF OF PROPOSITION 4.4.3. Just as in the proof of Proposition 4.4.2, we see that we may restrict our attention to the case when λ is a function. We have to prove that

$$\lim_{\delta \to 0} \int_{D^\delta_{IJ}(\varepsilon, \delta)} \lambda \psi = 0$$

and since by Theorem 4.3.1, the limit exists independently of ε we may assume ε to be regular for f and consider only those δ for which $\varepsilon(\delta)$ is a good value for f. By Lemma 4.4.6 we have

$$\pm \int_{D^\delta_{IJ}} \lambda\psi = \int_{\partial D^\delta_{I'J'}} \lambda\psi - \int_{\partial D^\delta_{I'J} \cap \{|f_i| > \varepsilon_i\}} \lambda\psi, \quad I' = I \setminus \{i\},\ J' = J \cup \{i\}. \tag{24}$$

Since

$$\partial D^\delta_{I'J} = \sum_{j \in J} \sigma_j D^\delta_{I' \cup \{j\}, J \setminus \{j\}}, \quad \sigma_j \text{ being } \pm 1, \tag{25}$$

the last term of (24) can be written as

$$-\sum_{j \in J} \sigma_j \int_{D^\delta_{I' \cup \{j\}, J \setminus \{j\}}} \lambda\psi.$$

We are performing the integrations on real-analytic manifolds so we may freely use the Stokes theorem. The middle term of (24) then becomes $\int_{D^\delta_{I'J'}} \lambda\bar{\partial}\psi$ and letting $\delta \to 0$ (in such a way that $\varepsilon(\delta)$ is always a good value) we get

$$\pm R^\lambda_{IJ} = \bar{\partial} R^\lambda_{I'J'} - \sum_{j \in J} \sigma_j R^\lambda_{I' \cup \{j\}, J' \setminus \{j\}}.$$

If we apply Proposition 4.4.2 this becomes

$$\pm R^\lambda_{IJ} = \bar{\partial} R^\lambda_{I'J} - \sum_{j \in J} \sigma_j R^\lambda_{I' \cup \{j\}, J \setminus \{j\}}. \tag{26}$$

When the last term acts on ψ it becomes

$$-\lim_{\delta \to 0} \int_{\sum \sigma_j D^\delta_{I' \cup \{j\}, J \setminus \{j\}}} \lambda\psi = -\lim_{\delta \to 0} \int_{\partial D^\delta_{I'J}} \lambda\psi$$

in view of (25). Making use of the Stokes theorem again we see that the right-hand side of (26) equals zero and the proposition follows.

Remark 4.4.7. One of the reasons for considering only good values $\varepsilon(\delta)$ is that we can then use the elementary manifold version of the Stokes theorem. There exist however more sophisticated Stokes formulas which are valid for analytic varieties with singularities. See e.g. Bungart [6, Part II], Herrera [18, Theorem 2.1] or Poly [25].

The following result is a special case of Theorem 1.7.6 (1) in Coleff and Herrera [7].

PROPOSITION 4.4.8. *Let D be a domain in $\mathbf{C}^n$ and $f: D \to \mathbf{C}^p$ a complete intersection. Let σ be a permutation of $\{1,\dots,p\}$ and $i_\sigma: \mathbf{C}^p \to \mathbf{C}^p$ given by*

$$i_\sigma(z) = (z_{\sigma(1)},\dots,z_{\sigma(p)}).$$

Then

$$R_{I\phi}^{\lambda,f} = \operatorname{sgn}\sigma R_{I\phi}^{\lambda, i_\sigma\circ f}.$$

When $I = \{1,\dots,p\}$ and $\lambda = (f_1\dots f_p)^{-1}$, we write

$$R_{I\phi}^{\lambda,f} = \bar\partial\frac{1}{f_1}\wedge\dots\wedge\bar\partial\frac{1}{f_p}. \tag{27}$$

Note that

$$\bar\partial\frac{1}{f_1}\wedge\bar\partial\frac{1}{f_2} = -\bar\partial\frac{1}{f_2}\wedge\bar\partial\frac{1}{f_1} \quad \text{etc.,}$$

by Proposition 4.4.8.

5. Weighted integral representation formulas and their limits.

5.1. REPRESENTATION OF HOLOMORPHIC FUNCTIONS AS INTEGRALS.

This section is devoted to the presentation of some of the main results of Berndtsson and Andersson [3] and Berndtsson [2]. We will not go into any details about the proofs.

D will always be a domain in $\mathbf{C}^n$ with ∂D of class C^2, and h a holomorphic function on D, belonging to $C^1(\bar D)$. We are going to construct a wide variety of formulas representing h as integrals over D and ∂D. To start we let $S: \bar D\times\bar D \to \mathbf{C}^n$ be a C^1 mapping defined by

$$(z,w) \mapsto (S_1(z,w),\dots,S_n(z,w))$$

and satisfying

$$|S(z,w)| \leqq C_1^K|z-w|$$

and

$$\left|\sum_{j=1}^n S_j(z,w)(z_j-w_j)\right| = |\langle S, z-w\rangle| \geqq C_2^K|z-w|^2,$$

for all $z\in\bar D$, $w\in K \subset\subset D$ and some positive constants C_1^K and C_2^K. Then we pick $M\in\mathbf{N}$ and make a number of choices:

For $j = 1,\dots,M$, choose C^1 functions

$$Q^j: \bar D\times\bar D \to \mathbf{C}^n, \quad (z,w) \mapsto (Q_1^j(z,w),\dots,Q_n^j(z,w)),$$

such that, for z fixed in $\bar{D}$, $Q^j(z,\cdot)$ is holomorphic in D.

For $j = 1,\ldots, M$, let U_j be a simply connected domain in C, containing the image of $\bar{D} \times \bar{D}$ under the map

$$(z,w) \mapsto \langle Q^j, w-z\rangle + 1 = \sum_{k=1}^{n} Q_k^j(z,w)(w_k - z_k) + 1.$$

Finally choose, for $j = 1,\ldots, M$, holomorphic functions $G_j : U_j \to \mathbf{C}$ satisfying $G_j(1) = 1$ and let $G_j^{(k)}$ denote the kth derivative of G_j.

To the mappings $Q^1,\ldots, Q^M$ and S we associate the (1,0)-forms

$$q^j(z,w) = \sum_{k=1}^{n} Q_k^j(z,w)dz_k$$

and

$$s(z,w) = \sum_{k=1}^{n} S_k(z,w)dz_k$$

respectively.

Using these more or less arbitrarily chosen mappings and forms we shall now define certain integral kernels which at first sight may seem somewhat awkward but which are really natural generalizations of classical ones. We need some multi-indices $\alpha = (\alpha_0,\ldots,\alpha_M) \in \mathbf{N}^{M+1}$, $\beta = (\beta_1,\ldots,\beta_M) \in \mathbf{N}^M$ and write as usual $|\alpha|$ for the length $\alpha_0 + \ldots + \alpha_M$ of α.

Put

$$(1)\quad K[Q^1,\ldots,Q^M; G_1,\ldots,G_M](z,w) = K(z,w)$$

$$= \sum_{|\alpha|=n-1} \frac{(n-1)!}{\alpha_1!\ldots\alpha_M!} G_1^{(\alpha_1)}(\langle Q^1, w-z\rangle + 1)\ldots G_M^{(\alpha_M)}(\langle Q^M, w-z\rangle + 1) \times$$

$$\times \frac{s(z,w) \wedge (\bar{\partial} s(z,w))^{\alpha_0} \wedge (\bar{\partial} q^1(z,w))^{\alpha_1} \wedge \ldots \wedge (\bar{\partial} q^M(z,w))^{\alpha_M}}{\langle S(z,w), z-w\rangle^{\alpha_0+1}}$$

and

$$(2)\quad P[Q^1,\ldots,Q^M; G_1,\ldots,G_M](z,w) = P(z,w)$$

$$= -\sum_{|\beta|=n} \frac{(n-1)!}{\beta_1!\ldots\beta_M!} G_1^{(\beta_1)}(\langle Q^1, w-z\rangle + 1)\ldots G_M^{(\beta_M)}(\langle Q^M, w-z\rangle + 1) \times$$

$$\times (\bar{\partial} q^1(z,w))^{\beta_1} \wedge \ldots \wedge (\bar{\partial} q^M(z,w))^{\beta_M}.$$

Here $\bar{\partial}$ operates on the variable z only. One can then prove the following representation theorem for C^1 functions on D.

THEOREM 5.1.1. *Let K and P be as above. Let φ be a function in $C^1(\bar{D})$.*

Then, for $w \in D$

$$\varphi(w) = C_n\Big(\int_{\partial D} \varphi(z)K(z,w) - \int_D \bar\partial\varphi(z) \wedge K(z,w) - \int_D \varphi(z)P(z,w)\Big),$$

where C_n *is a constant depending on the dimension n alone.*

PROOF. For $M = 1$, the theorem is contained as a special case in Theorem 1 of Berndtsson and Andersson [3] and we will assume that to be known. For $M = 2$ a proof is outlined in Berndtsson [2, p. 409], and the argument for general M is completely analogous.

COROLLARY 5.1.2. *Let* K *and* P *be given by* (1) *and* (2). *Let* h *be a holomorphic function on* D *such that* $h \in C^1(\bar D)$. *Then for* $w \in D$

$$h(w) = C_n \int_{\partial D} h(z)K(z,w) - C_n \int_D h(z)P(z,w), \tag{3}$$

where C_n *is a constant depending on the dimension n alone.*

PROOF. This is an immediate consequence of the preceding theorem, since $\bar\partial h = 0$ on D.

Since the above formulas remain valid for any choice of S, Q^j, and G_j, they are very flexible and useful. They also include some classical formulas as particular cases and may in fact be viewed as weighted versions of these well-known ones. We give a few examples (cf. Berndtsson and Andersson [3] and Berndtsson [2]).

EXAMPLE 5.1.3. Let $n = 1$ and Q^j holomorphic in both the variables z and w. Then (3) becomes

$$h(w) = c \int_{\partial D} h(z)G_1(\langle Q_1, w-z\rangle + 1)\dots G_M(\langle Q^M, w-z\rangle + 1)\frac{dz}{z-w}.$$

That is, we get a weighted Cauchy formula. Of course $G_j \equiv 1$ gives the ordinary Cauchy formula.

EXAMPLE 5.1.4. Let $G_j \equiv 1$ (or, equivalently, $Q^j \equiv 0$). Then P vanishes and K becomes

$$\frac{s \wedge (\bar\partial s)^{n-1}}{\langle S, z-w\rangle^n} = \frac{s \wedge (ds)^{n-1}}{\langle S, z-w\rangle^n} = (-1)^{\frac{1}{2}n(n-1)}(n-1)!\ \frac{\omega'(S) \wedge \omega(z-w)}{\langle S, z-w\rangle^n},$$

where the first equality follows by bidegree reasons (remember that the operators $\bar\partial$ and d act on the z variables only) and we have used the usual notation

$$\omega(a) = da_1 \wedge \ldots \wedge da_n, \quad \omega'(a) = \sum(-1)^{j+1} a_j da_1 \wedge \ldots \underset{j}{\wedge} \ldots \wedge da_n,$$

$a(z, w)$ being any smooth map $D \times D \to \mathbf{C}^n$ and the symbol $\underset{j}{\wedge}$ indicating that the differential da_j is omitted. If D is such that S can be chosen to be independent of w one thus obtains the Leray formula. More generally, if D is strictly pseudoconvex, it is known that S can be taken to be holomorphic in w and the corresponding formula is that of Henkin [16] and Ramirez [26].

EXAMPLE 5.1.5. Let D be convex and assume that ϱ is a defining function for D. That is, $D = \{z \in \mathbf{C}^n; \varrho(z) < 0\}$, with ϱ convex and of class C^2 on a neighborhood of ∂D. Using just one weight ($M = 1$) given by

$$q = \frac{\partial \varrho}{\varrho - \varepsilon}, \quad \varepsilon > 0;$$

$$G(\zeta) = \zeta^{-N}, \quad N > 0,$$

and letting $\varepsilon \to 0$, we get that $K = 0$ on ∂D and

$$P(z, w) = -\frac{1}{n}(-N)\ldots(-N-n+1) \times$$
$$\times \frac{\varrho(z)^{N+n}}{(\langle \partial \varrho(z), w - z\rangle + \varrho(z))^{N+n}} (\partial \bar{\partial} \log(-1/\varrho(z)))^n$$

and hence

$$h(w) = c_n \int_D h(z) P(z, w).$$

Note that the convexity of ϱ implies that the denominator never vanishes. In particular, if D is the unit ball B, we can take

$$\varrho(z) = \sum_{j=1}^{n} |z_j|^2 - 1 = |z|^2 - 1$$

and calculate

$$P(z, w) = c_n'(-N)\ldots(-N-n+1) \frac{(|z|^2 - 1)^{N-1}}{(\langle \bar{z}, w\rangle - 1)^{N+n}} (\sum dz_j \wedge d\bar{z}_j)^n.$$

If we then choose $N = 1$ we get

$$h(w) = c_n'' \int_D \frac{h(z)\omega(\bar{z}) \wedge \omega(z)}{(\langle \bar{z}, w\rangle - 1)^{n+1}},$$

the usual Bergman formula.

Let us see what happens if we instead let $N \to 0$. First note that

$$\bar{\partial}(1-|z|^2) \wedge \omega'(\bar{z}) \wedge \omega(z) = -|z|^2\omega(\bar{z}) \wedge \omega(z),$$

so if we put

$$c_n^N = (N+1)\dots(N+n-1)c_n'$$

we can write

$$h(w) = c_n^N \int_D h(z)N(1-|z|^2)^{N-1}\bar{\partial}(1-|z|^2) \wedge \frac{\omega'(\bar{z}) \wedge \omega(z)}{|z|^2(1-\langle \bar{z}, w\rangle)^{N+n}}$$

$$= c_n^N \int_D h(z)\bar{\partial}[(1-|z|^2)^N] \wedge \frac{\omega'(\bar{z}) \wedge \omega(z)}{|z|^2(1-\langle \bar{z}, w\rangle)^{N+n}}.$$

Since

$$(1-|z|^2)^N|_{\bar{D}} \underset{N\to 0}{\to} \chi_D|_{\bar{D}},$$

the characteristic function of D, we conclude that

$$\bar{\partial}(1-|z|^2)^N|_{\bar{D}} \underset{N\to 0}{\to} \bar{\partial}\chi_D|_{\bar{D}},$$

which acts on $(n, n-1)$-forms as integration over ∂D (cf. Section 5.2.).

We therefore end up with

$$h(w) = c_n^0 \int_{\partial D} h(z)\frac{\omega'(\bar{z}) \wedge \omega(z)}{(1-\langle \bar{z}, w\rangle)^n},$$

that is, the Leray formula again.

Finally we make a similar construction for the case, when D is a strictly pseudoconvex domain with C^N boundary, $N \geqq 2$. Let ϱ be a strictly plurisubharmonic defining function for D. It is shown in Fornæss [10] that one can find C^{N-1} functions $h_j: \bar{D} \times \bar{D} \to \mathbf{C}$, $j = 1, \dots, n$, such that

(i) $h_j(z, \cdot)$ is holomorphic on D for $z \in \bar{D}$ fixed;

(ii) if one defines

$$H(z, w) = \sum_{j=1}^{n} h_j(z, w)(z_j - w_j),$$

then $\operatorname{Re} H(z, w) \leqq \varrho(w) - \varrho(z) - \delta|z-w|^2$ for some $\delta > 0$.

We then choose

$$q^1 = \frac{\sum h_j dz_j}{\varrho - \varepsilon}, \quad \varepsilon > 0,$$

and

$$G_1(\zeta) = \zeta^{-N},$$

and, as before, let $\varepsilon \to 0$ to obtain

$$h(w) = c_n \int_D h(z)P(z, w), \quad \text{for} \quad w \in D,$$

with

$$P(z, w) = \sum_{|\beta| = n} C_{n,\beta} \frac{\varrho(z)^{N+\beta_1}}{(H(z, w) + \varrho(z))^{N+\beta_1}} \left(\bar{\partial} \frac{\sum h_j dz_j}{\varrho}\right)^{\beta_1} \wedge G_2^{(\beta_2)} \dots G_M^{(\beta_M)} \times$$

$$\times (\bar{\partial} q^2)^{\beta_2} \wedge \dots \wedge (\bar{\partial} q^M)^{\beta_M}.$$

If we, for $0 \leqq k \leqq n$, define the (k, k)-form $A^{N,k}(z, w)$ on $\mathbf{C}^n \times D$ by

$$(4) \qquad A^{N,k}(z, w) = \begin{cases} \dfrac{\varrho(z)^{N+k}}{(H(z, w) + \varrho(z))^{N+k}} \left(\bar{\partial} \dfrac{\sum h_j(z, w) dz_j}{\varrho(z)}\right)^k, & z \in \bar{D} \\ 0, & z \in \mathbf{C}^N \setminus \bar{D} \end{cases}$$

its coefficients will belong to $C^{N-2}(\mathbf{C}^n \times D)$.

This follows from the fact that the form

$$\left(\bar{\partial} \frac{\sum h_j dz_j}{\varrho}\right)^k$$

has coefficients which are bounded by $(-\varrho(z))^{-k-1}$ on $\bar{D}$. We have thus obtained the following result.

Proposition 5.1.6. *Let $D \subset \mathbf{C}^n$ be a strictly pseudoconvex domain with C^N boundary, $N \geqq 2$. Let $1 < M \in \mathbf{N}$ and, for $j = 2, \dots, M$, let $Q^j : \bar{D} \times \bar{D} \to \mathbf{C}^n$ be C^1 mappings such that for all $z \in \bar{D}$, $Q^j(z, \cdot)$ is holomorphic in D. Let, for $j = 2, \dots, M$, G_j be holomorphic functions of one variable such that $G_j(1) = 1$ and $G_j(\langle Q^j, w - z\rangle + 1)$ is defined on $\bar{D} \times \bar{D}$. Also put*

$$q^j = \sum_{k=1}^n Q_k^j(z, w) dz_k.$$

Then, if h is holomorphic on D and C^1 on $\bar{D}$, we have

$$h(w) = \sum_{|\beta| = n} C_{n,\beta} \int_D h(z) A^{N,\beta_1}(z, w) \wedge G_2^{(\beta_2)} \dots G_M^{(\beta_M)} (\bar{\partial} q^2)^{\beta_2} \wedge \dots \wedge (\bar{\partial} q^M)^{\beta_M},$$

where A^{N,β_1} are given by (4) and $w \in D$.

5.2. CURRENTS OF INTEGRATION OBTAINED AS LIMITS OF SMOOTH FORMS.

In order to establish a link between the currents of Chapter 4 and the integral formulas from the preceding section we want to regard integration over the tubes D^{δ}_{IJ} as limits of C^{∞} forms.

First we define certain characteristic functions and show how to approximate them.

DEFINITION 5.2.1. Let D be a domain in $\mathbf{C}^n$. Let $f: D \to \mathbf{C}^p$ be a holomorphic map and let $\varepsilon:]0,1] \to \mathbf{R}^p_>$ be an admissible path. Then, for $j = 1,\ldots,p$, we define the mappings $\chi^{\delta}_j(\varepsilon, f) = \chi_j: D \to [0,1]$ by

$$\chi_j(z) = \begin{cases} 1, & |f_j(z)| > \varepsilon_j(\delta) \\ 0, & |f_j(z)| \leqq \varepsilon_j(\delta). \end{cases}$$

Furthermore, for $r_j \in \mathbf{R}_>$, $j = 1,\ldots,p$, we choose C^{∞} mappings

$$\chi^{\delta, r_j}_j(\varepsilon, f) = \chi^{r_j}_j: D \to [0,1],$$

such that we have pointwise convergence

$$\chi^{r_j}_j \underset{r_j \to 0}{\to} \chi_j.$$

Let us now see how to approximate integration over tubes.

PROPOSITION 5.2.2. *Let D be a domain in $\mathbf{C}^n$ and let f be a holomorphic mapping $D \to \mathbf{C}^p$, $p \leqq n$. Let $I, J \subset \{1,\ldots,p\}$ be disjoint and given by $I = \{i_1,\ldots,i_s\}$, $J = \{j_1,\ldots,j_t\}$. Let $\varepsilon:]0,1] \to \mathbf{R}^p_>$ be an admissible path and $\delta \in]0,1]$ such that $\varepsilon(\delta)$ is a good valued for f. Then, for any compactly supported $(n, n-s)$-form ψ of class C^1 on D, we have*

$$(5) \qquad \lim_{r_p \to 0} \ldots \lim_{r_1 \to 0} \int_D \psi \chi^{r_{j_1}}_{j_1} \cdots \chi^{r_{j_t}}_{j_t} \wedge \bar{\partial}\chi^{r_{i_1}}_{i_1} \wedge \ldots \wedge \bar{\partial}\chi^{r_{i_s}}_{i_s} = \pm \int_{D^{\delta}_{IJ}} \psi,$$

independently of the particular choice of approximating functions $\chi^{r_k}_k$.

PROOF. We may of course assume that $I \cup J = \{1,\ldots,p\}$, since limits of the form $\lim_{r_j \to 0} A$ are trivial if A does not depend on r_j. The first limit will then be of one of the following two forms:

$$\text{(i)} \qquad \lim_{r_1 \to 0} \int_D \chi^{r_1}_1 \psi \wedge \varphi_s \qquad (\text{if } 1 \in J),$$

or

$$\text{(ii)} \qquad \lim_{r_1 \to 0} \int_D \bar{\partial}\chi^{r_1}_1 \wedge \psi \wedge \varphi_{s-1} \qquad (\text{if } 1 \in I),$$

where φ_k are smooth forms of bidegree $(0, k)$ defined in an obvious manner. In the first case we simply get

$$\lim_{r_1 \to 0} \int_D \chi_1^{r_1} \psi \wedge \varphi_s = \int_D \chi_1 \psi \wedge \varphi_s = \int_{D^\delta_{\emptyset,\{1\}}} \psi \wedge \varphi_s,$$

whereas in the second case one obtains

$$\lim_{r_1 \to 0} \int_D \bar\partial \chi_1^{r_1} \wedge \psi \wedge \varphi_{s-1} = \lim_{r_1 \to 0} \int_D d(\chi_1^{r_1} \psi \wedge \varphi_{s-1}) - \lim_{r_1 \to 0} \int_D \chi_1^{r_1} \bar\partial(\psi \wedge \varphi_{s-1})$$

$$= \int_{\partial D} \chi_1 \psi \wedge \varphi_{s-1} - \int_D \chi_1 \bar\partial(\psi \wedge \varphi_{s-1})$$

$$= \int_{\partial D \cap \{|f_1| > \varepsilon_1\}} \psi \wedge \varphi_{s-1} - \int_{\partial(D \cap \{|f_1| > \varepsilon_1\})} \psi \wedge \varphi_{s-1}$$

$$= \pm \int_{D^\delta_{\{1\},\emptyset}} \psi \wedge \varphi_{s-1},$$

by using the Stokes formula and the fact that

$$\partial(D \cap \{|f_1| > \varepsilon_1\}) = \partial D \cap \{|f_1| > \varepsilon_1\} \pm D \cap \{|f_1| = \varepsilon_1\}.$$

We have thus eliminated $\chi_1^{r_1}$.

Turning to the kth limit we have, similarly, two cases:

(i) $$\lim_{r_k \to 0} \int_{D^\delta_{I'J'}} \chi_k^{r_k} \psi \wedge \varphi_{s-s'} \qquad (\text{if } k \in J),$$

(ii) $$\lim_{r_k \to 0} \int_{D^\delta_{I'J'}} \bar\partial \chi_k^{r_k} \wedge \psi \wedge \varphi_{s-s'-1} \qquad (\text{if } k \in I),$$

where $s' = |I'|$. Once more we get

$$\lim_{r_k \to 0} \int_{D^\delta_{I'J'}} \chi_k^{r_k} \psi \wedge \varphi_{s-s'} = \int_{D^\delta_{I'J'}} \chi_k \psi \wedge \varphi_{s-s'} = \int_{D^\delta_{I',J' \cup \{k\}}} \psi \wedge \varphi_{s-s'}$$

and

$$\lim_{r_k \to 0} \int_{D^\delta_{I'J}} \bar\partial \chi_k^{r_k} \wedge \psi \wedge \varphi_{s-s'-1}$$

$$= \lim_{r_k \to 0} \int_{D^\delta_{I'J'}} d(\chi_k^{r_k} \psi \wedge \varphi_{s-s'-1}) - \lim_{r_k \to 0} \int_{D^\delta_{I'J'}} \chi_k^{r_k} \bar\partial(\psi \wedge \varphi_{s-s'-1})$$

$$= \int_{\partial D^\delta_{I'J'} \cap \{|f_k| > \varepsilon_k\}} \psi \wedge \varphi_{s-s'-1} - \int_{\partial(D^\delta_{I'J'} \cap \{|f_k| > \varepsilon_k\})} \psi \wedge \varphi_{s-s'-1}.$$

By Lemma 4.4.6 we have

$$\partial D^\delta_{I'J' \cup \{k\}} = \partial D^\delta_{I'J'} \cap \{|f_k| > \varepsilon_k\} \pm D^\delta_{I' \cup \{k\}, J'}$$

and it follows that

$$\lim_{r_k \to 0} \int_{D^\delta_{I'J'}} \bar\partial \chi_k^{r_k} \wedge \psi \wedge \varphi_{s-s'-1} = \pm \int_{D^\delta_{I' \cup \{k\}, J'}} \psi \wedge \varphi_{s-s'-1}.$$

Going through all the p steps and thereby eliminating all $\chi_k^{r_k}$ we end up with $\pm \int_{D^\delta_{IJ}} \psi$ and, since all we ever needed to know about $\chi_k^{r_k}$ was that it converges toward χ_k, the proposition is proved.

Let us now look at a more general situation. Let $I' = (i_r)_{r=1}^s$ and $J' = (j'_r)_{r=1}^t$ be arbitrary sequences of elements in $\{1, \ldots, p\}$ and write $I = \{i_1, \ldots, i_s\}$ and $J = \{j_1, \ldots, j_t\}$. We then consider once again the limit

$$\lim_{r_p \to 0} \ldots \lim_{r_1 \to 0} \int_D \psi \chi_{j_1}^{r_{j_1}} \ldots \chi_{j_t}^{r_{j_t}} \bar\partial \chi_{i_1}^{r_{i_1}} \wedge \ldots \wedge \bar\partial \chi_{i_s}^{r_{i_s}}. \tag{6}$$

We have three cases which are not included in the preceding proposition:

(i) Two of the indices $i_1, \ldots, i_s$ are equal: It follows that

$$\bar\partial \chi_{i_1}^{r_{i_1}} \wedge \ldots \wedge \bar\partial \chi_{i_s}^{r_{i_s}} = 0$$

so (6) is zero.

(ii) $i_1, \ldots, i_s$ are all different, $I \cap J = \emptyset$ but two of the indices $j_1, \ldots, j_t$ are equal: Since

$$(\chi_j^{r_j})^m \to \chi_j^m = \chi_j,$$

Proposition 5.2.2 implies that (6) equals $\pm \int_{D^\delta_{IJ}} \psi$.

(iii) $i_1, \ldots, i_s$ are all different, but $I \cap J \neq \emptyset$: Since

$$(\chi_j^{r_j})^m \bar{\partial} \chi_j^{r_j} = \frac{1}{m+1} \bar{\partial}[(\chi_j^{r_j})^{m+1}]$$

and since, as in (ii), we can replace $\chi_j^{r_j}$ by $(\chi_j^{r_j})^{m+1}$ (they both tend to χ_j), Proposition 5.2.2 implies that (6) equals

$$\pm \frac{1}{M_{IJ'}} \int_{D_{IJ}^\delta} \psi,$$

where $M_{IJ'} = \prod_{k \in I \cap J}(1+m_k)$, m_k = the number of j_r's which equal k.

The above argument shows that the following generalization of Proposition 5.2.2 holds.

PROPOSITION 5.2.3. *Let D be a domain in $\mathbf{C}^n$ and f a holomorphic mapping $D \to \mathbf{C}^p$, $p \leqq n$. Let $I' = (i_r)_{r=1}^s$ and $J' = (j_r)_{r=1}^t$ be sequences of elements in $\{1, \ldots, p\}$ and put $I = \{i_r;\ 1 \leqq r \leqq s\}$, $J = \{j_r;\ 1 \leqq r \leqq t\}$. Let $\varepsilon:]0,1] \to \mathbf{R}_>^p$ be an admissible path and $\delta \in]0,1]$ such that $\varepsilon(\delta)$ is good value for f. Then, for any compactly supported $(n, n-|I|)$-form ψ of class C^1 on D, one has*

$$\lim_{r_p \to 0} \ldots \lim_{r_1 \to 0} \int_D \psi \chi_{j_1}^{r_{j_1}} \ldots \chi_t^{r_{j_t}} \wedge \bar{\partial} \chi_{i_1}^{r_{i_1}} \wedge \ldots \wedge \bar{\partial} \chi_{i_s}^{r_{i_s}} = C_{I'J'} \int_{D_{IJ}^\delta} \psi,$$

where $C_{I'J'} = 0$, if $|I| < s$ and $C_{IJ'} = \pm 1/M_{IJ'}$, $M_{IJ'}$ being the integer given above, if $|I| = s$.

5.3. Representation of holomorphic functions by currents.

In this section we modify the integral formulas of Section 5.1 and obtain formulas, where a holomorphic function is represented as a sum of currents acting on certain test forms. The currents in question are the R_{IJ}^λ's from Chapter 4.

Our first theorem deals with respresentation on a strictly pseudoconvex domain. Recall that if T is a current and φ a smooth form, then $\varphi \wedge T$ is the current defined by $\varphi \wedge T(\psi) = T(\varphi \wedge \psi)$.

REMARK 5.3.1. In Chapter 4 we considered smooth test forms only. It is, however, clear that, if we use q_{IJ}^D to denote the (finite) order of the current R_{IJ}^λ on the relatively compact open set D, then R_{IJ}^λ can be extended to test forms ψ of class C^M, with $\operatorname{supp} \psi \subset \bar{D}$ and $M \geqq q_{IJ}^D$.

THEOREM 5.3.2. *Let D be a strictly pseudoconvex domain in $\mathbf{C}^n$ with smooth boundary. Let $\tilde{D}$ be a domain such that $\tilde{D} \supset\supset D$ and $f: \tilde{D} \to \mathbf{C}^p$ a holomorphic* *mapping. One can then find functions λ_{IJ}, meromorphic on $\tilde{D}$, smooth outside $V_{V \cup J}$ and independent of w, such that, if h is holomorphic on D and C^∞ on $\bar{D}$, we have, for each $N \geqq \max_{I,J} q_{IJ}^D + 2$ and $w \in D$*

$$h(w) = \sum_{I,J \subset \{1,\ldots,p\}} hR_{IJ}^{\lambda_{IJ}}(\psi_{IJ}^N(\cdot, w)),$$

for some C^{N-2} forms ψ_{IJ}^N on $\tilde{D}$, supported on D, of bidegree $(n, n-|I|)$ and depending holomorphically on the parameter w.

Before we prove the theorem we need a definition.

DEFINITION 5.3.3. Let D be a domain in $\mathbf{C}^n$ and $f: D \to \mathbf{C}$ a holomorphic function. A mapping $B: D \times D \to \mathbf{C}^n$ which is holomorphic in all variables is then called a *Hefer map for f* if

$$f(w) - f(z) = \sum_{j=1}^{n} B_j(z, w)(w_j - z_j)$$

for all $(z, w) \in D \times D$. (Cf. Hefer [14]).

PROOF OF THEOREM 5.3.2. We may assume that $\tilde{D}$ is pseudoconvex and hence, by Hefer's Lemma, we can find Hefer maps B^k for f_k, $k = 1, \ldots, p$. Then pick an admissible path ε which is regular for f and define, for $\delta \in]0, 1]$ and $r \in \mathbf{R}_>^p$, smooth maps $Q^{\delta,r}: \tilde{D} \times \tilde{D} \to \mathbf{C}^n$ by

$$Q_j^{\delta,r}(z, w) = \frac{1}{p} \sum_{k=1}^{p} \frac{\chi_k^{\delta,r_k}(z) B_j^k(z, w)}{f_k(z)}, \tag{7}$$

where the smooth functions χ_k^{δ,r_k} are chosen so that

$$\chi_k^{\delta,r_k} \to \chi_k^\delta$$

just as in Section 5.2, and also so that for all r_k,

$$\chi_k^{\delta,r_k}(z) = 0 \quad \text{on} \quad \{z; |f_k(z)| \leqq \tfrac{1}{2}\varepsilon_k(\delta)\}.$$

With $Q^{\delta,r}$ we associate the (1, 0)-form $q^{\delta,r}$ given by

$$q^{\delta,r}(z, w) = \sum_{j=1}^{n} Q_j^{\delta,r}(z, w) dz_j = \frac{1}{p} \sum_{k=1}^{p} \frac{\chi_k^{\delta,r_k}(z) b^k(z, w)}{f_k(z)},$$

where

$$b^k(z, w) = \sum_{j=1}^{n} B_j^k(z, w) dz_j.$$

Finally we let G be a polynomial of degree p in one complex variable such that $G(1) = 1$. If we now apply Proposition 5.1.6 with $M = 2$, $G_2 = G$, and

$Q_2 = Q^{\delta,r}$ we obtain for $w \in D$ and $N \geqq 2$,

$$h(w) = \sum_{s=0}^{n} C_{n,s} \int_D h(z) A^{N,n-s}(z,w) \wedge G^{(s)}(\langle Q^{\delta,r}, w-z\rangle + 1)(\bar{\partial} q^{\delta,r})^s,$$

where $A^{N,n-\beta}$ are of class C^{N-2} in z, holomorphic in w and of bidegree $(n-s, n-s)$. Since the $(1,1)$-form $\bar{\partial} q^{\delta,r}$ can be written

$$\bar{\partial} q^{\delta,r} = \frac{1}{p} \sum_{k=1}^{p} \frac{\bar{\partial}\chi_k^{\delta,r_k} b^k}{f_k},$$

it follows that

$$(\bar{\partial} q^{\delta,r})^m = 0 \quad \text{for} \quad m > p$$

and we conclude that

$$h(w) = \sum_{s=0}^{p} C_{n,s} \int_D h(z) A^{N,n-s}(z,w) \wedge G^{(s)}(\langle Q^{\delta,r}, w-z\rangle + 1)(\bar{\partial} q^{\delta,r})^s, \quad w \in D. \tag{8}$$

Recalling that $G^{(s)}$ is a polynomial of degree $p-s$ and observing that the argument may be written

$$\begin{aligned}\langle Q^{\delta,r}, w-z\rangle + 1 &= \frac{1}{p} \sum_{k=1}^{p} \left(\frac{\chi_k^{\delta,r_k} \langle B^k, w-z\rangle}{f_k} + 1 \right) \\ &= \frac{1}{p} \sum_{k=1}^{p} \left(\frac{\chi_k^{\delta,r_k}(z) f_k(w)}{f_k(z)} + (1 - \chi_k^{\delta,r_k}(z)) \right)\end{aligned}$$

we find that each term of (8) is of the form

$$L = \sum_{I,J'} \int_D H \Lambda_{IJ'} \Psi_{IJ'}^N(\cdot, w) \wedge \chi_{j_1}^{r_{j_1}} \cdots \chi_{j_t}^{r_{j_t}} \bar{\partial}\chi_{i_1}^{r_{i_1}} \wedge \ldots \wedge \bar{\partial}\chi_{i_s}^{r_{i_s}}, \tag{9}$$

where $I = \{i_1, \ldots, i_s\}$, $J' = (j_r)_{r=1}^t$, $t \leqq n-s$, $J = J(J') = \{j_1, \ldots, j_t\}$, $\Lambda_{IJ'}$ are monomials in $1/f_1, \ldots, 1/f_p$ and $\Psi_{IJ'}^N$ are $(n, n-s)$-forms of class C^{N-2} on $\tilde{D}$ and holomorphic in w.

Put

$$F_{IK} = \prod_{k \in I \cup K} f_k^{p-s+1}.$$

One checks that $F_{IK} \Lambda_{IJ'}$ has no poles when $J(J') \subset K$. Next, for $J \subset \{1, \ldots, p\}$ such that $I \cap J = \emptyset$, define

$$\lambda_{IK} = 1/F_{IK}$$

and

$$\psi_{IK}^{N}(\cdot,w) = F_{IK} \sum_{\substack{J' \\ J(J')=K}} C_{IJ'} \Psi_{IJ'}^{N}(\cdot,w) \Lambda_{IJ'},$$

where $C_{IJ'} = \pm 1/M_{IJ'}$ are the coefficients occurring in Proposition 5.2.3.

We see that the ψ_{IK}^{N} are $(n, n-s)$-forms of class C^{N-2} on $\tilde{D}$ and the λ_{IK} are meromorphic on $\tilde{D}$. After re-grouping the terms in (9) and taking iterated limits we get (by the same Proposition 5.2.3)

$$\lim_{r_p \to 0} \ldots \lim_{r_1 \to 0} L = \sum_{\substack{|I|=s \\ I \cap J = \emptyset}} \int_{D_{IJ}^{\delta}} h \lambda_{IJ} \psi_{IJ}^{N}(\cdot, w).$$

Now we recall that (9) is a typical term of the right-hand side of (8) and since the left-hand side of the same equation does not depend on $r_1, \ldots, r_p$ we conclude that, for $w \in D$

$$h(w) = \sum_{I \cdot J \subset \{1,\ldots,p\}} \int_{D_{IJ}^{\delta}} h(z) \lambda_{IJ}(z) \psi_{IJ}^{N}(z,w).$$

Finally, letting $\delta \to 0$ (avoiding points such that $\varepsilon(\delta)$ is not good for f), we obtain, for $w \in D$ and $N \geqq \max_{I,J} q_{IJ}^{D} + 2$

$$h(w) = \sum_{I,J \subset \{1,\ldots,p\}} R_{IJ}^{\lambda_{IJ}}(h \psi_{IJ}^{N}(\cdot,w)) = \sum_{I,J \subset \{1,\ldots,p\}} h R_{IJ}^{\lambda_{IJ}}(\psi_{IJ}^{N}(\cdot,w)).$$

The theorem follows.

REMARK 5.3.4. Note that the λ_{IJ}'s and the ψ_{IJ}^{N}'s depend on our choice of the polynomial G in the proof. However, when $I = \{1, \ldots, p\}$, so that $s = p$ and $J = \emptyset$, we have

$$\lambda_{IJ} = (F_{IJ})^{-1} = (f_1 \cdots f_p)^{-1}$$

and, since $G^{(p)} \equiv \text{const.}$, the corresponding term in (8) is

$$C_{n,p,G} \int_{D} h A^{N,n-p}(\cdot,w) \wedge b(\cdot,w) \wedge \bar{\partial}\chi_1^{r_1} \wedge \ldots \wedge \bar{\partial}\chi_p^{r_p},$$

where

$$b = b^1 \wedge \ldots \wedge b^p.$$

It follows that, up to a multiplicative constant, we have

$$\psi_{IJ}^{N} = A^{N,n-p} \wedge b.$$

That is, the term with $|I| = p$ is independent of G. In Chapter 6 we will choose G in a particular way and we will then be able to draw further conclusions about the remaining terms ($|I| < p$) in the representation of Theorem 5.3.2.

Next we are going to consider entire functions. The result which we obtain will be used in Section 6.2.

Choose a smooth, real-valued function ξ on $\mathbf{C}^n$ such that

$$(10)\qquad \begin{cases} \xi(z) \equiv 1, & |z| \leqq 2 \\ \xi(z) \equiv 0, & |z| \geqq 3 \end{cases}$$

and define, for $R > 0$,

$$\xi_R(z) = \xi(z/R).$$

We then have the following result.

Proposition 5.3.5. *Let $f: \mathbf{C}^n \to \mathbf{C}^p$ and $h: \mathbf{C}^n \to \mathbf{C}$ be entire maps. For $R > 0$, let ξ_R be given as above. Then, for $|w| \leqq R$, one has a representation formula as follows:*

$$h(w) = \sum_{I,J \subset \{1,\ldots,p\}} hR_{IJ}^{\tilde{\lambda}_{IJ}}(\bar{\partial}\xi_R \wedge \tilde{\psi}_{IJ}(\cdot, w)) + \sum_{I,J \subset \{1,\ldots,p\}} hR_{IJ}^{\lambda_{IJ}}(\xi_R \psi_{IJ}(\cdot, w)),$$

where the R_{IJ}^{λ} are the currents defined in Chapter 4; $\tilde{\lambda}_{IJ}$ and λ_{IJ} are monomials in $1/f_1, \ldots, 1/f_p$; $\tilde{\psi}_{IJ}$ and ψ_{IJ} are smooth forms of bidegree $(n, n-|I|-1)$ and $(n, n-|I|)$, respectively, and ψ_{IJ} is holomorphic in w.

Proof. Applying Theorem 5.1.1 to the compactly supported function $\xi_R h$ we get, for $|w| \leqq 2R$

$$(11)\qquad h(w) = C_n \int_{\mathbf{C}^n} h(z)\bar{\partial}\xi_R(z) \wedge K^{\delta,r}(z, w) + C_n \int_{\mathbf{C}^n} h(z)\xi_R(z)P^{\delta,r}(z, w),$$

where $K^{\delta,r}$ and $P^{\delta,r}$ are given by (1) and (2) with $M = 2$, $G_2 = G$ a polynomial of degree p, $Q^2 = Q^{\delta,r}$ (as in the proof of Theorem 5.3.1) and $S = \bar{z} - \bar{w}$. That is, we have

$$(12)\qquad K^{\delta,r} = \sum_{|\alpha| = n-1} c_{\alpha,n} G_1^{(\alpha_1)}(\bar{\partial}q^1)^{\alpha_1} \wedge \frac{(\sum(\bar{z}_j - \bar{w}_j)dz_j) \wedge (\sum dz_j \wedge d\bar{z}_j)^{\alpha_0}}{|z-w|^{2\alpha_0+2}} \wedge$$
$$\wedge\ G^{(\alpha_2)}(\bar{\partial}q^{\delta,r})^{\alpha_2}$$

and

$$P^{\delta,r} = \sum_{|\beta| = n} c_{\beta,n} G_1^{(\beta_1)}(\bar{\partial} q^1)^{\beta_1} \wedge G^{(\beta_2)}(\bar{\partial} q^{\delta,r})^{\beta_2}, \tag{13}$$

where

$$q^{\delta,r} = \sum Q_j^{\delta,r} dz_j$$

and $Q_j^{\delta,r}$ as in (7). If we restrict our attention to w for which $|w| \leqq R$ it follows that $|z-w| \geqq R$ for $z \in \operatorname{supp} \bar{\partial}\xi_R$. Hence the only singularities which occur in (11) are those coming from $Q^{\delta,r}$.

We can therefore let $r_1, \ldots, r_p$ and δ tend to zero precisely as when proving Theorem 5.3.2 and the proposition follows.

6. Division in rings of holomorphic functions.

6.1. Ideals in the ring of holomorphic functions on a strictly pseudoconvex domain.

Recall from Section 4.4 that if the holomorphic mapping $f: D \to \mathbf{C}^p$ is a complete intersection then we write $\bar{\partial}(1/f_1) \wedge \ldots \wedge \bar{\partial}(1/f_p)$ for the current R_{IJ}^{λ}, $I = \{1, \ldots, p\}$, $J = \emptyset$, $\lambda = (f_1 \ldots f_p)^{-1}$. We have the following theorem.

Theorem 6.1.1. *Let $D \subset \mathbf{C}^n$ be a strictly pseudoconvex domain with smooth boundary and let $\tilde{D}$ be a domain such that $\tilde{D} \supset\supset D$. Let $f: \tilde{D} \to \mathbf{C}^p$ be a complete intersection. Then, if h is holomorphic on D and belongs to $C^\infty(\bar{D})$, we have for $w \in D$,*

$$h(w) = \sum_{\substack{I,J \subset \{1,\ldots,p\} \\ |I| < p}} h R_{IJ}^{\lambda_{IJ}}(\psi_{IJ}^N(\cdot, w)) + h\bar{\partial}\frac{1}{f_1} \wedge \ldots \wedge \bar{\partial}\frac{1}{f_p}(\psi^N(\cdot, w)), \tag{1}$$

where each $(n,n-|I|)$-form ψ_{IJ}^N can be written

$$\psi_{IJ}^N(\cdot, w) = \sum f_k(w) \psi_{IJ}^{N,k}(\cdot, w), \tag{2}$$

with $\psi_{IJ}^{N,k}$ depending holomorphically on w.

That is, one can find functions g_k, $1 \leqq k \leqq p$, holomorphic in D and belonging to $C^\infty(\bar{D})$ such that

$$h(w) = \sum_{k=1}^{p} g_k(w) f_k(w) + h\bar{\partial}\frac{1}{f_1} \wedge \ldots \wedge \bar{\partial}\frac{1}{f_p}(\psi^N(\cdot, w)).$$

We shall use the following notation: Let $f: D \to \mathbf{C}^p$ be a holomorphic mapping and put

$$I_f = \{\sum g_k f_k ; g_k \quad \text{are holomorphic on } D,\ 1 \leqq k \leqq p\},$$

i.e. the ideal in the ring of holomorphic functions on D generated by $f_1, \ldots, f_p$.

Corollary 6.1.2. *Under the assumptions given in the theorem the following two statements are equivalent:*

a) $h \in I_f$

b) *the current* $h\bar{\partial}\frac{1}{f_1} \wedge \ldots \wedge \bar{\partial}\frac{1}{f_p}$ *is zero.*

Proof. It follows immediately from Theorem 6.1.1 that b) implies a). In order to prove the opposite implication we assume that $h \in I_f$, that is $h = \sum g_k f_k$. We then have

$$h\bar{\partial}\frac{1}{f_1} \wedge \ldots \wedge \bar{\partial}\frac{1}{f_k} = \sum g_k f_k R^{\lambda}_{I\phi},$$

where $\lambda = 1/f_1 \ldots f_k$ and $I = \{1, \ldots, p\}$. But since

$$f_k R^{\lambda}_{I\phi} = R^{\lambda \cdot f_k}_{I\phi}$$

and $\lambda \cdot f_k$ is smooth outside $V_{I'}$, where $I' = I \setminus \{k\}$, we conclude by Proposition 4.4.3 that each $f_k R^{\lambda}_{I\phi}$ vanishes and the corollary follows.

Proof of Theorem 6.1.1. In view of Theorem 5.3.2 and Remark 5.3.4 we only have to prove that each ψ^N_{IJ}, $|I| < p$, is of the form (2). We start by recalling that we obtained Theorem 5.3.2 by taking limits

$$h(w) = \lim_{\delta \to 0} \lim_{r_p \to 0} \ldots \lim_{r_1 \to 0} \quad \text{(right-hand side of Chapter 5 (8))}.$$

Now, the right-hand side of Chapter 5 (8) is a sum of terms of the form

$$\int_D G^{(s)}(\langle Q^{\delta,r}, w-z\rangle + 1)\omega_s^{\delta,r}(z,w),$$

where $G^{(s)}$ is the sth derivative of the polynomial G, G is of degree p, $G(1) = 1$, $\omega_s^{\delta,r}$ is a smooth (n,n)-form and

$$\langle Q^{\delta,r}, w-z\rangle + 1 = \frac{1}{p}\sum_{k=1}^{p}\left(\frac{\chi_k^{\delta,r_k}(z)f_k(w)}{f_k(z)} + (1-\chi_k^{\delta,r_k}(z))\right) = x + y,$$

with

$$x = \frac{1}{p}\sum \frac{\chi_k^{\delta,r_k}(z)f_k(w)}{f_k(z)}$$

and

$$y = \frac{1}{p} \sum (1 - \chi_k^{\delta, r_k}(z)).$$

For $s = p$ we obtain the last term of (1) as limit (Remark 5.3.4). For $s < p$ we can write

$$G^{(s)}(x+y) = G^{(s)}(y) + x\tilde{G}_s(x, y),$$

where $\tilde{G}_s$ is a polynomial of degree $p-s-1$. Since

$$\int_D x\tilde{G}_s(x, y)\omega_s^{\delta, r}(z, w) = \sum_k \int_D f_k(w) \frac{\chi_k^{\delta, r_k}(z)}{f_k(z)} \tilde{G}_s \omega_s^{\delta, r}$$

it follows that

$$\lim_{\delta \to 0} \lim_{r_p \to 0} \ldots \lim_{r_1 \to 0} \int_D G^{(s)} \omega_s^{\delta, r} = \text{terms of the desired kind} +$$
$$+ \lim_{\delta \to 0} \lim_{r_p \to 0} \ldots \lim_{r_1 \to 0} \int_D G^{(s)}(y) \omega_s^{\delta, r}$$

so we will be done if we show that, for an appropriate choice of G, this latter limit becomes zero. Examining Chapter 5 (8) again we find that $\omega_s^{\delta, r}$ is of the form

$$\omega_s^{\delta, r} = \sum_{|I| = s} \omega_I^{\delta, r},$$

with

$$\omega_I^{\delta, r} = A_I^N \wedge \frac{\bar{\partial} \chi_{i_1}^{\delta, r_{i_1}}}{f_{i_1}} \wedge \ldots \wedge \frac{\bar{\partial} \chi_{i_s}^{\delta, r_{i_s}}}{f_{i_s}}, \tag{3}$$

A_I^N being an $(n, n-s)$-form of class C^{N-2} on $\tilde{D}$. It will of course be enough to prove that

$$\int_D G^{(s)}(y) \omega_I^{\delta, r} \to 0, \quad \text{for each } I \subset \{1, \ldots, p\} \text{ with } |I| = s.$$

To simplify the notation we assume that $I = S = \{1, \ldots, s\}$. We also put

$$y = \frac{1}{p}(y_1 + \ldots + y_p), \quad y_j = 1 - \chi_j^{\delta, r_j}.$$

Let us first see that every term of $G^{(s)}(y)$ which contains some y_j, $j \notin S$, say

$y_j M(y_1, \ldots, y_p)$, where M is a monomial, satisfies

$$\int_D y_j M(y_1, \ldots, y_p) \omega_S^{\delta,r} \to 0, \quad \text{as} \quad r_1, \ldots, r_p \quad \text{and} \quad \delta \to 0.$$

Since

$$y_k = 1 - \chi_k^{\delta, r_k}$$

we have

$$M(y_1, \ldots, y_p) = P(\chi_1^{\delta, r_1}, \ldots, \chi_p^{\delta, r_p}),$$

where P is a polynomial a typical term of which is of the form

$$\text{const.}\ \chi_{j_1}^{\delta, r_{j_1}} \ldots \chi_{j_t}^{\delta, r_{j_t}}, \quad j_k \in \{1, \ldots, p\}$$

and we shall show that

$$\int_D y_j \chi_{j_1}^{\delta, r_{j_1}} \ldots \chi_{j_t}^{\delta, r_{j_t}} \omega_S^{\delta,r} \to 0,$$

or what amounts to the same thing

(4) $$\lim_{\delta, r \to 0} \int_D \chi_{j_1}^{\delta, r_{j_1}} \ldots \chi_{j_t}^{\delta, r_{j_t}} \omega_S^{\delta,r} = \lim_{\delta, r \to 0} \int_D \chi_j^{\delta, r_j} \chi_{j_1}^{\delta, r_{j_1}} \ldots \chi_{j_t}^{\delta, r_{j_t}} \omega_S^{\delta,r}.$$

If we recall (3) and apply Proposition 5.2.3, (4) becomes

$$\lim_{\delta \to 0} C \int_{D_{SJ'}^{\delta}} \lambda A_S^N = \lim_{\delta \to 0} C \int_{D_{S,J' \cup \{j\}}^{\delta}} \lambda A_S^N,$$

where $\lambda = (f_1 \ldots f_s)^{-1}$ and $J' = \{j_1, \ldots, j_t\}$.

After dividing by the constant C, this can be rewritten as

(5) $$R_{SJ'}^{\lambda}(A_S^N) = R_{S,J' \cup \{j\}}^{\lambda}(A_S^N).$$

But, since λ is smooth outside V_S (and hence outside $V_{S \cup J'}$), (5) is a consequence of Proposition 4.4.2. We may thus neglect all the terms of $G^{(s)}(y)$ containing some y_j, $j \notin S$ and it remains to be shown that G can be picked so that

(6) $$\lim_{\delta, r \to 0} \int_D G^{(s)}\left(\frac{1}{p}(y_1 + \ldots + y_s)\right) \omega_S^{\delta,r} = 0, \quad s = 0, \ldots, p-1.$$

Now,

$$G^{(s)}\left(\frac{1}{p}(y_1+\ldots+y_s)\right)\omega_S^{\delta,r} = \lambda A_S^N \wedge \hat{\omega},$$

where

$$\hat{\omega} = G^{(s)}\left(\frac{1}{p}(y_1+\ldots+y_s)\right)\bar{\partial}\chi_1^{\delta,r_1} \wedge \ldots \wedge \bar{\partial}\chi_s^{\delta,r_s}.$$

We also have, for $k \geqq 0$,

$$y_j^k\bar{\partial}\chi_j^{\delta,r_j} = -(1-\chi_j^{\delta,r_j})^k\bar{\partial}(1-\chi_j^{\delta,r_j}) = -\frac{1}{k+1}\bar{\partial}(1-\chi_j^{\delta,r_j})^{k+1}$$

and letting $r_j \to 0$ we get (since $\lim \chi_j^{\delta,r_j} = \chi_j^\delta = (\chi_j^\delta)^m$ for all $m > 0$)

$$\begin{aligned}\lim y_j^k\bar{\partial}\chi_j^{\delta,r_j} &= -\frac{1}{k+1}\lim \bar{\partial}(1-\chi_j^{\delta,r_j})^{k+1}\\ &= -\frac{1}{k+1}\lim \bar{\partial}(1-\chi_j^{\delta,r_j}) = \frac{1}{k+1}\lim \bar{\partial}\chi_j^{\delta,r_j}.\end{aligned}$$

It follows that

$$\lim_{r\to 0} \hat{\omega} = C_s \lim_{r\to 0} \bar{\partial}\chi_1^{\delta,r_1} \wedge \ldots \wedge \bar{\partial}\chi_s^{\delta,r_s},$$

where C_s is a constant. In fact, since

$$\frac{1}{k+1} = \int_0^1 y_j^k dy_j, \quad \text{for} \quad k \geqq 0,$$

we get

$$\begin{aligned}C_s &= \int_{[0,1]^s} G^{(s)}\left(\frac{1}{p}(y_1+\ldots+y_s)\right)dy_1\ldots dy_s\\ &= p_s \int_{[0,1]^s} \partial^s/\partial y_1\ldots\partial y_s G\left(\frac{1}{p}(y_1+\ldots+y_s)\right)dy_1\ldots dy_s\\ &= p^s \sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}G(j/p).\end{aligned}$$

Hence, if we choose

$$G(\zeta) = \frac{1}{p!} \prod_{j=0}^{p-1} (p\zeta - j),$$

we get $C_s = 0$, $s = 0, \ldots, p-1$, so (6), and hence the theorem, follows. (Note that G is of degree p and $G(1) = 1$, as required.)

6.2. IDEALS IN RINGS OF ENTIRE FUNCTIONS.

Let $\varphi: \mathbf{C}^n \to \mathbf{R}_+$ be a smooth convex function and define a ring A_φ by

$$A_\varphi = \{h: \mathbf{C}^n \to \mathbf{C}\,;\, h \text{ is holomorphic and } |h(z)| \leqq C \exp(C\varphi(z)), \text{ for some } C\}.$$

If $f: \mathbf{C}^n \to \mathbf{C}^p$ is an entire mapping such that $f_j \in A_\varphi$ we denote by $I_{\varphi,f}$ the ideal in A_φ generated by the f_j's.

We shall assume that φ satisfies the following three conditions:

(i) A_φ contains all polynomials (and hence $\lim_{|z| \to \infty} \varphi(z) = \infty$),
(ii) A_φ is closed under differentiation,
(iii) for all $\alpha, \beta \in \mathbf{N}^n$ one can find constants $C_{\alpha\beta}$ such that $|(\partial/\partial z)^\alpha (\partial/\partial \bar{z})^\beta \varphi(z)| \leqq C_{\alpha\beta} \exp(C_{\alpha\beta}\varphi(z))$.

REMARK 6.2.1. The alternative condition

(iv) $|w - z| \leqq 1 \Rightarrow \varphi(w) \leqq C\varphi(z)$ (for some constant C independent of z) implies (ii) (cf. Hörmander [21, Lemma 2]). Moroever, if we let χ be a non-negative, smooth function on $\mathbf{C}^n$, whose support is contained in the unit ball and which is such that

$$\int_{\mathbf{C}^n} \chi(z)dm(z) = 1,$$

and we define

$$\tilde{\varphi}(z) = \varphi * \chi(z),$$

where φ satisfies (iv), then

$$|(\partial/\partial z)^\alpha (\partial/\partial \bar{z})^\beta \tilde{\varphi}(z)| \leqq C\varphi(z) \left| \int_{\mathbf{C}^n} (\partial/\partial z)^\alpha (\partial/\partial \bar{z})^\beta \chi(z)dm(z) \right|,$$

so $\tilde{\varphi}$ satisfies (iii). Since we also have

$$0 \leqq \tilde{\varphi}(z) - \varphi(z) = \int_{\mathbf{C}^n} \chi(z-w)(\varphi(w) - \varphi(z))dm(w) \leqq (C-1)\varphi(z),$$

it follows that

$$\varphi(z) \leqq \tilde{\varphi}(z) \leqq C\varphi(z)$$

and hence $A_{\tilde{\varphi}} = A_{\varphi}$.

It is clear from the definition of the current R_{IJ}^{λ} (see Chapter 4) that it is invariant under biholomorphic coordinate changes. Indeed, if μ is such a coordinate transformation, we have

$$R_{IJ}^{\lambda,f}(\psi) = R_{IJ}^{\mu^*\lambda,\mu^*f}(\mu^*\psi).$$

Hence, if M is a complex manifold, $f: M \to \mathbf{C}^p$ a holomorphic mapping and λ a semimeromorphic form on M with poles contained in V_f, there is no problem in defining the current $R_{IJ}^{\lambda,f}$ on M.

Suppose now that $f_1,\ldots,f_p$ are polynomials in the variables $(z_1,\ldots,z_n) \in \mathbf{C}^n$ and that they are of degrees $m_1,\ldots,m_p$, respectively. Writing

$$z_k = Z_k/Z_0, \quad k = 1,\ldots,n,$$

and putting

$$F_j(Z_0,\ldots,Z_n) = Z_0^{m_j}f_j(z),$$

we obtain a homogeneous polynomial defined on $\mathbf{C}^{n+1}$. If we let

$$U_k = \{Z \in \mathbf{CP}^n; Z_k \neq 0\}, \quad k = 0,\ldots,n,$$

denote the usual coordinate neighborhoods on $\mathbf{CP}^n$ with local coordinates w^k given by

$$(w_1^k,\ldots,w_n^k) = (Z_0/Z_k,\ldots,Z_{k-1}/Z_k, Z_{k+1}/Z_k,\ldots,Z_n/Z_k),$$

we see that $U_k \cong \mathbf{C}^n$ and $w^0 = z$. We denote by V_j the algebraic variety on $\mathbf{CP}^n$ induced by $F_j = 0$ and $V = \bigcap_{j=1}^p V_j$. Then we put

$$f_j^k(w^k) = Z_k^{-m_j}F_j(Z)$$

and it follows that on $U_0 \cap U_k \cong \mathbf{C}^n \setminus \{z_k = 0\}$ we have two different mappings $f_j(z)$ and $f_j^k(w^k) = z_k^{-m_j}f_j(z)$ defining the variety $V_j \cap U_0 \cap U_k$.

Let $\tilde{f}: \mathbf{C}^n \setminus \{z_k = 0\} \to \mathbf{C}^p$ be the holomorphic map defined by

$$\tilde{f}_j = z_k^{-m_j}f_j(z)$$

and let ψ be a test form on $\mathbf{C}^n \setminus \{z_k = 0\}$. We want to compare $R_{IJ}^{\lambda,f}(\psi)$ and $R_{IJ}^{\lambda,\tilde{f}}(\psi)$, where $I,J \subset \{1,\ldots,p\}$ and λ is a rational function with its poles contained in $V_{I \cup J}$. After a partition of unity we may assume ψ to have small enough support for Hironaka's theorem to apply on $\operatorname{supp}\psi$ (see Section 4.3), i.e. we may assume that

$$f_j(z) = u_j(z)z^{\alpha_j},$$

with u_j nonvanishing. It follows that

$$\tilde{f}_j(z) = \tilde{u}_j(z)z^{\alpha_j},$$

where

$$\tilde{u}_j(z) = u_j(z)/z_k^{m_j}.$$

Now, we can write $\lambda = z^{-\lambda}a$, where $a \in C^\infty(\operatorname{supp}\psi)$, and it is not too hard to prove that

$$\lim_{\delta \to 0} \int_{D_{IJ}^\delta(u_1 z^{\alpha_1}, \ldots, u_p z^{\alpha_p})} z^{-\gamma}\psi' = \lim_{\delta \to 0} \int_{D_{IJ}^\delta(\tilde{u}_1 z_1^{\alpha_1}, \ldots, \tilde{u}_p z^{\alpha_p})} z^{-\gamma}\psi'$$
$$= \lim_{\delta \to 0} \int_{D_{IJ}^\delta(z^{\alpha_1}, \ldots, z^{\alpha_p})} z^{-\gamma}\psi',$$

where $\psi' = a\psi$.

This is carried out in Coleff and Herrera [7, Proposition 2.15] for the case when $J = \phi$ or $\{p\}$ and the argument in the general case is similar. (That is, one uses the coordinate transformation μ (see p. 22) to conclude that

$$\int_{D_{IJ}^\delta(u_1 z^{\alpha_1}, \ldots, u_p z^{\alpha_p})} z^{-\gamma}\psi' = \int_{D_{UJ}^\delta(w^{\alpha_1}, \ldots, w^{\alpha_p})} (\mu^{-1})^*(z^{-\gamma}\psi')$$

and then one shows that

$$\lim_{\delta \to 0} \int_{D_{IJ}^\delta(w^{\alpha_1}, \ldots, w^{\alpha_p})} \{(\mu^{-1})^*(z^{-\gamma}\psi'(z)) - w^{-\gamma}\psi'(w)\} = 0.)$$

Hence we have

$$R_{IJ}^{\lambda, f}(\psi) = R_{IJ}^{\lambda, \tilde{f}}(\psi).$$

That is, it is immaterial whether we use the functions $\tilde{f}_j$ or just their numerators f_j to define our tube. It is therefore meaningful to consider the current $R_{IJ}^\lambda = R_{IJ}^{\lambda, V}$ on $\mathbf{CP}^n$, defined by

$$R_{IJ}^{\lambda, V} = \begin{cases} R_{IJ}^{\lambda, f} & \text{on } U_0 = \mathbf{C}^n \\ R_{IJ}^{\lambda, \tilde{f}} & \text{on } U_k, \text{ etc.} \end{cases}$$

This is the situation which we will consider in this section.

THEOREM 6.2.2. *Let $f: \mathbf{C}^n \to \mathbf{C}^p$ be a complete intersection given by $f(z) = (f_1(z), \ldots, f_p(z))$ with the f_j's being polynomials. Let $h \in A_\varphi$. Then for $w \in \mathbf{C}^n$,*

$$h(w) = \sum_{\substack{I,J \\ |I| < p}} hR_{IJ}^{\lambda_{IJ}}(\psi_{IJ}(\cdot, w)) + h\bar{\partial}\frac{1}{f_1} \wedge \ldots \wedge \bar{\partial}\frac{1}{f_p}(\psi(\cdot, w)),$$

where each λ_{IJ} is a rational function (monomial in $1/f_1, \ldots, 1/f_p$), ψ_{IJ} and ψ are C^q forms on $\mathbf{CP}^n$ of bidegree $(n, n-|I|)$ and $(n, n-p)$ respectively ($q \geqq$ the (finite) orders of the $R_{IJ}^{\lambda_{IJ}}$). Moreover, each ψ_{IJ} is holomorphically parametrized by w in such a way that

$$\psi_{IJ}(\cdot, w) = \sum_k f_k(w)\psi_{IJ}^k(\cdot, w) \quad (\textit{for some } C^q \textit{ forms } \psi_{IJ}^k)$$

and

$$|hR_{IJ}^{\lambda_{IJ}}(\psi_{IJ}^k(\cdot, w))| \leqq C_k \exp(C_k\varphi(w)). \tag{7}$$

That is, one can find functions $g_k \in A_\varphi$, $1 \leqq k \leqq p$, such that

$$h(w) = \sum_{k=1}^{p} g_k(w)f_k(w) + h\bar{\partial}\frac{1}{f_1} \wedge \ldots \wedge \bar{\partial}\frac{1}{f_p}(\psi(\cdot, w)), \quad w \in \mathbf{C}^n.$$

COROLLARY 6.2.3. *Under the assumptions given in the theorem, the following two statements are equivalent:*

a) $h \in I_{\varphi, f}$,
b) *the current* $h\bar{\partial}\frac{1}{f_1} \wedge \ldots \wedge \bar{\partial}\frac{1}{f_p} = 0$.

PROOF. See the proof of Corollary 6.1.2.

PROOF OF THEOREM 6.2.2. By Propositions 5.3.5 we have, for $|w| \leqq R$

$$h(w) = \tilde{H}_R(w) + H_R(w), \tag{8}$$

where

$$\tilde{H}_R(w) = \sum_{I,J} hR_{IJ}^{\tilde{\lambda}_{IJ}}(\bar{\partial}\xi_R \wedge \tilde{\psi}_{IJ}(\cdot, w))$$

and

$$H_R(w) = \sum_{I,J} hR_{IJ}^{\lambda_{IJ}}(\xi_R\psi_{IJ}(\cdot, w)), \tag{9}$$

with $\tilde{\psi}_{IJ}(\cdot, w)$ being a smooth form on $\mathbf{C}^n \setminus \{w\}$ of bidegree $(n, n-|I|-1)$, $\psi_{IJ}(\cdot, w)$ a smooth form on $\mathbf{C}^n$ of bidegree $(n, n-|I|)$, and $\tilde{\lambda}_{IJ}$, λ_{IJ} being monomials in $1/f_1, \ldots, 1/f_p$. Recalling that $\tilde{H}_R$ and H_R are obtained as limits

$$\tilde{H}_R(w) = \lim_{\delta \cdot r \to 0} C_n \int_{\mathbf{C}^n} h\bar{\partial}\xi_R \wedge K^{\delta,r}(\cdot, w)$$

and

$$H_R(w) = \lim_{\delta \cdot r \to 0} C_n \int_{\mathbf{C}^n} h\xi_R P^{\delta,r}(\cdot, w),$$

where $H^{\delta,r}$ and $P^{\delta,r}$ are the kernels in Chapter 5 (12, 13), we will make the following choices

$$Q^1(z, w) = (A(\partial\varphi/\partial z_1)(z), \ldots, A(\partial\varphi/\partial z_n)(z)),$$

where A is a positive constant to be specified later,

$$G_1(\zeta) = e^{\zeta},$$

$$G(\zeta) = \frac{1}{p!} \prod_{j=0}^{p-1} (p\zeta - j).$$

The same argument as in the proof of Theorem 6.1.1 shows that each ψ_{IJ} (with $|I| < p$), occuring in the expression (9), is of the form

$$\psi_{IJ}(\cdot, w) = \sum f_k(w)\psi_{IJ}^k(\cdot, w),$$

where the ψ_{IJ}^k are smooth forms on $\mathbf{C}^n$, depending holomorphically on w. The term in (9) corresponding to $|I| = p$ becomes

$$h\bar{\partial}\frac{1}{f_1} \wedge \ldots \wedge \bar{\partial}\frac{1}{f_p}(\xi_R\psi),$$

where

$$\psi = \psi_{I\phi}, \quad I = \{1, \ldots, p\}.$$

We shall show that for each $q \in \mathbf{N}$ one may extend $\tilde{\psi}_{IJ}$ and ψ_{IJ} to C^q forms on all of $\mathbf{CP}^n$ by putting them equal to zero on $\mathbf{CP}^n \setminus \mathbf{C}^n$. (In fact, $\tilde{\psi}_{IJ}$ will have a singularity at w but it will be killed by $\bar{\partial}\xi_R$ when R is large enough.) Since, in particular, the coefficients of $\tilde{\psi}_{IJ}$ tend to zero uniformly (for w in a compact set) as $|z| \to \infty$, it will follow that

$$\lim_{R \to \infty} \bar{\partial}\xi_R \wedge \tilde{\psi}_{IJ}(\cdot, w) = 0$$

and hence (assuming that q is larger than the order of $R_{IJ}^{\bar{\lambda}_{IJ}}$ on $\mathbf{CP}$)

$$\lim_{R \to \infty} \tilde{H}_R(w) = 0.$$

Moreover, we will have (for q larger than the order of $R_{IJ}^{\lambda_{IJ}}$ on $\mathbf{CP}^n$)

$$\lim_{R\to\infty} H_R(w) = \sum_{\substack{I,J,k\\ |I|<p}} f_k(w)hR_{IJ}^{\lambda_{IJ}}(\psi_{IJ}^k(\cdot,w)) + h\bar{\partial}\frac{1}{f_1} \wedge \ldots \wedge \bar{\partial}\frac{1}{f_p}(\psi)$$

and, in view of (8), the theorem (except for the estimate (7)) will be proved.

First, we observe that, since $f_1,\ldots,f_p$ are polynomials, the Hefer maps $B^1,\ldots,B^p$, which occur in $Q^{\delta,r}$ (Chapter 5 (7)) may be taken to be polynomials as well.

Examining Chapter 5 (13) more carefully one sees that each $\psi_{IJ}^k(z,w)$ can be written as a finite sum of terms of the form

$$c\exp(A\langle\partial\varphi, w-z\rangle)(h\Phi)(z)z^a w^b dz \wedge d\bar{z}(B) \tag{10}$$

where $\Phi(z)$ is a product of derivatives of φ, a and b belong to $\mathbf{N}^n$, c is a complex constant and $B \subset \{1,\ldots,n\}$ satisfies $|B| = |I|$. Writing

$$\Psi = c\exp(A\langle\partial\varphi, w-z\rangle)h\Phi z^a w^b,$$

we find that $(\partial/\partial z)^\alpha(\partial/\partial\bar{z})^\beta\Psi$ becomes a finite sum of terms of the form

$$\tilde{c}\exp(A\langle\partial\varphi, w-z\rangle)(\partial/\partial z)^\gamma(h)\tilde{\Phi}z^{\tilde{a}}w^{\tilde{b}},$$

where $\tilde{a},\tilde{b},\tilde{c}$ and $\tilde{\Phi}$ are of the same kind as a,b,c, and Φ respectively. We now use what we know about φ:

By property (i) there is a constant C such that $|\tilde{c}z^{\tilde{a}}| \leqq C\exp(C\varphi(z))$.

By property (ii) there is a constant C such that

$$|(\partial/\partial z)^\gamma(h)| \leqq C\exp(C\varphi(z)).$$

By property (iii) there is a constant C such that

$$|\tilde{\Phi}(z)| \leqq C\exp(C\varphi(z)).$$

It follows that one can in fact find constants $C_{\alpha\beta}$ (which are independent of w) such that

$$|(\partial/\partial z)^\alpha(\partial/\partial\bar{z})^\beta\Psi(z,w)| \leqq \pi_{\alpha\beta}(w)C_{\alpha\beta}|\exp(A\langle\partial\varphi, w-z\rangle + C_{\alpha\beta}\varphi(z))|,$$

where $\pi_{\alpha\beta}$ is a polynomial.

The convexity of φ implies that

$$\varphi(w)-\varphi(z) \geqq 2\,\mathrm{Re}\langle\partial\varphi, w-z\rangle$$

and we obtain

$$|(\partial/\partial z)^\alpha(\partial/\partial\bar{z})^\beta\Psi(z,w)| \leqq \pi_{\alpha\beta}(w)C_{\alpha\beta}\exp(A\varphi(w)/2 + (C_{\alpha\beta}-A/2)\varphi(z)). \tag{11}$$

Assuming (which we may) that $A > 2C_{\alpha\beta}$ for all α,β with $|\alpha|+|\beta| \leqq q$, we find that

$$\lim_{|z| \to \infty} |\pi(z)(\partial/\partial z)^{\alpha}(\partial/\partial \bar{z})^{\beta}\Psi(z, w)| = 0,$$

where π is any polynomial and, recalling that ψ_{IJ}^{k} was decomposed into a sum of terms of the type (10), we conclude that all derivatives up to order q of its coefficients tend to zero quicker than any polynomial as $|z| \to \infty$. Since the different coordinate systems on $\mathbf{CP}^n$ are related by rational transformations it follows that ψ_{IJ} becomes a C^q form on all of $\mathbf{CP}^n$ if we let it equal zero on the hyperplane at infinity, i.e. on $\mathbf{CP}^n \setminus \mathbf{C}^n$.

Except for the harmless singularity at w (recall the factor $|z-w|^{-2\alpha_0-2}$ in Chapter 5 (12)), the same argument as above shows that ψ_{IJ} can be extended to a C^q form on $\mathbf{CP}^n \setminus \{w\}$ by taking A large enough (independently of w). Finally, to prove the estimate (7), we just have to observe that the order of $R_{IJ}^{\lambda_{IJ}}$ on $\mathbf{CP}^n$ is less than q and that it is continuous in the usual seminorms. Since the derivatives of the coefficients of ψ_{IJ} satisfy estimates like (11) we get

$$|hR_{IJ}^{\lambda_{IJ}}(\psi_{IJ}^{k}(\cdot, w))| \leqq \sum_{|\alpha|+|\beta| \leq q} \tilde{\pi}_{\alpha\beta}(w)\exp(A\varphi(w)/2) = \tilde{\pi}(w)\exp(A\varphi(w)/2),$$

for some polynomials $\tilde{\pi}_{\alpha\beta}$ and $\tilde{\pi}$. The desired estimate (7) follows and so does the theorem.

6.3. The local version.

In this section we consider the following local question: Let D be a domain in $\mathbf{C}^n$, $f: D \to \mathbf{C}^p$ a holomorphic mapping and w_0 an arbitrary point in D. Under what conditions may then a holomorphic function h be represented as

$$h(w) = \sum_{k=1}^{p} g_k^{w_0}(w) f_k(w), \tag{12}$$

where w belongs to a neighborhood D_{w_0} of w_0 and $g_k^{w_0}: D_{w_0} \to \mathbf{C}$ are holomorphic? When D_{w_0} is strictly pseudoconvex, the question is answered by Corollary 6.1.2, but we shall see below that the local result may be proved with less effort than the global formula of Theorem 6.1.1 and we also obtain a connection to the cohomological residue of Chapter 3.

Letting $\gamma_{w_0}(h)$ denote the germ of h at w_0, we will write $\gamma_{w_0}(h) \in I_f^{w_0}$ when (12) is satisfied, thereby defining the local ideal $I_f^{w_0}$.

Theorem 6.3.1. *Let D be a domain in $\mathbf{C}^n$, $f: D \to \mathbf{C}^p$ a complete intersection and $h: D \to \mathbf{C}$ a holomorphic function. Then the following three conditions are equivalent:*

a) $\gamma_{w_0}(h) \in I_f^{w_0}$ for all $w_0 \in D$,

b) $h\bar\partial\frac{1}{f_1} \wedge \ldots \wedge \bar\partial\frac{1}{f_p} = 0$,

c) $\operatorname{Res}[h\Omega_f] = 0$,

where

$$\Omega_f(z) = |f|^{2p}\left(\sum_{j=1}^{p}(-1)^{j+1}\bar f_j(z)d\bar f_1(z) \wedge \ldots \underset{j}{\wedge} \ldots \wedge d\bar f_p(z)\right).$$

REMARK 6.3.2. By the general theory of coherent analytic sheaves (Theorem A by Cartan) it follows that if D is pseudoconvex, then condition a) above implies that $h \in I_f$ so Corollary 6.1.2 is a special case of Theorem 6.3.1.

PROOF. c) $\Rightarrow$ a). Put as before $V = \{z \in D; f_1(z) = \ldots = f_p(z) = 0\}$. We always have $\gamma_{w_0}(h) \in I_f^{w_0}$ if $w_0 \notin V$ so we assume $w_0 \in V$. After a linear change of coordinates we obtain the following situation $w_0 = 0$ and

$$V \cap \{|z'| = 0\} \cap \{|z''| < 1\} = \{0\},$$

where $z' = (z_1, \ldots, z_{n-p})$, $z'' = (z_{n-p+1}, \ldots, z_n)$. It follows that we can find $r' > 0$ such that

$$V \cap \{|z'| < r'\} \cap \{|z''| < 1\} \subset \{|z''| < 1/3\}.$$

Take $r = \min(r', 1/3)$ and let D_0 be a smooth, convex domain such that

$$\{|z'| < r\} \cap \{|z''| < 2/3\} \subset D_0 \subset \{|z'| < r\} \cap \{|z''| < 1\}.$$

Then let ϱ_0 be a smooth convex function such that

$$D_0 = \{z \in D; \varrho_0(z) < 0\}$$

and

$$\varrho_0(z) = |z'|^2 - r^2 \quad \text{for} \quad z \in W = D_0 \cap \{|z''| < 1/3\}.$$

(See Figure 1.) Next, define the (k,k)-form $A_0^{N,k}$ as in Chapter 5 (4) with

$$H(z,w) = (\partial\varrho_0/\partial z_1(z), \ldots, \partial\varrho_0/\partial z_n(z))$$

and notice that its coefficients will be of class C^{N-2}. Applying Proposition 5.1.6 with $M = 2$,

$$G_2(\zeta) = \zeta^p \quad \text{and} \quad Q^2 = Q^\varepsilon = (Q_1^\varepsilon, \ldots, Q_n^\varepsilon),$$

$$Q_j^\varepsilon = \sum_{k=1}^{p}\overline{f_k(z)}B_j^k(z,w)\Big/\left(\varepsilon + \sum_{k=1}^{p}|f_k(z)|^2\right),$$

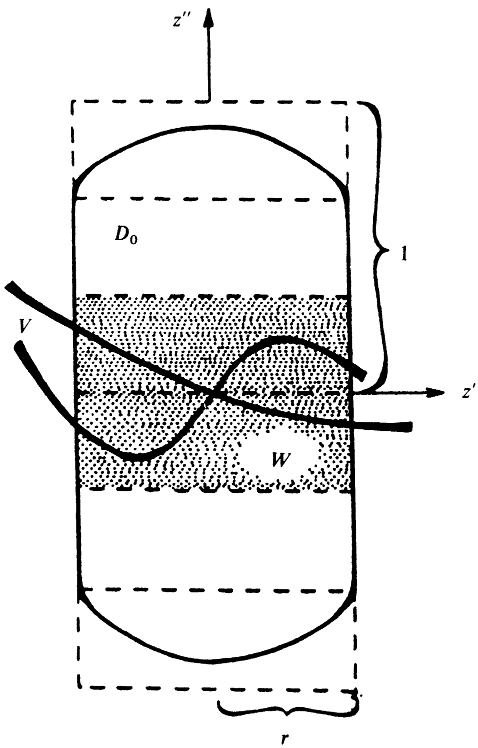

Figure 1.

B_k being a Hefer map for f_k, we obtain for $w \in D_0$ (recall that $q^\varepsilon = \sum_{j=1}^{n} Q_j^\varepsilon dz_j$),

$$h(w) = \sum_{s=0}^{n} C_{n,s} \int_{D_0} h(z) A_0^{N,n-s}(z,w) \wedge G^{(s)}(\langle Q^\varepsilon, w-z\rangle + 1)(\bar{\partial} q^\varepsilon)^s.$$

Since $\bar{\partial} q^\varepsilon$ is of the form $\sum_{j=1}^{p} d\bar{f}_k \wedge \alpha_k$ we have $(\bar{\partial} q^\varepsilon)^s = 0$ when $p < s$ and hence

$$h(w) = \sum_{s=0}^{p} C_{n,s} \int_{D_0} h(z) A_0^{N,n-s}(z,w) \wedge G^{(s)}(\langle Q^\varepsilon, w-z\rangle + 1)(\bar{\partial} q^\varepsilon)^s.$$

Now, on W we know that ϱ_0 depends on the first $n-p$ coordinates only. Hence so does the (k,k)-form $A_0^{N,k}$ and it follows that $A^{N,n-s} = 0$ on W if $s < p$. Thus we get

$$h(w) = \sum_{s=0}^{p} C_{n,s} \int_{D_0 \setminus W} hA^{N,n-s} \wedge G^{(s)}(\bar{\partial} q^\varepsilon)^s + C'_{n,p} \int_W hA^{N,n-p} \wedge (\bar{\partial} q^\varepsilon)^p$$
$$= I^\varepsilon + I^\varepsilon_W.$$

For the same reason as above $A^{N,n-p}$ is $\bar{\partial}$-closed on W and we have

$$I^\varepsilon_W = C'_{n,p} \int_{\partial W} hA^{N,n-p} \wedge q^\varepsilon \wedge (\bar{\partial} q^\varepsilon)^{p-1}$$
$$= C'_{n,p} \int_{\partial W \setminus \partial D_0} hA^{N,n-p} \wedge q^\varepsilon \wedge (\bar{\partial} q^\varepsilon)^{p-1}.$$

On $\partial W \setminus \partial D_0$, we can safely let $\varepsilon \to 0$ and by a straightforward calculation one sees that

$$\lim_{\varepsilon \to 0} q^\varepsilon \wedge (\bar{\partial} q^\varepsilon)^{p-1} = \text{const.}\, \Omega_f \wedge b,$$

where

$$b = \left(\sum_{j=1}^{n} B^1_j dz_j \right) \wedge \ldots \wedge \left(\sum_{j=1}^{n} B^p_j dz_j \right).$$

Writing ψ^N for the $(n, n-p)$-form $A^{N,n-p} \wedge b$ we therefore obtain

$$\lim_{\varepsilon \to 0} I^\varepsilon_W = \text{const.} \int_{\partial W \setminus \partial D_0} h\Omega_f \wedge \psi^N = \text{const.} \int_{D_0 \setminus W} h\Omega_f \wedge \bar{\partial} \psi^N$$
$$= \text{const.} \int_{D_0} h\Omega_f \wedge \bar{\partial} \psi^N = \text{const.}\, \text{Res}[h\Omega_f]([\omega^N]),$$

where the second equality follows from the Stokes theorem and the fact that $\psi^N = 0$ on ∂D_0 and we have written $\omega^N = \pi(\psi^N)$, π being the projection in Chapter 3 (1). But we are assuming $\text{Res}[h\Omega_f] = 0$ and are thus left with

$$h(w) = I^\varepsilon \quad \text{for all} \quad \varepsilon > 0.$$

Hence, letting ε tend to zero, we obtain

$$h(w) = \lim_{\varepsilon \to 0} I^\varepsilon$$

and all that remains is to verify (by direct computation) that

$$\lim_{\varepsilon \to 0} (\bar{\partial} q^\varepsilon)^p = \text{const.}\, \bar{\partial}(\Omega_f \wedge b) = 0$$

outside V and that

$$\lim_{\varepsilon\to 0} G^{(s)}(\langle Q^{\varepsilon}, w-z\rangle + 1) = \lim_{\varepsilon\to 0} \text{const.} \left(\frac{\overline{f(z)}f(w)+\varepsilon}{|f(z)|^2+\varepsilon}\right)^{p-s}$$

is of the form

$$\sum_{k=1}^{p} g_k^s(w) f_k(w) \quad \text{for} \quad s < p.$$

a) $\Rightarrow$ b). This follows form Proposition 4.4.3 precisely as in the proof of Corollary 6.1.2.

b) $\Rightarrow$ c). Pick an arbitrary class $[\omega] \in H_V^{n,n-p}(D)$ and take $\psi \in \mathscr{D}^{n,n-p}(D)$ such that $\pi(\psi) = \omega$, where π is the mapping in Chapter 3 (1). By Remark 3.2.2 we have

$$\operatorname{Res}[h\Omega_f]([\omega]) = \text{const.} \int_{D_f^{\varepsilon}} \frac{h\psi}{f_1 \cdots f_p},$$

where $\varepsilon \in \mathbf{R}_>^p$ and

$$D_f^{\varepsilon} = \{z \in D : |f_j(z)| = \varepsilon_j, \quad j = 1, \dots, p\}.$$

Choosing an admissible path $\varepsilon :]0,1] \to \mathbf{R}_>^p$ we find that for $\delta \in]0,1]$

$$D_f^{\varepsilon(\delta)} = \pm D_{I\phi}^{\delta}$$

with $I = \{1, \dots, p\}$. (The sign depends on how we choose to orient the tubes D_{IJ}^{δ}.)

Since ω is $\bar\partial$-closed it follows that ψ is $\bar\partial$-closed in some neighborhood of V.

$$\int_{D_{I\phi}^{\delta}} \frac{h\psi}{f_1 \cdots f_p} = \lim_{\delta\to 0} \int_{D_{I\phi}^{\delta}} \frac{h\psi}{f_1 \cdots f_p} = h\bar\partial\frac{1}{f_1} \wedge \dots \wedge \bar\partial\frac{1}{f_p}(\psi) = 0,$$

(where the first equality follows by using the Stokes theorem p times, once for each component of $\varepsilon(\delta)$) and finally

$$\operatorname{Res}[h\Omega_f]([\omega]) = \text{const.} \int_{D_{I\phi}^{\delta}} \frac{h\psi}{f_1 \cdots f_p} = 0.$$

But $[\omega]$ was arbitrary and the theorem follows.

Remark 6.3.3. When proving the last implication we used the fact that

$\mathrm{Res}[\omega_f]$ is essentially equal to the resdiue current $\bar\partial(1/f_1) \wedge \ldots \wedge \bar\partial(1/f_p)$ restricted to test forms which are closed near V. This explains why it is much easier to define the cohomological residue than the resdidue current - the limiting process becomes trivial. (Compare the amount of work needed in Chapters 3 and 4.)

COROLLARY 6.3.4. *Let $D \subset \mathbf{C}^n$ be a pseudoconvex domain, $f: D \to \mathbf{C}^p$, $p > 1$, a complete intersection and $h: D \to \mathbf{C}$ a holomorphic function. Then if the equivalent conditions of Theorem* 6.3.1 *are fulfilled the $(0, p-1)$-form $h\Omega_f$ is $\bar\partial$-exact on $D \setminus V$.*

That is, $[h\Omega_f] = 0$.

PROOF. By Remark 6.3.2 we know that $h = \sum_k g_k f_k$ for some g_k, holomorphic on D. Define the $(0, p-2)$-form ω_f^h on $D \setminus V$ by

$$\omega_f^h = |f|^{-2(p-1)}\left(\sum_{1 \le j < k \le p} (-1)^{j+k+1}(g_j \bar f_k - g_k \bar f_j) d\bar f_1 \wedge \ldots \underset{j}{\wedge} \underset{k}{\wedge} \ldots \wedge d\bar f_p\right).$$

We shall see that

$$(p-1)^{-1}\bar\partial\omega_f^h = h\Omega_f.$$

Writing $\omega_f^h = |f|^{-2(p-1)}T$ we get

$$|f|^{2p}\bar\partial\omega_f^h = |f|^2\bar\partial T - (p-1)\left(\sum_i f_i d\bar f_i\right) \wedge T$$

and we have to show that the right-hand side equals

$$(p-1)h \sum_{j=1}^{p} (-1)^{j+1} \bar f_j d\bar f_1 \wedge \ldots \underset{j}{\wedge} \ldots \wedge d\bar f_p.$$

First we calculate $\bar\partial T$

$$\begin{aligned}\bar\partial T &= \sum_{j<k} (-1)^{j+k+1}(g_j d\bar f_k - g_k d\bar f_j) \wedge d\bar f_1 \wedge \ldots \underset{j}{\wedge} \underset{k}{\wedge} \ldots \wedge d\bar f_p \\ &= \sum_j (-1)^{j+1}(p-j) g_j d\bar f_1 \wedge \ldots \underset{j}{\wedge} \ldots \wedge d\bar f_p - \\ &\quad - \sum_k (-1)^k (k-1) g_k d\bar f_1 \wedge \ldots \underset{k}{\wedge} \ldots \wedge d\bar f_p \\ &= (p-1) \sum_j (-1)^{j+1} g_j d\bar f_1 \wedge \ldots \underset{j}{\wedge} \ldots \wedge d\bar f_p.\end{aligned}$$

We then obtain

$$\Big(\sum_i f_i d\bar{f}_i\Big) \wedge T = \sum_{j<k} (-1)^{j+k+1}(f_j d\bar{f}_j + f_k d\bar{f}_k)(g_j\bar{f}_k - g_k\bar{f}_j) d\bar{f}_1 \wedge \dots \underset{j}{\wedge} \underset{k}{\wedge} \dots \wedge d\bar{f}_p$$
$$= \textstyle\sum_1 + \sum_2$$

with

$$\textstyle\sum_1 = \displaystyle\sum_{j<k} (-1)^{j+k+1}(g_j|f_k|^2 d\bar{f}_k - g_k|f_j|^2 d\bar{f}_j) \wedge d\bar{f}_1 \wedge \dots \underset{j}{\wedge} \underset{k}{\wedge} \dots \wedge d\bar{f}_p$$
$$= \sum_j (-1)^{j+1} \Big(\sum_{k\neq j} |f_k|^2\Big)\bar{g}_j df_1 \wedge \dots \underset{j}{\wedge} \dots \wedge d\bar{f}_p$$

and

$$\textstyle\sum_2 = \displaystyle\sum_{j<k} (-1)^{j+k+1}(f_j g_j \bar{f}_k d\bar{f}_j - f_k g_k \bar{f}_j d\bar{f}_k) \wedge d\bar{f}_1 \wedge \dots \underset{j}{\wedge} \underset{k}{\wedge} \dots \wedge d\bar{f}_p$$
$$= \sum_j (-1)^{j+1} \Big(\sum_{k\neq j} -\bar{f}_k g_k\Big)\bar{f}_j d\bar{f}_1 \wedge \dots \underset{j}{\wedge} .. \wedge d\bar{f}_p.$$

Hence

$$|f|^2\bar{\partial}T - (p-1)\Big(\sum_i f_i d\bar{f}_i\Big) \wedge T$$
$$= (p-1)\sum_j (-1)^{j+1}\Big(|f|^2 g_j - \sum_{k\neq j} (|f_k|^2 g_j - f_k g_k \bar{f}_j)\Big) d\bar{f}_1 \wedge \dots \underset{j}{\wedge} \dots \wedge d\bar{f}_p$$
$$= (p-1)\sum_j (-1)^{j+1}\Big(|f_j|^2 g_j + \sum_{k\neq j} f_k g_k \bar{f}_j\Big) d\bar{f}_1 \wedge \dots \underset{j}{\wedge} \dots \wedge d\bar{f}_p$$
$$= (p-1)h \sum_j (-1)^{j+1}\bar{f}_j d\bar{f}_1 \wedge \dots \underset{j}{\wedge} \dots \wedge d\bar{f}_p$$

and we are done.

7. Some examples and final remarks.

7.1. Illustrations in two dimensions.

Confining our attention to holomorphic mappings $f: \mathbf{C}^2 \to \mathbf{C}^2$, we calculate explicitly the associated residue currents in a few cases. For simplicity, we fix an admissible path by putting

$$(\varepsilon_1, \varepsilon_2) = (e^{-1/\delta}, \delta) \quad \text{for} \quad \delta \in]0, 1].$$

Example 7.1.1. Take $f_1 = z_1^m$, $f_2 = z_2^n$ and let $\psi = \psi(z) dz_1 \wedge dz_2$ be a test form. By Taylor's formula we have

$$\psi(z) = \sum_{\substack{0 \leq p+r \leq m-1 \\ 0 \leq s+t \leq n-1}} a_{prst} z_1^p \bar{z}_1^r z_2^s \bar{z}_2^t + O(|z_1|^m |z_2|^n),$$

where

$$a_{prst} = \frac{1}{p!r!s!t!} (\partial^{p+r+s+t} / \partial z_1^p \partial \bar{z}_1^r \partial z_2^s \partial \bar{z}_2^t)(\psi)(0,0).$$

Since

$$\left| \int_{\substack{|z_1| = \varepsilon_1 \\ |z_2| = \varepsilon_2}} \frac{O(|z_1|^m |z_2|^n) dz_1 \wedge dz_2}{z_1^m z_2^n} \right| \leq M(2\pi\varepsilon_1)(2\pi\varepsilon_2), \quad \text{for some constant } M,$$

we have

$$\text{(1)} \qquad \bar{\partial}\frac{1}{f_1} \wedge \bar{\partial}\frac{1}{f_2}(\psi) = \lim_{\delta \to 0} \sum_{\substack{0 \leq p+r \leq m-1 \\ 0 \leq s+t \leq n-1}} a_{prst} \int_{\substack{|z_1| = \varepsilon_1 \\ |z_2| = \varepsilon_2}} z_1^{p-m} \bar{z}_1^r z_2^{s-n} \bar{z}_2^t dz_1 \wedge dz_2.$$

Now, using polar coordinates, we get

$$\int_{\substack{|z_1| = \varepsilon_1 \\ |z_2| = \varepsilon_2}} z_1^{p-m} \bar{z}_1^r z_2^{s-n} \bar{z}_2^t dz_1 \wedge dz_2$$

$$= i^2 \int_{\substack{0 \leq \theta_1 \leq 2\pi \\ 0 \leq \theta_2 \leq 2\pi}} \varepsilon_1^{1+p+r-m} \varepsilon_2^{1+s+t-n} \exp(i(1+p-r-m)\theta_1 + i(1+s-t-n)\theta_2) d\theta_1 \wedge d\theta_2$$

$$= \begin{cases} (2\pi i)^2 \varepsilon_1^{2r} \varepsilon_2^{2t}, & \text{if } \; 1+p-r-m = 1+s-t-n = 0 \\ 0, & \text{otherwise.} \end{cases}$$

But, since we also have $p+r \leqq m-1$ and $s+t \leqq n-1$, the only term of (1) which does not vanish is the one corresponding to $p = m-1$, $r = 0$, $s = n-1$, $t = 0$, and we conclude that

$$\bar{\partial}\frac{1}{z^m} \wedge \bar{\partial}\frac{1}{z^n}(\psi) = \frac{(2\pi i)^2}{(m-1)!(n-1)!} (\partial^{m+n-2} / \partial z_1^{m-1} \partial z_2^{n-1})(\psi)(0,0).$$

EXAMPLE 7.1.2. Take $f_1 = z_2 - z_1^2$, $f_2 = z_2^2$ and let $\psi = \psi(z) dz_1 \wedge dz_2$ be a test form. Consider the change of coordinates given by

$$\begin{cases} z_1 = w_1 \\ z_2 = w_1^2 w_2. \end{cases}$$

The mapping $w \mapsto z$ is an isomorphism when looked upon as a map

$$\mathbf{C}^2 \setminus \{w_1 = 0\} \to \mathbf{C}^2 \setminus \{z_1 = 0\}.$$

(Using a double blow-up of the origin one obtains a manifold X and a proper mapping $X \to \mathbf{C}^2$ which looks like the above in certain local coordinates on X. This is how Hironaka's theorem applies in this simple case.)

Since $z_2 - z_1^2 = w_1^2(w_2 - 1)$ and $z_2^2 = w_1^4 w_2^2$, we have obtained normal crossings (cf. Section 4.2). We have $\varepsilon_1 < \varepsilon_2 \leqq 1$, so it follows that the conditions $|w_1^2(w_2-1)| = \varepsilon_1$ and $|w_1^4 w_2^2| = \varepsilon_2$ imply that

$$\varepsilon_1^2 = |w_1^2(w_2-1)|^2 < |w_1^4 w_2^2| = \varepsilon_2$$

and hence

$$|w_2 - 1|^2 < |w_2|^2.$$

It follows that $\operatorname{Re} w_2 > 1/2$ on the tube $D^\delta = \{|f_1| = \varepsilon_1, |f_2| = \varepsilon_2\}$. We can therefore choose a branch of $\sqrt{w_2}$ on D^δ and define new coordinates by

$$\begin{cases} t_1 = w_1\sqrt{w_2} \\ t_2 = 1 - 1/w_2. \end{cases}$$

The residue current becomes

$$\bar{\partial}\frac{1}{f_1} \wedge \bar{\partial}\frac{1}{f_2}(\psi) = \lim_{\delta \to 0} \int_{\substack{|f_1| = \varepsilon_1 \\ |f_2| = \varepsilon_2}} \frac{\psi(z)dz_1 \wedge dz_2}{(z_2 - z_1^2)z_2^2}$$

$$= \lim_{\delta \to 0} \int_{\substack{|w_1^2(w_2-1)| = \varepsilon_1 \\ |w_1^4 w_2^2| = \varepsilon_2}} \frac{\psi(w_1, w_1^2 w_2)dw_1 \wedge dw_2}{w_1^4 w_2^2(w_2 - 1)}$$

$$= \lim_{\delta \to 0} \int_{\substack{|t_1^2 t_2| = \varepsilon_1 \\ |t_1^4| = \varepsilon_2}} \frac{\psi(t_1\sqrt{1-t_2}, t_1^2)dt_1 \wedge dt_2}{t_1^4 t_2 \sqrt{1-t_2}},$$

where $\sqrt{1-t_2}$ is the branch given by $\sqrt{1-t_2} = 1/\sqrt{w_2}$.

Writing $\varepsilon_1' = \varepsilon_1/\sqrt{\varepsilon_2}$ and $\varepsilon_2' = \sqrt[4]{\varepsilon_2}$ (note that $(\varepsilon_1', \varepsilon_2')$ is an admissible path) we get

$$\bar\partial\frac{1}{f_1}\wedge\bar\partial\frac{1}{f_2}(\psi)=\lim_{\delta\to 0}\int_{\substack{|t_2|=\varepsilon_1'\\|t_1|=\varepsilon_2'}}\frac{\psi(t_1\sqrt{1-t_2},t_1^2)dt_1\wedge dt_2}{t_1^4t_2\sqrt{1-t_2}}$$

$$=-(2\pi i)^2\frac{1}{6}\frac{\partial^3}{\partial t_1^3}\left(\frac{\psi(t_1\sqrt{1-t_2},t_1^2)}{\sqrt{1-t_2}}\right)\Bigg|_{t_1=t_2=0}$$

$$=-(2\pi i)^2\left\{\frac{1}{6}\frac{\partial^3}{\partial z_1^3}\psi(0,0)+\frac{\partial^2}{\partial z_1\partial z_2}\psi(0,0)\right\},$$

where the second equality follows from Example 7.1.1.

EXAMPLE 7.1.3. (Coleff and Herrera [7, p. VII]). Take $f_1=z_1z_2$, $f_2=z_2$ and let $\psi=\psi(z)dz_1\wedge dz_2$ be a test form. Since $V=\{z_2=0\}$ has codimension one, this is not a complete intersection. Putting $\varepsilon_1'=\varepsilon_1/\varepsilon_2$ and $\varepsilon_2'=\varepsilon_2$ we get

$$R^{(f_1f_2)^{-1}}_{\{1,2\},\emptyset}(\psi)=\lim_{\delta\to 0}\int_{\substack{|z_1z_2|=\varepsilon_1\\|z_2|=\varepsilon_2}}\frac{\psi(z)dz_1\wedge dz_2}{z_1z_2^2}$$

$$=\lim_{\delta\to 0}\int_{\substack{|z_1|=\varepsilon_1'\\|z_2|=\varepsilon_2'}}\frac{\psi(z)dz_1\wedge dz_2}{z_1z_2^2}=(2\pi i)^2\frac{\partial\psi}{\partial z_2}(0,0).$$

Since $(z_1z_2^2)^{-1}$ is smooth outisde $V_1=\{f_1=0\}$, it follows that the condition of complete intersections is necessary in Proposition 4.4.3 (and hence in Proposition 4.4.2 as well). If we now write $g_1=f_2$ and $g_2=f_1$ we obtain

$$R^{(g_1g_2)^{-1}}_{\{1,2\},\emptyset}(\psi)=\lim_{\delta\to 0}\int_{\substack{|z_2|=\varepsilon_1\\|z_1z_2|=\varepsilon_2}}\frac{\psi(z)dz_1\wedge dz_2}{z_1z_2^2}=0,$$

where the last equality follows from the fact that if

$$M=\sup\{|z_1|;(z_1,z_2)\in\operatorname{supp}\psi\},$$

then $\delta<1/M$ implies that

$$\varepsilon_1=e^{-1/\delta}<\delta/M=\varepsilon_2/M$$

and hence

$$\operatorname{supp}\psi \cap \{|z_2| = \varepsilon_1\} \cap \{|z_1 z_2| = \varepsilon_2\} = \emptyset.$$

This shows that the conclusion in Proposition 4.4.8 is not true without the assumption of complete intersections.

7.2. Concluding comments.

Remark 7.2.1. The condition $h\bar\partial(1/f_1) \wedge \dots \wedge \bar\partial(1/f_p) = 0$ which (by Corollary 6.1.2 and Theorem 6.3.1) is fulfilled precisely when $h \in I_f$ may be interpreted in a more concrete way:

Example 7.1.1 shows that if $f_1 = z_1^m$, $f_2 = z_2^n$, then $h\bar\partial(1/f_1) \wedge \bar\partial(1/f_2) = 0$ can be reformulated as

$$(\partial^{m+n-2}/\partial z_1^{m-1}\partial z_2^{n-1})(h\psi)(0,0) = 0, \quad \text{for all } \psi.$$

Hence, by Leibniz' formula,

$$(\partial^{j+k}/\partial z_1^j \partial z_2^k)h(0,0) = 0, \quad j = 0,\dots,m-1, \quad k = 0,\dots,n-1.$$

If $f_1 = z_2 - z_1^2$, $f_2 = z_2^2$ we get from Example 7.1.2 that

$$(\tfrac{1}{6}\partial^3/\partial z_1^3 + \partial^2/\partial z_1 \partial z_2)(h\psi)(0,0) = 0, \quad \text{for all } \psi.$$

That is,

$$\begin{aligned} h(0,0) = \partial h/\partial z_1(0,0) &= (\tfrac{1}{2}\partial^2/\partial z_1^2 + \partial/\partial z_2)h(0,0) \\ &= (\tfrac{1}{6}\partial^3/\partial z_1^3 + \partial^2/\partial z_1 \partial z_2)h(0,0) = 0. \end{aligned}$$

(See the examples in Ehrenpreis [9, p. 37].)

More generally, from Theorems 1.8.3 and 4.2.2 of Coleff and Herrera [7] it follows that, if the test form $\psi = \sum_{|I|=p}\psi_I(z)dz \wedge d\bar z(I)$ has small enough support, we have

$$h\bar\partial\frac{1}{f_1} \wedge \dots \wedge \bar\partial\frac{1}{f_p}(\psi) = \sum_{|I|=p} \lim_{\delta\to 0} \int_{V \cap \{|k_I| > \delta\}} D_I(h\psi_I)dz \wedge dz(I),$$

where the k_I are holomorphic functions on V and the D^I are differential operators on V with meromorphic coefficients whose poles are contained in $\{k_I = 0\}$. This means that $h \in I_f$ if and only if h satisfies certain differential equations on V. Notice the resemblance to the Noetherian operators in e.g. Björk [4, Section 8.4].

Remark 7.2.2. Let $D \subset \mathbf{C}^n$ and $f: D \to \mathbf{C}^p$ a complete intersection. There is a simple relation between the usual current of integration

$$C_p(\partial\bar{\partial}\log(|f_1|^2+\ldots+|f_p|^2))^p$$

(see Lelong [23]) and the residue current $\bar{\partial}(1/f_1) \wedge \ldots \wedge \bar{\partial}(1/f_p)$. Indeed, from Coleff and Herrera [7, Section 1.9], it follows that

$$\text{(2)} \qquad \sum m_k[V_k] = (2\pi i)^{-p} df_1 \wedge \ldots \wedge df_p \wedge \bar{\partial}\frac{1}{f_1} \wedge \ldots \wedge \bar{\partial}\frac{1}{f_p},$$

where m_k denotes the multiplicity of f along the irreducible component V_k of V. This relation is also suggested by the following observation: We saw in Section 6.3 that $\bar{\partial}(1/f_1) \wedge \ldots \wedge \bar{\partial}(1/f_p)$ is essentially equal to $\mathrm{Res}[\Omega_f]$ which (see Chapter 3) can be viewed as $\bar{\partial}\Omega_f$. A calculation shows that

$$\partial\log|f|^2 \wedge (\partial\bar{\partial}\log|f|^2)^{p-1} = \text{const.}\, df_1 \wedge \ldots \wedge df_p \wedge \Omega_f$$

and (2) is (in a vague sense) obtained by taking $\bar{\partial}$ of this equation. Furthermore, we see that if $df_1 \wedge \ldots \wedge df_p \neq 0$, then the residue current has measure coefficients and the condition $h\bar{\partial}(1/f_1) \wedge \ldots . \wedge \bar{\partial}(1/f_p) = 0$ is just $h|_V = 0$.

Example 7.2.3. Consider the ideal I_f generated by $f_1 = z_1 z_2$ and $f_2 = z_2^2$. Since $f = (f_1, f_2)$ is not a complete intersection, we can not use the theorems of Chapter 6 directly. However, if we write

$$f' = (f'_1, f'_2) = (z_1, z_2^2)$$

and

$$f'' = z_2,$$

we get that

$$I_f = I_{f'} \cap I_{f''},$$

and, since f' and f'' are complete intersections, it follows that

$$h \in I_{f'} \Leftrightarrow h\bar{\partial}\frac{1}{z_1} \wedge \bar{\partial}\frac{1}{z_2^2} = 0 \Leftrightarrow h(0,0) = \partial h/\partial z_2(0,0) = 0$$

and

$$h \in I_{f''} \Leftrightarrow h\bar{\partial}\frac{1}{z_2} = 0 \Leftrightarrow h(z_1, 0) = 0 \quad \text{for all } z_1.$$

We conclude that $h \in I_f$ if and only if

$$h(z_1, 0) = \partial h/\partial z_2(0,0) = 0 \quad \text{for all } z_1.$$

This can clearly be generalized to arbitrary intersections of ideals whose generators are complete intersections.

REMARK 7.2.4. The residue current $\bar{\partial}(1/f)$ is in a sense discontinuous in f. Indeed, we have for $r > 0$

$$\begin{aligned}\bar{\partial}\frac{1}{z^2-r^2}(\psi(z)dz) &= \lim_{\varepsilon\to 0}\int_{|z^2-r^2|=\varepsilon}\frac{\psi(z)dz}{z^2-r^2}\\ &= \lim_{\varepsilon\to 0}\frac{1}{2r}\left(\int_{|z^2-r^2|=\varepsilon}\frac{\psi(z)dz}{z-r}-\int_{|z^2-r^2|=\varepsilon}\frac{\psi(z)dz}{z+r}\right)\\ &= \frac{\pi i}{r}(\psi(r)-\psi(-r)).\end{aligned}$$

It follows that

$$\lim_{r\to 0}\bar{\partial}\frac{1}{z^2-r^2} = -2\pi i\frac{\partial\delta_0}{\partial x}d\bar{z},$$

where $x = \operatorname{Re} z$ and δ_0 is the Dirac measure at the origin. On the other hand,

$$\bar{\partial}\frac{1}{z^2} = -2\pi i\frac{\partial\delta_0}{\partial z}d\bar{z},$$

(cf. Example 7.1.1) and so

$$\bar{\partial}\frac{1}{z^2-r^2}\quad\text{tends to}\quad\bar{\partial}\frac{1}{z^2}$$

as an analytic functional – but not in terms of currents.

REFERENCES

1. C. A. Berenstein and B. A. Taylor, *Interpolation problems in* $\mathbf{C}^n$ *with applications to harmonic analysis*, J. Analyse Math. 38 (1980), 188–254.
2. B. Berndtsson, *A formula for interpolation and division in* $\mathbf{C}^n$, Math. Ann. 263 (1983), 399–418.
3. B. Berndtsson and M. Andersson, *Henkin-Ramirez formulas with weight factors*, Ann. Inst. Fourier (Grenoble) 32 (1982), 91–110.
4. J.-E. Björk, *Rings of Differential Operators* (North-Holland Math. Library 21), North-Holland Publ. Co., Amsterdam - New York, 1979.
5. P. Bonneau, A. Cumenge, and A. Zériahi, *Division dans les espaces de Lipschitz de fonctions holomorphes*, C.R. Acad. Sci. Paris Sér. I, Math. 297 (1983), 517–520.
6. L. Bungart, *Integration on real analytic varieties* I, II, J. Math. Mech. 15 (1966), 1039–1054.
7. N. R. Coleff and M. E. Herrera, *Les Courants Résiduels Associés à une Forme Méromorphe* (Lecture Notes in Mathematics 633), Springer-Verlag, Berlin - Heidelberg - New York, 1978.
8. P. Dolbeault, *General theory of multidimensional residues*, Itogi Nauki i Techniki, Serija Sovremennye Problemy Matematiki, Fundamental'nye Napravlenija 7 (1985), 227–251.
9. L. Ehrenpreis, *Fourier Analysis in Several Complex Variables* (Pure and Appl. Math. 17), Wiley-Interscience Publishers, New York - London, 1970.

10. J. E. Fornaess, *Embedding strictly pseudoconvex domains in convex domains*, Amer. J. Math. 98 (1976), 529–569.
11. G. L. Gordon, *The residue calculus in several complex variables*, Trans. Amer. Math. Soc. 213 (1975), 127–176.
12. P. A. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience [John Wiley and Sons], New York, 1978.
13. R. Harvey, *Integral formulae connected by Dolbeault's isomorphism* in *Complex Analysis* (Proc. Conf., Rice Univ., Houston, Texas, 1969). Rice Univ. Stud. 56 (1970), 77–97.
14. H. Hefer, *Zur Funktionentheorie mehrerer Veränderlichen. Über eine Zerlegung analytischer Funktionen und die Weilsche Integraldarstellung*, Math. Ann. 122 (1950), 276–278.
15. G. M. Henkin, *Integral representation of functions which are holomorphic in strictly pseudo-convex domains, and some applications* (Russian), Mat. Sb. (N.S.) 78 (120) (1969), 611–632.
16. G. M. Henkin, *Integral representation in strongly pseudoconvex domains, and applications to the $\bar\partial$-problem* (Russian), Mat. Sb. (N.S.) 82 (124) (1970), 300–308.
17. G. M. Henkin, *The Lewy equation and analysis on pseudoconvex manifolds*, Russian Math. Surveys 32 (1977), 59–130.
18. M. E. Herrera, *Integration on a semianalytic set*, Bull. Soc. Math. France 94 (1966), 141–180.
19. M. E. Herrera and D. I. Lieberman, *Residues and principal values on complex spaces*, Math. Ann. 194 (1971), 259–294.
20. H. Hironaka, *Resolution of singularities of an algebraic variety over a filed of characteristic zero* I, II, Ann. of Math. 79 (1964), 109–326.
21. L. Hörmander, *Generators for some rings of analytic functions*, Bull. Amer. Math. Soc. 73 (1967), 943–949.
22. L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, 2nd ed. (North-Holland Math. Library 7), North-Holland Publ. Co., Amsterdam - New York - London, 1973.
23. P. Lelong, *Fonctions Plurisousharmoniques et Formes Différentielles Positives*, Gordon and Breach, Paris - London - New York, 1968.
24. N. Øvrelid, *Integral representation formulas and L^p-estimates for the $\bar\partial$-equation*, Math. Scand. 29 (1971), 137–160.
25. J. B. Poly, *Formule de Résidus et Intersection des Chaînes Sous-analytiques, Thèsis*, Université Poitiers, 1974.
26. E. Ramirez de Arellano, *Ein Divisionsproblem und Randintegraldarstellungen in der komplexen Analysis*, Math. Ann. 184 (1969/70), 172–187.
27. H. Skoda, *Application des techniques L^2 à la théorie des idéaux d'une algèbre de fonctions holomorphes avec poids*, Ann. Sci. École Norm. Sup. 5 (1972), 545–579.

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