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ON THE TORELLI PROBLEM OF ABELIAN COMPLEX LIE GROUPS

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0. Preliminaries.

We consider the Torelli problem to be the question, whether the objects of a certain category of complex spaces are determined by the Hodge structures on their cohomology groups. Of course this presumes, that on the given category there is a functor, which assigns to each object a Hodge structure on the cohomology. In the case of compact kählerian manifolds this is well-known since long, and the existence of a functorially depending Hodge structure on arbitrary complex algebraic varieties (possibly singular and noncompact) is due to Deligne (cf. [3]). However on complex manifolds, which are not necessarily assumed to be compact or algebraic, such a Hodge structure functor does not exist (even if we assume our manifolds to be kählerian). So if we want to consider non-compact complex manifolds, we have to impose additional structures. In the present paper we deal with abelian complex Lie groups, which we endow with "pseudo-algebraic" structures, which are modelled in analogy to commutative algebraic groups. In this category we are able to define natural Hodge structures, so the Torelli question makes sense.

As to the Torelli problem there are many positive as well as negative results, but most of them concern the case of compact objects. Apart from the classical result for polarized compact Riemann surfaces a positive answer was obtained for K3-surfaces and also for some classes of singular varieties. For the case of noncompact, but smooth objects, there seem to be no important results.

For the above mentioned category of pseudo-algebraic commutative groups we are able to answer the Torelli problem. More precisely, on the moduli space $\mathrm{Ext}(Y, Z)$ of pseudo-algebraic commutative groups with fixed compact quotient Y and fixed linear subgroup Z the Torelli mapping, which assigns

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to each pseudo-algebraic commutative group X a natural Hodge structure on the cohomology, is a holomorphic homomorphism of groups, which is injective if and only if Z is an algebraic torus $(\mathbf{C}^*)^k$.

In the first two sections we repeat the well-known results on Hodge structures and extensions of Hodge structures, which are needed in the sequel. The third section concerns with the definition of the category of pseudo-algebraic abelian complex Lie groups; in particular we describe moduli spaces of such groups. The results are very similar to the facts known in the algebraic case, but our calculations are explicit and very simple. In section 4, we prove the existence of Hodge structures on pseudo-algebraic groups and we analyse the Torelli mapping, and the last section states some applications of the theory, as are the description of the Néron-Severi group and the Picard group.

Throughout this paper we will assume all our complex Lie groups to be connected. Furthermore we will use the following notations:

$H^* \quad = \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$ for any finitely generated abelian group H;
$V^*_{\mathbf{R}} \quad = \mathrm{Hom}_{\mathbf{R}}(V, \mathbf{C})$, $V^*_{\mathbf{C}} = \mathrm{Hom}_{\mathbf{C}}(V, \mathbf{C})$,
$V^*_{\bar{\mathbf{C}}} \quad = \mathrm{Hom}_{\bar{\mathbf{C}}}(V, \mathbf{C}) = \{f: V \to \mathbf{C}; f \text{ R-linear}, f(iz) = -if(z)\}$
for any complex vector space V;
$G_a(\mathbf{C}) = \mathbf{C}$ as additive algebraic group;
$G_m(\mathbf{C}) = \mathbf{C}^* = \mathbf{C} \setminus \{0\}$ as multiplicative group.

1. Hodge structures.

1.1. *Pure Hodge structures.*

A pure Hodge structure H of weight $m \in \mathbf{Z}$ consists of a finitely generated abelian group $H_{\mathbf{Z}}$ (or a finite-dimensional $\mathbf{Q}$-vector space $H_{\mathbf{Q}}$) and a decreasing filtration $F^{\cdot}(H) = (F^p(H))_{p \in \mathbf{Z}}$ of $H_{\mathbf{C}} := H_{\mathbf{Z}} \otimes \mathbf{C}$ (respectively $H_{\mathbf{Q}} \otimes \mathbf{C}$), such that

$$H_{\mathbf{C}} = F^p(H) \oplus \bar{F}^q(H) \quad \text{for all } p,q \in \mathbf{Z} \text{ with } p+q = m+1,$$

where the bar denotes complex conjugation (note that $H_{\mathbf{C}}$ is defined over $\mathbf{R}$). The filtration $F^{\cdot}(H)$ then induces a decomposition

$$H_{\mathbf{C}} = \bigoplus_{p+q=m} H^{p,q},$$

where $H^{p,q} = F^p(H) \cap \bar{F}^q(H)$, $p+q = m$, such that $H^{q,p} = \bar{H}^{p,q}$ (Hodge decomposition).

If X is a compact kählerian manifold or a complete nonsingular algebraic variety over $\mathbf{C}$, then the cohomology groups $H^m(X, \mathbf{Z})$, $m \in \mathbf{N}$, carry pure

Hodge structures of weight m, which behave functorially with respect to holomorphic mappings or regular algebraic morphisms respectively, and which are compatible with cup-products and Künneth formula (cf. [5, Chapter 0.6.], [2]).

If for example Y is a compact complex torus, given as the quotient W/Λ of a complex vector space W by a lattice $\Lambda \subset W$ of maximal rank $2\dim_{\mathbf{C}} W$, then $H^1(Y, \mathbf{C})$ is isomorphic to $\mathrm{Hom}_{\mathbf{R}}(W, \mathbf{C}) = W^*_{\mathbf{R}}$, and the Hodge structure H^1_Y on $H^1(Y, \mathbf{Z})$ is given by the decomposition:

$$W^*_{\mathbf{R}} \xrightarrow{\sim} W^*_{\mathbf{C}} \oplus W^*_{\bar{\mathbf{C}}}, \quad f \mapsto (g, h)$$

with $g(w) = \frac{1}{2}(f(w) - if(iw))$, $h(w) = \frac{1}{2}(f(w) + if(iw))$. The Hodge structure on $H^m(Y, \mathbf{C}) \cong \wedge^m W^*_{\mathbf{R}}$ is then given by the decomposition

$$\wedge^m W^*_{\mathbf{R}} \cong \bigoplus_{p+q=m} (\wedge^p W^*_{\mathbf{C}} \otimes_{\mathbf{C}} \wedge^q W^*_{\bar{\mathbf{C}}}).$$

1.2. *Mixed Hodge structures.*

A (*mixed*) *Hodge structure* H consists of

(i) a finitely generated abelian group $H_{\mathbf{Z}}$,
(ii) an incressing filtration $W.(H) = (W_d(H))_{d \in \mathbf{Z}}$ of $H_{\mathbf{Q}} := H_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{Q}$ (weight filtration),
(iii) a decreasing filtration $F^{\cdot}(H) = (F^p(H))_{p \in \mathbf{Z}}$ of $H_{\mathbf{C}} = H_{\mathbf{Z}} \otimes \mathbf{C}$ (Hodge filtration),

such that the filtration $F^{\cdot}(H)$ induces on each of the $\mathbf{Q}$-vector spaces $\mathrm{gr}^W_d(H) = W_d(H)/W_{d-1}(H)$ a pure Hodge structure of weight d. As usual one defines

$$H^{p,q} := (\mathrm{gr}^W_{p+q}(H))^{p,q}.$$

If the weight filtration $W.(H)$ is induced by a filtration on $H_{\mathbf{Z}}$, we say that the Hodge structure is integrally defined.

A *morphism of Hodge structures* φ: $H' \to H''$ is a homomorphism $\varphi_{\mathbf{Z}}: H'_{\mathbf{Z}} \to H''_{\mathbf{Z}}$ between the integral lattices, which is compatible with the filtrations $W.$ and $F^{\cdot}$ (and consequently with $\bar{F}^{\cdot}$). Thus the Hodge structures together with their morphisms define a category, which is abelian and admits duals and tensor products.

If H' and H'' are Hodge structures, then the abelian group $\mathrm{Hom}_{\mathbf{Z}}(H'_{\mathbf{Z}}, H''_{\mathbf{Z}})$ carries a natural Hodge structure $\mathrm{Hom}(H', H'')$, which is determined by the filtrations:

$$W_m(\mathrm{Hom}(H', H'')) = \{\varphi \in \mathrm{Hom}_{\mathbf{Q}}(H'_{\mathbf{Q}}, H''_{\mathbf{Q}}) : \varphi(W_r(H')) \subset W_{r+m}(H'') \,\forall\, r \in \mathbf{Z}\}$$

$$F^p(\mathrm{Hom}(H', H'')) = \{\varphi \in \mathrm{Hom}_{\mathbf{C}}(H'_{\mathbf{C}}, H''_{\mathbf{C}}) : \varphi(F^r(H')) \subset F^{r+p}(H'') \,\forall\, r \in \mathbf{Z}\}.$$

By the results of Deligne (cf. [3]) the cohomology groups $H^r(X, \mathbf{Z})$ of an arbitrary complex algebraic variety carry mixed Hodge structures, which depend functorially on the algebraic variety. Moreover the Hodge structures are compatible with cup-product and Künneth formula.

Now let Z be a commutative Stein group. Then Z is isomorphic (as a complex Lie group) to

$$(G_a(\mathbf{C}))^k \times (G_m(\mathbf{C}))^l = \mathbf{C}^k \times (\mathbf{C}^*)^l$$

for some uniquely determined integers $k, l \in \mathbf{N}$, and is hence a linear algebraic group (cf. [7] or [4]). An easy computation shows, that the canonical Hodge structure H^r_Z on $H^r(Z, \mathbf{Z})$ is pure of weight $2r$, and is in fact of type (r, r) (that is $(H^r_Z)^{p,q} = 0$ for $(p, q) \neq (r, r)$). In particular the Hodge filtration on $H^r(Z, \mathbf{C})$ is given by

$$H^r(Z, \mathbf{C}) = F^r(H^r_Z) \supset F^{r+1}(H^r_Z) = 0.$$

1.3. *Jacobians.*

Let H be a mixed Hodge structure, Then

$$\mathrm{Jac}^p(H) := H_{\mathbf{C}}/(F^p(H) + H_{\mathbf{Z}}), \quad p \in \mathbf{Z}$$

is called the pth *Jacobian* of H (here as usual we identify $H_{\mathbf{Z}}$ with its image in the complexification $H_{\mathbf{C}}$).

If $W_r(H)/W_{r-1}(H) = 0$ for all $r > 2p$, then the image of $H_{\mathbf{Z}}$ in the complex vector space $H_{\mathbf{C}}/F^p(H)$ is a discrete subgroup, and consequently $\mathrm{Jac}^p(H)$ has a natural structure of an abelian complex Lie group. Note that $\mathrm{Jac}^p(H)$ depends functorially on H (cf. [1]).

We compute an example, which will be needed in the next sections.

Let $Y = W/\Lambda$ be a complex torus and let $Z = \mathbf{C}^k \times (\mathbf{C}^*)^l$ be a commutative Stein group. We represent Z as a quotient $\mathbf{C}^{k+l}/\Delta$, where $\Delta = \{0\} \times \mathbf{Z}^l \subset \mathbf{C}^{k+l}$. Then $H^1(Z, \mathbf{Z}) \cong \Delta^*$ and its complexification

$$H^1(Z, \mathbf{C}) \cong \mathrm{Hom}_{\mathbf{Z}}(\Delta, \mathbf{C})$$

can be identified with $U^*_{\mathbf{C}}$, where

$$U = \Delta \otimes_{\mathbf{Z}} \mathbf{C} \cong \{0\} \times \mathbf{C}^l \subset \mathbf{C}^{k+1}.$$

Then the Hodge structure $\mathrm{Hom}(H^1_Z, H^1_Y)$ is pure of weight -1, and is given by the Hodge filtration

$$F^{-1}(\mathrm{Hom}(H^1_Z, H^1_Y)) = \mathrm{Hom}_{\mathbf{C}}((H^1_Z)_{\mathbf{C}}, (H^1_Y)_{\mathbf{C}}) = \mathrm{Hom}_{\mathbf{C}}(U^*_{\mathbf{C}}, W^*_{\mathbf{R}}) \supset$$
$$\supset F^0(\mathrm{Hom}(H^1_Z, H^1_Y)) = \mathrm{Hom}_{\mathbf{C}}(U^*_{\mathbf{C}}, W^*_{\mathbf{C}}) \supset F^1(\mathrm{Hom}(H^1_Z, H^1_Y)) = 0.$$

According to the decomposition $W^*_{\mathbf{R}} = W^*_{\mathbf{C}} \oplus W^*_{\bar{\mathbf{C}}}$ we can identify the quotient $\mathrm{Hom}_{\mathbf{C}}(U^*_{\mathbf{C}}, W^*_{\mathbf{R}})/\mathrm{Hom}_{\mathbf{C}}(U^*_{\mathbf{C}}, W^*_{\mathbf{C}})$ with the space $\mathrm{Hom}_{\mathbf{C}}(U^*_{\mathbf{C}}, W^*_{\bar{\mathbf{C}}})$. The projection

$$\mathrm{Hom}_{\mathbf{C}}(U^*_{\mathbf{C}}, W^*_{\mathbf{R}}) \to \mathrm{Hom}_{\mathbf{C}}(U^*_{\mathbf{C}}, W^*_{\bar{\mathbf{C}}})$$

maps the lattice $\mathrm{Hom}_{\mathbf{Z}}(\Delta^*, \Lambda^*)$ onto the lattice

$$\mathrm{Hom}_{\mathbf{Z}}(\Delta^*, \bar{\Lambda}^*) := \{\varphi \in \mathrm{Hom}_{\mathbf{C}}(U^*_{\mathbf{C}}, W^*_{\bar{\mathbf{C}}}) : (2\,\mathrm{Re}\,\varphi)(\delta \otimes_{\mathbf{Z}} \mathbf{C})|\Lambda \in \Lambda^* \quad \text{for all } \delta \in \Delta^*\}.$$

Consequently for the 0th Jacobian of $\mathrm{Hom}(H^1_Z, H^1_Y)$ we have the natural isomorphisms:

$$\begin{aligned}\mathrm{Jac}^0(\mathrm{Hom}(H^1_Z, H^1_Y)) &= \mathrm{Hom}_{\mathbf{C}}(U^*_{\mathbf{C}}, W^*_{\mathbf{R}})/(\mathrm{Hom}_{\mathbf{C}}(U^*_{\mathbf{C}}, W^*_{\mathbf{C}}) + \mathrm{Hom}_{\mathbf{Z}}(\Delta^*, \Lambda^*)) \\ &\cong \mathrm{Hom}_{\mathbf{C}}(U^*_{\mathbf{C}}, W^*_{\bar{\mathbf{C}}})/\mathrm{Hom}_{\mathbf{Z}}(\Delta^*, \bar{\Lambda}^*).\end{aligned}$$

2. Extensions of Hodge structures.

Let H', H'' be Hodge structures. An *extension of* H'' *by* H' is an exact sequence of mixed Hodge structures

$$0 \to H' \xrightarrow{j} H \xrightarrow{p} H'' \to 0.$$

A *morphism* of two such extensions is a commutative diagram of morphisms of Hodge structures

$$\begin{array}{ccccccccc} 0 \to & H' & \to & H & \to & H'' & \to 0 \\ & \downarrow a & & \downarrow c & & \downarrow b & \\ 0 \to & \tilde{H}' & \to & \tilde{H} & \to & \tilde{H}'' & \to 0. \end{array}$$

The notions of epi-, mono-, isomorphisms are evident. An isomorphism of extensions with $a = \mathrm{id}_{H'}$, $b = \mathrm{id}_{H''}$ is called a *congruence* and the extensions H and $\tilde{H}$ of H'' by H' are called *congruent*. An extension

$$0 \to H' \xrightarrow{j} H \xrightarrow{p} H'' \to 0$$

is called a *split extension*, if there exists a *section*, i.e. a morphism $s: H'' \to H$, such that $p \circ s = \mathrm{id}_{H''}$. Any split extension is congruent to the trivial extension, which is given by the direct sum of H' and H''. We define $\mathbf{Ext}(H'', H')$ to be the set of congruence classes of extensions of H'' by H'.

A standard argument on abelian categories yields, that $\mathbf{Ext}(H'', H')$ is an abelian group with Baer summation as composition law and with the class of split extensions as neutral element. $\mathbf{Ext}$ defines a bifunctor on the category of mixed Hodge structures with values in the category of abelian groups, which is contravariant in the first, covariant in the second variable.

We say, that the Hodge structure H' is *separated* from the Hodge structure H'', if the highest weight of H' is less than the lowest weight of H'', i.e. if for all $p \in \mathbf{Z}$, such that $W_q(H')/W_{q-1}(H') = 0$ for all $q \geqq p$, we have $W_p(H'') = 0$.

REMARK. If H' is separated from H'', then for any extension

$$0 \to H' \to H \to H'' \to 0$$

the weight filtration $W.(H)$ is uniquely determined.

In fact the weight filtration on the trivial extension $H' \oplus H''$ is uniquely given by

$$W_p(H' \oplus H'') = W_p(H') \oplus W_p(H'');$$

but since the sequence $0 \to H'_{\mathbf{Q}} \to H_{\mathbf{Q}} \to H''_{\mathbf{Q}} \to 0$ splits and is thus congruent to the trivial one, the weight filtration on H is also unique.

In the case of separated Hodge structures the group $\mathbf{Ext}(H'', H')$ carries a natural complex structure:

PROPOSITION. *Let H', H'' be mixed Hodge structures, such that the extension group* $\operatorname{Ext}(H''_{\mathbf{Z}}, H'_{\mathbf{Z}})$ *of the integral lattices is trivial, and such that H' is separated from H''. Then the* 0th *Jacobian* $\operatorname{Jac}^0(\operatorname{Hom}(H', H''))$ *is an abelian complex Lie group and there is a canonical isomorphism*

$$\Psi : \operatorname{Jac}^0(\operatorname{Hom}(H'', H')) \to \mathbf{Ext}(H'', H').$$

For the proof we refer to [1]. Since we need it in the sequel, we will briefly recall the construction.

Given $\psi \in \operatorname{Hom}_{\mathbf{C}}(H''_{\mathbf{C}}, H'_{\mathbf{C}})$ we consider the automorphism

$$G(\psi) : H_{\mathbf{C}} := H'_{\mathbf{C}} \oplus H''_{\mathbf{C}} \to H_{\mathbf{C}}, \quad G(\psi)(h', h'') := (h' + \psi(h''), h'').$$

Now we define the extension $\Psi(\psi)$ by taking $H_{\mathbf{Z}} := H'_{\mathbf{Z}} \oplus H''_{\mathbf{Z}}$ as integral lattice,

$$W_m(H) := W_m(H') \oplus W_m(H''), \quad m \in \mathbf{Z},$$

as weight filtration, and

$$F^p(H) := G(\psi)(F^p(H') \oplus F^p(H'')), \quad p \in \mathbf{Z},$$

as Hodge filtration.

One easily shows (using the separation hypothesis and the triviality of the integral extensions), that any extension of H'' by H' is congruent to an extension of the form $\Psi(\psi)$. Furthermore two extentions $\Psi(\psi)$ and $\Psi(\tilde{\psi})$ are congruent iff

$$\psi - \tilde{\psi} \in F^0(\mathrm{Hom}(H'', H')) + \mathrm{Hom}_{\mathbf{Z}}(H''_{\mathbf{Z}}, H'_{\mathbf{Z}}).$$

Consequently:

$$\begin{aligned}\mathrm{Ext}(H'', H') &\cong \mathrm{Hom}_{\mathbf{C}}(H''_{\mathbf{C}}, H'_{\mathbf{C}})/(F^0(\mathrm{Hom}(H'', H')) + \mathrm{Hom}_{\mathbf{Z}}(H''_{\mathbf{Z}}, H'_{\mathbf{Z}}))\\ &\cong \mathrm{Jac}^0(\mathrm{Hom}(H'', H')).\end{aligned}$$

We consider again the example of 1.3. So let H^r_Y (respectively H^r_Z) be the canonical Hodge structures on the cohomology groups of the complex torus $Y = W/\Lambda$ (respectively on the linear group $Z = \mathbf{C}^k \times (\mathbf{C}^*)^l$). Here H^r_Y is separated from H^r_Z and thus $\mathbf{Ext}(H^r_Z, H^r_Y)$ is isomorphic to $\mathrm{Jac}^0(\mathrm{Hom}(H^r_Z, H^r_Y))$.

In particular for $r = 1$ we have

$$F^0(H^1_Z) = (H^1_Z)_{\mathbf{C}},\ F^0(H^1_Y) = (H^1_Y)_{\mathbf{C}},\ F^2(H^1_Z) = 0 = F^2(H^1_Y),$$

and hence any extension H of H^1_Z by H^1_Y is already determined by the subspace $F^1(H) \subset H_{\mathbf{C}}$. If H is given by $\psi \in \mathrm{Hom}_{\mathbf{C}}((H^1_Z)_{\mathbf{C}}, (H^1_Y)_{\mathbf{C}})$, then

$$\begin{aligned}F^1(H) &= G(\psi)(F^1(H^1_Y) \oplus F^1(H^1_Z))\\ &= \{(f + \psi(g), g) \in (H^1_Y)_{\mathbf{C}} \oplus (H^1_Z)_{\mathbf{C}} : f \in W^*_{\mathbf{C}}, g \in U^*_{\mathbf{C}}\},\end{aligned}$$

where $U = \{0\} \times \mathbf{C}^1 \subset \mathbf{C}^k \times \mathbf{C}^1$.

3. Pseudo-algebraic structures on abelian complex Lie groups.

3.1. Let X be an abelian complex Lie group. Then X is isomorphic to

$$(G_a(\mathbf{C}))^k \times (G_m(\mathbf{C}))^l \times T,$$

where $k, l \in \mathbf{N}$ are uniquely determined integers and T is a toroidal group (i.e. $\mathcal{O}(T) = \mathbf{C}$), which is uniquely determined up to isomorphy (cf. [7], [8]).

We may represent X as a quotient $X = V/\Gamma$ of a complex vector space V by a discrete subgroup $\Gamma \subset V$.

REMARKS.

a) X is compact iff $\Gamma \subset V$ is of maximal rank $2n$, $n = \dim_{\mathbf{C}} V$.
b) If X is a toroidal group, then $n + 1 \leqq \mathrm{rank}(\Gamma) \leqq 2n$.
c) X is Stein iff $T = 0$.

For further results we refer to [11].

3.2. By the well-known theorem of Chevalley (cf. [9]) every commutative algebraic group X is an extension $0 \to L \to X \to A \to 0$ of an abelian variety A by a linear group L; moreover the extension is unique up to congruence.

In analogy to the algebraic case we define a *pseudo-algebraic structure* on the abelian Lie group X to be a congruence class of strict extensions

$$0 \to Z \xrightarrow{\varepsilon} X \xrightarrow{\pi} Y \to 0$$

of a compact complex torus Y by a Stein group Z (here "strict" means, that the homomorphisms ε and π are assumed to be holomorphic).

A *morphism* between pseudo-algebraic groups X and X' is given by a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z & \xrightarrow{\varepsilon} & X & \xrightarrow{\pi} & Y & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & Z' & \xrightarrow{\varepsilon'} & X' & \xrightarrow{\pi'} & Y' & \longrightarrow & 0, \end{array}$$

where α, β, γ are holomorphic homomorphisms. So the pseudo-algebraic abelian complex Lie groups together with their morphisms form a category, which we will denote by $\mathcal{G}\imath_{\mathrm{alg}}$.

Via the exponential mapping an extension $0 \to Z \xrightarrow{\varepsilon} X \xrightarrow{\pi} Y \to 0$ defines a commutative diagram

$$(*) \qquad \begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Delta & \longrightarrow & \Gamma & \longrightarrow & \Lambda & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & U & \xrightarrow{\hat{\varepsilon}} & V & \xrightarrow{\hat{\pi}} & W & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z & \xrightarrow{\varepsilon} & X & \xrightarrow{\pi} & Y & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where $0 \to U \xrightarrow{\hat{\varepsilon}} V \xrightarrow{\hat{\pi}} W \to 0$ is an exact sequence of complex vector spaces and

$$0 \to \Delta \to \Gamma \to \Lambda \to 0$$

is an exact sequence of free abelian groups, such that $Z \cong U/\Delta$, $Y \cong W/\Lambda$ $X \cong V/\Gamma$.

Conversely, any diagram of the form (*) defines a strict extension of Y by Z.

Remarks. a) Any abelian complex Lie group X carries at least one pseudo-algebraic structure.

To see this, we represent X as quotient $X \cong V/\Gamma$. Let $V' \subset V$ be the complex vector space generated by Γ and let $U' \subset V$ be a complex direct complement of V' in V; then $U' \cap \Gamma = 0$. Furthermore, let $W \subset V'$ be the maximal complex vector space, which is contained in $\Gamma \otimes_{\mathbf{Z}} \mathbf{R} \subset V'$, and let $U'' \subset V'$ be a complex direct complement of W in V'. Then $\Delta := \Gamma \cap U'' \subset U''$ is a lattice in $U := U' \oplus U''$ with $\operatorname{rank} \Delta = \dim_{\mathbf{C}} U''$ and $Z := U/\Delta$ is a Stein group, which is isomorphic to $\mathbf{C}^k \times (\mathbf{C}^*)^l$, $k = \dim_{\mathbf{C}} U'$, $l = \dim_{\mathbf{C}} U''$. If $\hat{\pi}: V \to W$ is the projection according to the decomposition $V = U \oplus W$, then the lattice $\Gamma \subset V$ projects onto a lattice $\Lambda \subset W$ of maximal rank $2\dim_{\mathbf{C}} W$, and the sequence

$$0 \to U \to V \xrightarrow{\hat{\pi}} W \to 0$$

defines an extension $0 \to Z \to X \to Y = W/\Lambda \to 0$ of the desired form.

b) If X is neither compact nor Stein, then there are many non-congruent pseudo-algebraic structures on X. Moreover it may happen, that some of those structures are algebraic (which means quasi-projective in our case), whereas others are not. Note that the pseudo-algebraic structure

$$0 \to Z \to X \to Y \to 0$$

on X is algebraic iff Y is an abelian variety.

c) If $0 \to Z \to X \to Y \to 0$ is a pseudo-algebraic structure on X, then X carries a natural structure of a holomorphic Z-principal fibre bundle over Y. For details on this viewpoint we refer to [11].

3.3. Let Z be a commutative Stein group and Y a compact complex torus. As usual we define $\operatorname{Ext}(Y, Z)$ to be the set of congruence classes of strict extensions of Y by Z. Similarly as in the algebraic case (for which we refer to [10]), $\operatorname{Ext}(Y, Z)$ carries a natural structure of an abelian complex Lie group, which we will describe more precisely. First we note that for compact complex tori Y, Y' and commutative Stein groups Z, Z' there are canonical isomorphisms

$$\operatorname{Ext}(Y \times Y, Z) \cong \operatorname{Ext}(Y, Z) \times \operatorname{Ext}(Y', Z),$$

$$\operatorname{Ext}(Y, Z \times Z') \cong \operatorname{Ext}(Y, Z) \times \operatorname{Ext}(Y, Z').$$

So if Z is a commutative Stein group, isomorphic to

$$(G_a(\mathbf{C}))^k \times (G_m(\mathbf{C}))^l = \mathbf{C}^k \times (\mathbf{C}^*)^l,$$

we have

$$\operatorname{Ext}(Y, Z) \cong (\operatorname{Ext}(Y, \mathbf{C}))^k \times (\operatorname{Ext}(Y, \mathbf{C}^*))^l$$

for any compact complex torus Y. The structure on $\operatorname{Ext}(Y, \mathbf{C})$ and $\operatorname{Ext}(Y, \mathbf{C}^*)$

is given by the following proposition, which is essentially due to Serre (cf. [10], Chapter VII).

PROPOSITION. *Let $Y = W/\Lambda$ be a compact complex torus. Then:*

a) $\mathrm{Ext}(Y, \mathsf{C})$ *is canonically isomorphic to*

$$H^1(Y, \mathcal{O}_Y) \cong \mathrm{Hom}_{\mathsf{R}}(W, \mathsf{C})/\mathrm{Hom}_{\mathsf{C}}(W, \mathsf{C}).$$

b) $\mathrm{Ext}(Y, \mathsf{C}^*)$ *is canonically isomorphic to the dual torus*

$$\hat{Y} := \mathrm{Pic}^0(Y) \cong H^1(Y, \mathcal{O}_Y)/H^1(Y, \mathsf{Z})$$
$$\cong \mathrm{Hom}_{\mathsf{R}}(W, \mathsf{C})/(\mathrm{Hom}_{\mathsf{C}}(W, \mathsf{C}) + \mathrm{Hom}_{\mathsf{Z}}(\Lambda, \mathsf{Z})).$$

PROOF. a) Every extension of Y by $G_a(\mathsf{C}) = \mathsf{C}$ is given by a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \Gamma & \xrightarrow{p} & \Lambda & \longrightarrow & 0 \\ & & \cap\downarrow & & \cap\downarrow & & \cap\downarrow & & \\ 0 & \longrightarrow & \mathsf{C} & \xrightarrow{\hat{\varepsilon}} & \mathsf{C} \oplus W & \xrightarrow{\hat{\pi}} & W & \longrightarrow & 0 \end{array}$$

where $\hat{\varepsilon}$ and $\hat{\pi}$ are the natural injection, respectively projection. The isomorphism p induces a homomorphism

$$p^{-1} \otimes \mathsf{R} =: \tau : W = \Lambda \otimes_{\mathsf{Z}} \mathsf{R} \to \Gamma \otimes_{\mathsf{Z}} \mathsf{R} \hookrightarrow \mathsf{C} \oplus W$$

such that $\hat{\pi} \circ \tau = \mathrm{id}_W$. Thus τ is of the form (φ, id_W) with $\varphi \in \mathrm{Hom}_{\mathsf{R}}(W, \mathsf{C})$, and we have $\Gamma = \tau(\Lambda) \subset \mathsf{C} \oplus W$.

So we constructed a surjective homomorphism

$$\mathrm{Hom}_{\mathsf{R}}(W, \mathsf{C}) \to \mathrm{Ext}(Y, \mathsf{C}), \quad \varphi \mapsto \Phi(\varphi),$$

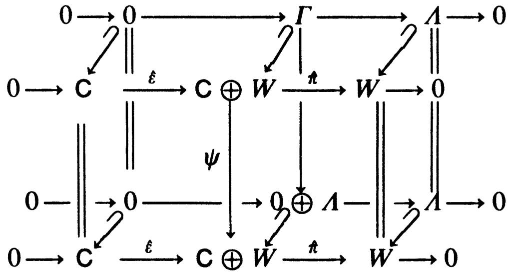
where the extension $\Phi(\varphi)$ is given by the sequence

$$0 \to \mathsf{C} \to \mathsf{C} \oplus W/\Gamma \to W/\Lambda \to 0$$
$$\text{with } \Gamma = (\varphi, \mathrm{id}_W)(\Lambda) \subset \mathsf{C} \oplus W.$$

Now if $\Phi(\varphi)$, $\varphi \in \mathrm{Hom}_{\mathsf{R}}(W, \mathsf{C})$, represents the trivial extension, which is given by

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{C} & \xrightarrow{\hat{\varepsilon}} & \mathsf{C} \oplus W & \xrightarrow{\hat{\pi}} & W & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathsf{C} & \xrightarrow{\varepsilon} & \mathsf{C} \oplus W/\{0\} \oplus \Lambda & \xrightarrow{\pi} & W/\Lambda & \longrightarrow & 0 \\ & & & & \|\cong & & & & \\ & & & & \mathsf{C} \times W\Lambda & & & & \end{array}$$

then there is a C-linear isomorphism $\psi : \mathsf{C} \oplus W \to \mathsf{C} \oplus W$, such that the following diagram commutes:



Now $\psi(z, w) = (z + \sigma(w), w)$, where $\sigma : W \to \mathbf{C}$ is a $\mathbf{C}$-linear mapping. Since

$$(\varphi, \mathrm{id}_W)(\lambda) = \psi^{-1}(0, \lambda) = (-\sigma(\lambda), \lambda) = (-\sigma, \mathrm{id}_W)(\lambda), \quad \forall \lambda \in \Lambda,$$

and since $\Lambda \subset W$ is a lattice of maximal rank, we have $\varphi = -\sigma$; in particular φ is $\mathbf{C}$-linear.

Conversely every $\mathbf{C}$-linear homomorphism $\varphi : W \to \mathbf{C}$ defines an extension congruent to the trivial one. So we get the isomorphism

$$\mathrm{Ext}(Y, \mathbf{C}) \cong \mathrm{Hom}_{\mathbf{R}}(W, \mathbf{C})/\mathrm{Hom}_{\mathbf{C}}(W, \mathbf{C}) \cong H^1(Y, \mathcal{O}_Y).$$

b) An extension of Y by $\mathbf{C}^*$ is determined by a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z} & \xrightarrow{j} & \Gamma & \xrightarrow{p} & \Lambda & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{C} & \xrightarrow{\varepsilon} & \mathbf{C} \oplus W & \xrightarrow{\pi} & W & \longrightarrow & 0 \end{array}$$

where the rows are exact and $\mathbf{Z} \hookrightarrow \mathbf{C}$ is the natural inclusion. In particular Γ is an extension of Λ by $\mathbf{Z}$, and so (since $\mathrm{Ext}(\Lambda, \mathbf{Z}) = 0$) the upper sequence splits. We choose a section $s : \Lambda \to \Gamma$, that is $p \circ s = \mathrm{id}_\Lambda$. Then the $\mathbf{R}$-linear mapping

$$\sigma := s \otimes \mathbf{R} : W = \Lambda \otimes_{\mathbf{Z}} \mathbf{R} \to \Gamma \otimes_{\mathbf{Z}} \mathbf{R} \hookrightarrow \mathbf{C} \oplus W$$

is of the form $\sigma(w) = (\varphi(w), w)$, where $\varphi : W \to \mathbf{C}$ is a $\mathbf{R}$-linear mapping.

If we set

$$g(\varphi) : \mathbf{C} \oplus W \to \mathbf{C} \oplus W, \quad g(\varphi)(z, w) := (z + \varphi(w), w),$$

then one checks easily, that $g(\varphi)(\mathbf{Z} \oplus \Lambda) = \Gamma$.

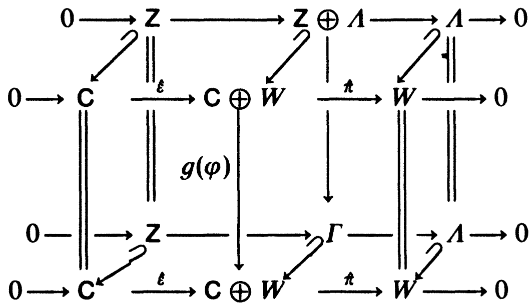
Conversely for $\varphi \in \mathrm{Hom}_{\mathbf{R}}(W, \mathbf{C})$, the setting

$$\Gamma := g(\varphi)(\mathbf{Z} \oplus \Lambda) \subset \mathbf{C} \oplus W$$

defines an extension of Y by $\mathbf{C}^* \cong \mathbf{C}/\mathbf{Z}$. So we have got a surjective homomorphism

$$\Phi : \mathrm{Hom}_{R}(W, C) \to \mathrm{Ext}(Y, C^{*}).$$

If $\varphi \in \mathrm{Hom}_{C}(W, C)$, then in the commutative diagram



the mapping $g(\varphi)$ is C-linear, and so determines a congruence from the trivial extension (upper rows) to the one given by $\Phi(\varphi)$.

Now let $f \in \mathrm{Hom}_{Z}(\Lambda, Z)$, and consider

$$\varphi := f \otimes R : \Lambda \otimes_{Z} R = W \to C.$$

Then

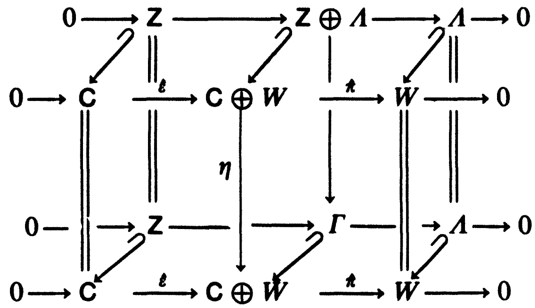
$$g(\varphi)(Z \oplus \Lambda) = Z \oplus \Lambda,$$

so that $\Phi(\varphi)$ defines the trivial extension. Hence we have

$$\mathrm{Hom}_{C}(W, C) + \mathrm{Hom}_{Z}(\Lambda, Z) \subset \mathrm{Ker}(\Phi)$$

(where as usual we identify $\mathrm{Hom}_{Z}(\Lambda, Z)$ with its image in $\mathrm{Hom}_{R}(W, C)$).

On the other hand consider $\varphi \in \mathrm{Hom}_{R}(W, C)$, such that $\Phi(\varphi)$ is the trivial extension, $\Gamma := g(\varphi)(Z \oplus \Lambda)$. Then there is a commutative diagram, which gives a congruence from the trivial extension to the one defined by Γ:



Here η is C-linear, $\eta(Z \oplus \Lambda) = \Gamma$, and η is necessarily of the form

$$\eta(z, w) = (z + \zeta(w), w) = g(\zeta)(z, w),$$

where $\zeta : W \to \mathbf{C}$ is $\mathbf{C}$-linear. Hence

$$g(\varphi)(\mathbf{Z} \oplus \Lambda) = \Gamma = g(\zeta)(\mathbf{Z} \oplus \Lambda).$$

This implies $g(\varphi - \zeta)(\mathbf{Z} \oplus \Lambda) = \mathbf{Z} \oplus \Lambda$ and consequently $(\varphi - \zeta)(\Lambda) \subset \mathbf{Z}$; so there is a homomorphism $r \in \mathrm{Hom}_{\mathbf{Z}}(\Lambda, \mathbf{Z})$ with $\varphi - \zeta = r \otimes_{\mathbf{Z}} \mathbf{R}$. It follows, that

$$\varphi = \zeta + (\varphi - \zeta) \in \mathrm{Hom}_{\mathbf{C}}(W, \mathbf{C}) + \mathrm{Hom}_{\mathbf{Z}}(\Lambda, \mathbf{Z}).$$

So we have a natural isomorphism

$$\mathrm{Ext}(Y, \mathbf{C}^*) = \mathrm{Hom}_{\mathbf{R}}(W, \mathbf{C})/(\mathrm{Hom}_{\mathbf{C}}(W, \mathbf{C}) + \mathrm{Hom}_{\mathbf{Z}}(\Lambda, \mathbf{Z})).$$

The right hand side being isomorphic to $H^1(Y, \mathcal{O}_Y)/H^1(Y, \mathbf{Z})$ (where as always we identify $H^1(Y, \mathbf{Z})$ with its image in $H^1(Y, \mathcal{O}_Y)$ by the projection

$$H^1(Y, \mathbf{Z}) \hookrightarrow H^1(Y, \mathbf{C}) \twoheadrightarrow H^1(Y, \mathcal{O}_Y)$$

according to the Hodge decomposition $H^1(Y, \mathbf{C}) \cong H^{1,0}(Y) \oplus H^{0,1}(Y)$), which is nothing else but the dual torus $\hat{Y} = \mathrm{Pic}^0(Y)$, the proof of the proposition is finished.

Corollary. *If $Y = W/\Lambda$ is a compact complex torus and Z is a commutative Stein group, isomorphic to $\mathbf{C}^k \times (\mathbf{C}^*)^1 \cong U/\Delta$, with $U = \mathbf{C}^{k+1}$, $\Delta = \{0\} \times \mathbf{Z}^1 \subset U$, then*

$$\mathrm{Ext}(Y, Z) \cong \mathrm{Hom}_{\mathbf{R}}(W, U)/(\mathrm{Hom}_{\mathbf{C}}(W, U) + \mathrm{Hom}_{\mathbf{Z}}(\Lambda, \Delta))$$
$$\cong (H^1(Y, \mathcal{O}_Y))^k \times (\hat{Y})^1.$$

4. Hodge structures and Torelli problem on pseudo-algebraic commutative Lie groups.

4.1. We consider abelian complex Lie groups X provided with pseudo-algebraic structures, which are given by exact sequences

$$0 \to Z \xrightarrow{\varepsilon} X \xrightarrow{\pi} Y \to 0,$$

where $Z \cong \mathbf{C}^k \times (\mathbf{C}^*)^1$ is a commutative Stein group and $Y \cong W/\Lambda$ is a compact complex torus.

If we write $Z = U/\Delta$ with $U = U' \oplus U''$, $U' = \mathbf{C}^k$, $U'' = \mathbf{C}^1$, $\Delta = \{0\} \times \Delta'' \subset U$,

$\Delta'' = \mathbf{Z}^1 \subset U''$, then X is determined as quotient $X = V/\Gamma_\varphi$, where

$$\varphi = (\varphi', \varphi'') \in \mathrm{Hom}_{\mathbf{R}}(W, U) \cong \mathrm{Hom}_{\mathbf{R}}(W, U') \times \mathrm{Hom}_{\mathbf{R}}(W, U''),$$

$$\Gamma_\varphi = g(\varphi)(\Delta \oplus \Lambda) = \{(\delta + \varphi(\lambda), \lambda); \delta \in \Delta, \lambda \in \Lambda\} \subset V = U \oplus W,$$

and ε, π are induced by the canonical injection $U \hookrightarrow V$, respectively projection $V \twoheadrightarrow W$.

REMARKS. a) The homomorphism $\varphi'' \in \mathrm{Hom}_{\mathbf{R}}(W, U'')$ defines an extension $X'' := V''/\Gamma_{\varphi''}$, $V'' = U'' \oplus W$, $\Gamma_{\varphi''} = g(\varphi'')(\Delta'' \oplus \Lambda) \subset V''$, of Y by $Z'' := U''/\Delta'' = (\mathbf{C}^*)^1$:

$$0 \to Z'' \xrightarrow{\varepsilon''} X'' \xrightarrow{\pi''} Y \to 0.$$

If $p: U \to U''$ denotes the projection to the second factor, then p maps Γ_φ bijectively onto $\Gamma_{\varphi''}$; hence p induces a morphism of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z & \xrightarrow{\varepsilon} & X & \xrightarrow{\pi} & Y & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \| & & \\ 0 & \longrightarrow & Z'' & \longrightarrow & X'' & \longrightarrow & Y & \longrightarrow & 0, \end{array}$$

which induces isomorphisms in cohomology, $r \geqq 1$:

$$\begin{array}{ccccccccc} 0 & \longleftarrow & H^r(Z, \mathbf{Z}) & \xleftarrow{\varepsilon^*} & H^r(X, \mathbf{Z}) & \xleftarrow{\pi^*} & H^r(Y, \mathbf{Z}) & \longleftarrow & 0 \\ & & \uparrow{\scriptstyle\alpha^*}\ \cong & & \uparrow \cong & & \| & & \\ 0 & \longleftarrow & H^r(Z'', \mathbf{Z}) & \xleftarrow{\varepsilon''^*} & H^r(X'', \mathbf{Z}) & \xleftarrow{\pi''^*} & H^r(Y, \mathbf{Z}) & \longleftarrow & 0 \end{array}$$

where α^* is even an isomorphism of the Hodge structures $H^r_{Z''}$ and H^r_Z.

(b) Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z & \xrightarrow{\varepsilon} & X & \xrightarrow{\pi} & Y & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\varrho} & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\sigma} & & \\ 0 & \longrightarrow & \tilde{Z} & \xrightarrow{\tilde{\varepsilon}} & \tilde{X} & \xrightarrow{\tilde{\pi}} & \tilde{Y} & \longrightarrow & 0 \end{array}$$

be a morphism of pseudo-algebraic Lie groups, where

$$\tilde{Z} = \tilde{U}' \oplus \tilde{U}''/\{0\} \times \Delta'', \quad \tilde{Y} = \tilde{W}/\tilde{\Lambda}, \quad \tilde{X} = \tilde{V}/\Gamma_{\tilde{\varphi}},$$

$$\tilde{\varphi} = (\tilde{\varphi}', \tilde{\varphi}'') \in \mathrm{Hom}_{\mathbf{R}}(\tilde{W}, \tilde{U}' \oplus \tilde{U}'').$$

Then ϱ, σ, τ are induced by complex linear mappings

$$\hat{\varrho}: U' \oplus U'' \to \tilde{U}' \oplus \tilde{U}'', \quad \hat{\sigma}: W \to \tilde{W}, \quad \hat{\tau}: V \to \tilde{V},$$

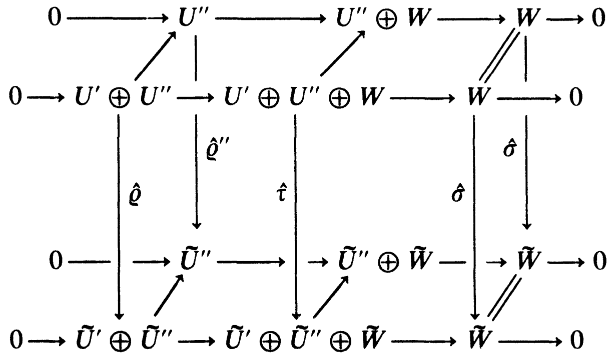
such that

$$\hat{\varrho}(\{0\}\times\Delta'')\subset\{0\}\times\Delta'',\quad \hat{\sigma}(\Lambda)\subset\tilde{\Lambda},\quad \hat{\tau}(\Gamma_{\varphi})\subset\Gamma_{\tilde{\varphi}}.$$

Since

$$U''=\Delta''\otimes_{\mathbf{Z}}\mathbf{C},\quad \tilde{U}''=\tilde{\Lambda}''\otimes_{\mathbf{Z}}\mathbf{C},$$

$\hat{\varrho}$ induces a complex linear mapping $\hat{\varrho}''=U''\to\tilde{U}''$, such that the following diagram commutes:



An easy computation shows, that $\hat{\tau}(U')\subset\tilde{U}'$; hence there is a complex linear mapping

$$\hat{\tau}'':U''\oplus W\to\tilde{U}''\oplus\tilde{W},$$

which completes the above diagram, such that $\tau''(\Gamma_{\varphi''})\subset\Gamma_{\tilde{\varphi}''}$. Clearly $(\hat{\varrho}'',\hat{\tau}'',\hat{\sigma})$ defines a morphism $(\varrho'',\tau'',\sigma)$ between the extensions given by φ'' and $\tilde{\varphi}''$. This shows, that the assignment, which associates to a pseudo-algebraic group

$$0\to Z\xrightarrow{\varepsilon}X\xrightarrow{\pi}Y\to 0$$

the sequence

$$0\to Z''\xrightarrow{\varepsilon''}X''\xrightarrow{\pi''}Y\to 0,$$

determines a functor $\Phi:\mathcal{G}r_{\text{alg}}\to\mathcal{G}r_{\text{alg}}$ such that the transformation $H^*\to H^*\circ\Phi$ (where H^* denotes the cohomology functor $H^*:\mathcal{G}r_{\text{alg}}\to\mathcal{S}ets$) is an isomorphism of functors.

4.2. Theorem. *Let X be an abelian complex Lie group, provided with a pseudo-algebraic structure given by the exact sequence*

$$0\to Z\xrightarrow{\varepsilon}X\xrightarrow{\pi}Y\to 0,$$

where $Y=W/\Lambda$ is a complex torus, $Z\cong\mathbf{C}^k\times(\mathbf{C}^*)^1$ is a commutative Stein *group. Then there is a unique extension H^1_X of the Hodge structure H^1_Z by H^1_Y, which depends functorially on $X \in |\mathcal{G}r_{\mathrm{alg}}|$, such that the underlying sequence of integral lattices is given by the cohomology sequence*

$$0 \to H^1(Y, \mathbf{Z}) \xrightarrow{\pi^*} H^1(X, \mathbf{Z}) \xrightarrow{\varepsilon^*} H^1(Z, \mathbf{Z}) \to 0.$$

If

$$Z = U' \times (U''/\Delta'') \cong (U' \oplus U'')/\Delta,$$

where $U' = \mathbf{C}^k$, $\Delta'' = \mathbf{Z}^1 \subset \mathbf{C}^l = U''$, $\Delta = \{0\} \times \Delta'' \subset U = U' \oplus U''$, and if X is given by

$$\varphi = (\varphi', \varphi'') \in \mathrm{Hom}_{\mathbf{R}}(W, U) \cong \mathrm{Hom}_{\mathbf{R}}(W, U') \times \mathrm{Hom}_{\mathbf{R}}(W, U''),$$

then the extension $H^1_X \in \mathbf{Ext}(H^1_Z, H^1_Y)$ is determined by the homomorphism

$$(-\varphi'')^* : (U'')^*_{\mathbf{C}} \to W^*_{\mathbf{R}}, \quad f \mapsto -f \circ \varphi''.$$

PROOF. The uniqueness assertion is clear from functioriality. Note that the remarks in 4.1. allow us to consider only the case $Z \cong (\mathbf{C}^*)^1$. So we will fix our notations as follows:

$$Z = U/\Delta, \text{ where } U = \mathbf{C}^l, \Delta = \mathbf{Z}^l \subset \mathbf{C}^l;$$

$$X = V/\Gamma_\varphi, \text{ where } V = U \oplus W, \varphi \in \mathrm{Hom}_{\mathbf{R}}(W, U), \quad \Gamma_\varphi = g(\varphi)(\Delta \oplus \Lambda).$$

It is enough to show, that the extension H^1_X determined by $-\varphi^* : U^*{}_{\mathbf{C}} \to W^*_{\mathbf{R}}$ has the required properties.

First we claim, that the Hodge structure H^1_X is well-defined. In fact if

$$\varphi \in \mathrm{Hom}_{\mathbf{C}}(W, U) + \mathrm{Hom}_{\mathbf{Z}}(\Lambda, \Delta),$$

then

$$-\varphi^* \in \mathrm{Hom}_{\mathbf{C}}(U^*_{\mathbf{C}}, W^*_{\mathbf{C}}) + \mathrm{Hom}_{\mathbf{Z}}(\Delta^*, \Lambda^*),$$

so $-\varphi^*$ defines the trivial extension.

Observe, that the cohomology sequence

$$(+) \qquad \begin{array}{ccccccccc} & & (H^1_Y)_{\mathbf{Z}} & & & & (H^1_Z)_{\mathbf{Z}} & & \\ & & \| & & & & \| & & \\ 0 & \to & H^1(Y, \mathbf{Z}) & \xrightarrow{\pi^*} & H^1(X, \mathbf{Z}) & \xrightarrow{\varepsilon^*} & H^1(Z, \mathbf{Z}) & \to & 0 \\ & & \|\cong & & \|\cong & & \|\cong & & \\ & & \mathrm{Hom}_{\mathbf{Z}}(\Lambda, \mathbf{Z}) & & \mathrm{Hom}_{\mathbf{Z}}(\Gamma_\varphi, \mathbf{Z}) & & \mathrm{Hom}_{\mathbf{Z}}(\Delta, \mathbf{Z}) & & \end{array}$$

arises by dualizing the exact sequence

$$(++) \qquad \begin{array}{ccccc} 0 \to \Delta \xrightarrow{\hat{\varepsilon}} & \Gamma_\varphi & \xrightarrow{\hat{\pi}} & \Lambda \to 0 \\ & (\delta+\varphi(\lambda), \lambda) & \mapsto & \lambda \end{array}$$

which admits a section $\sigma : \Lambda \to \Gamma_\varphi$, $\sigma(\lambda) = (\varphi(\lambda), \lambda)$.

Since H^1_Z is separated from H^1_Y, the weight filtration on H^1_X is uniquely determined:

$$W_0(H^1_X) = 0 \subset W_1(H^1_X) = \pi^*(W_1(H^1_Y)) = \pi^*((H^1_Y)_{\mathbf{Q}}).$$

Note that $W.(H^1_X)$ is integrally defined.

We set

$$\tilde{U} := \Delta \otimes_{\mathbf{Z}} \mathbf{R} \subset U = \Delta \otimes_{\mathbf{Z}} \mathbf{C},$$

$$\tilde{V} := \Gamma_\varphi \otimes_{\mathbf{Z}} \mathbf{R} = \{(u+\varphi(w), w);\ u \in \tilde{U}, w \in W\} \subset V.$$

The sequence $(++)$ induces the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \\ & & \alpha \uparrow \cup & & \uparrow \beta_\varphi & & \| & & \\ 0 & \longrightarrow & \tilde{U} & \longrightarrow & \tilde{V} & \longrightarrow & W & \longrightarrow & 0 \\ & & \alpha \downarrow \cap & & \downarrow \beta_0 & & \| & & \\ 0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \end{array}$$

where the upper and lower sequences are given by the natural injection, respectively projection, the homomorphism β_0 is the inclusion and β_φ is the homomorphism $\beta_\varphi(u+\varphi(w), w) = (u, w)$.

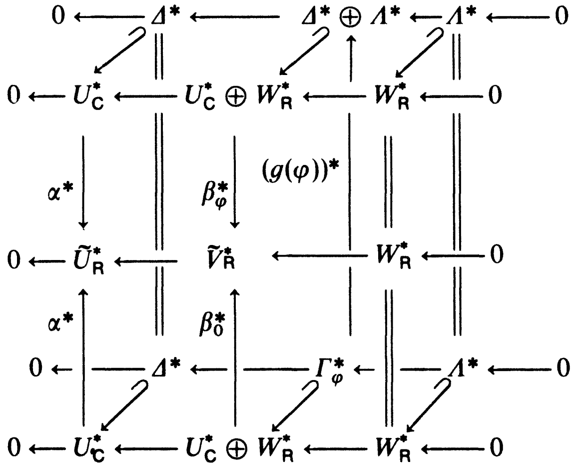
Applying the functor $\mathrm{Hom}_{\mathbf{R}}(-, \mathbf{C})$ we obtain the following diagram

$$\begin{array}{ccccccccc} & & & & U^*_{\mathbf{R}} \oplus W^*_{\mathbf{R}} & & & & \\ & & & & \| \cong & & & & \\ 0 & \longleftarrow & U^*_{\mathbf{R}} & \longleftarrow & V^*_{\mathbf{R}} & \longleftarrow & W^*_{\mathbf{R}} & \longleftarrow & 0 \\ & & \downarrow \alpha^* & & \downarrow \beta^*_\varphi & & \| & & \\ 0 & \longleftarrow & \tilde{U}^*_{\mathbf{R}} & \longleftarrow & \tilde{V}^*_{\mathbf{R}} & \longleftarrow & W^*_{\mathbf{R}} & \longleftarrow & 0 \\ & & \uparrow \alpha^* & & \uparrow \beta^*_0 & & \| & & \\ 0 & \longleftarrow & U^*_{\mathbf{R}} & \longleftarrow & V^*_{\mathbf{R}} & \longleftarrow & W^*_{\mathbf{R}} & \longleftarrow & 0 \end{array}$$

where α^*, β^*_0 are the restriction homomorphisms.

If we restrict α^*, β^*_0, β^*_φ to the subspaces $U^*_{\mathbf{C}} \subset U^*_{\mathbf{R}}$, respectively

$U_{\mathbb{C}}^* \oplus W_{\mathbb{R}}^* \subset V_{\mathbb{R}}^*$, then by calculating dimension α^*, β_0^*, β_φ^* become isomorphisms. Thus we obtain the commutative diagram:



Here the upper rectangle defines the trivial extension of the Hodge structure H_Z^1 by H_Y^1 and the morphism

$$(\beta_0^*)^{-1} \circ \beta_\varphi^* : U_{\mathbb{C}}^* \oplus W_{\mathbb{R}}^* \longrightarrow U_{\mathbb{C}}^* \oplus W_{\mathbb{R}}^*$$

maps the trivial extension to the extension of H_Z^1 by H_Y^1, given by the diagram:

$$\begin{array}{ccccccccc}
 & & H^1(Z,\mathbb{Z}) & & H^1(X,\mathbb{Z}) & & H^1(Y,\mathbb{Z}) & & \\
 & & \| \cong & & \| \cong & & \| \cong & & \\
0 & \longleftarrow & \Delta^* & \overset{\varepsilon^*}{\longleftarrow} & \Gamma_\varphi^* & \overset{\pi^*}{\longleftarrow} & \Lambda^* & \longleftarrow & 0 \\
 & & \cap\downarrow & & \cap\downarrow & & \cap\downarrow & & \\
0 & \longleftarrow & U_{\mathbb{C}}^* & \longleftarrow & U_{\mathbb{C}}^* \oplus W_{\mathbb{R}}^* & \longleftarrow & W_{\mathbb{R}}^* & \longleftarrow & 0 \\
 & & \| \cong & & \| \cong & & \| \cong & & \\
 & & H^1(Z,\mathbb{C}) & & H^1(X,\mathbb{C}) & & H^1(Y,\mathbb{C}) & &
\end{array}$$

But since β_0^* is induced by the inclusion β_0 and β_φ^* is determined by $g(\varphi)$, the homomorphism $(\beta_0^*)^{-1} \circ \beta_\varphi^*$ is nothing else but

$$(\beta_0^*)^{-1} \circ \beta_\varphi^* = (g(\varphi)^*)^{-1} = (g(\varphi)^{-1})^* = (g(-\varphi))^* = G(-\varphi^*).$$

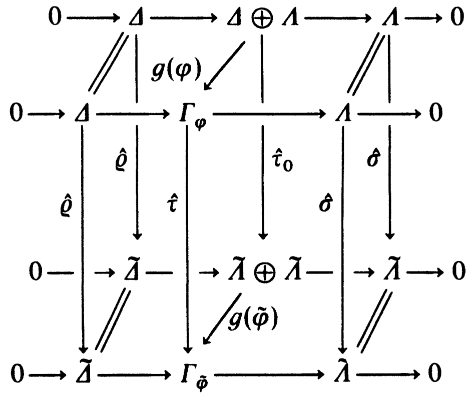
So the Hodge structure defined by $-\varphi^*$ is of the desired form.

To obtain functioriality we consider a morphism of pseudo-algebraic commutative Lie groups:

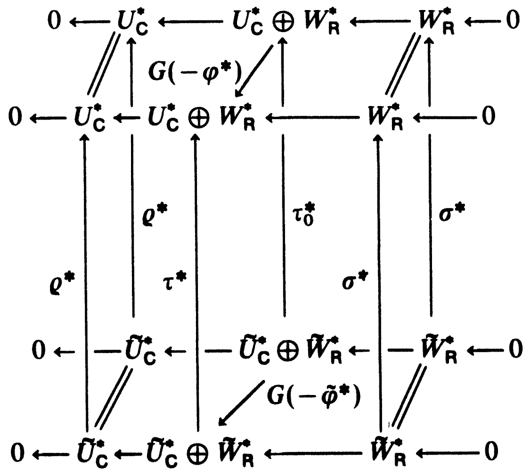
$$\begin{array}{ccccccccc} 0 & \to & Z & \xrightarrow{\varepsilon} & X & \xrightarrow{\pi} & Y & \to & 0 \\ & & \downarrow{\scriptstyle\varrho} & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\sigma} & & \\ 0 & \to & \tilde{Z} & \to & \tilde{X} & \to & \tilde{Y} & \to & 0 \end{array}$$

By the remarks in 4.1. it is enough to treat the case, where both Z and $\tilde{Z}$ are of pure multiplicative type. We use notations analogous to the ones before.

Now we look at the commutative diagram



where $\hat{\tau}_0$ is the homomorphism $\hat{\tau}_0(\delta, \lambda) = (\hat{\varrho}(\delta), \hat{\sigma}(\lambda))$. With the above considerations we obtain the following diagram in cohomology:



where

$$\tau_0^*(f, g) = (\varrho^*(f), \sigma^*(g)) \quad \text{for} \quad (f, g) \in \tilde{U}_{\mathbf{C}}^* \oplus \tilde{W}_{\mathbf{R}}^*.$$

Since ϱ^* and σ^* define morphisms of the canonical Hodge structures $H_{\mathbf{Z}}^1$,

$H^1_{\tilde{Z}}$, respectively $H^1_{\tilde{Y}}$, H^1_Y, τ_0^* induces a morphism of the trivial extension of $H^1_{\tilde{Z}}$ by $H^1_{\tilde{Y}}$ to the trivial extension of H^1_Z by H^1_Y, and since $G(-\varphi^*)$ and $G(-\tilde{\varphi}^*)$ transport the trivial extensions to the Hodge structures H^1_X, respectively $H^1_{\tilde{X}}$, τ^* is a morphism of $H^1_{\tilde{X}}$ to H^1_X. Hence $(\varrho^*, \tau^*, \sigma^*)$ defines a morphism of extensions of Hodge structures and so functoriality is obtained.

Since the higher cohomology groups of X are generated by the first one, we get as an immediate consequence:

COROLLARY. *The cohomology groups $H^r(X, \mathbf{Z})$ of abelian Lie groups X with pseudo-algebraic structures $0 \to Z \xrightarrow{\varepsilon} X \xrightarrow{\pi} Y \to 0$ carry uniquely determined natural mixed Hodge structures H^r_X, which behave functorially in the category $\mathscr{G}\mathscr{r}_{\mathrm{alg}}$ of pseudo-algebraic abelian Lie groups, and which are compatible with cup-products. H^r_X is extension of the canonical Hodge structure H^r_Z by H^r_Y.*

REMARKS. a) The Hodge structures H^r_X are integrally defined.

b) The Hodge structure H^1_X is of type $\{1, 2\}$, that is

$$W_p(H^1_X)/W_{p-1}(H^1_X) = 0 \quad \text{for } p \notin \{1, 2\}.$$

Consequently the Hodge structures H^r_X are of type $\{r, \ldots, 2r\}$.

c) If

$$0 \to Z \xrightarrow{\varepsilon} X \xrightarrow{\pi} Y \to 0 \quad \text{and} \quad 0 \to \tilde{Z} \xrightarrow{\tilde{\varepsilon}} \tilde{X} \xrightarrow{\tilde{\pi}} \tilde{Y} \to 0$$

are two pseudo-algebraic abelian Lie groups, then the *product*

$$0 \to Z \times \tilde{Z} \xrightarrow{\varepsilon \times \tilde{\varepsilon}} X \times \tilde{X} \xrightarrow{\pi \times \tilde{\pi}} Y \times \tilde{Y} \to 0$$

can be viewed as an extension in

$$\mathrm{Ext}(Y, Z) \times \mathrm{Ext}(\tilde{Y}, \tilde{Z}) \subset \mathrm{Ext}(Y \times \tilde{Y}, Z \times \tilde{Z}).$$

Since the Hodge structures on complex tori and on linear algebraic groups are compatible with Künneth formula, the Hodge structures on pseudo-algebraic Lie groups defined above are also compatible with Künneth formula.

d) Every pseudo-algebraic abelian complex Lie group X is determined by a *Serre fibration*

$$0 \to Z \xrightarrow{\varepsilon} X \xrightarrow{\pi} Y \to 0$$

(cf. [6]).

Hence we may calculate $H^r(X, \mathbf{Z})$ by the *Serre-Leray spectral sequence*. It turns out, that the filtration on $H^r(X, \mathbf{Z})$ induced by the spectral sequence coincides with the weight filtration of the Hodge structure H^r_X.

This remark may be useful for generalisations of the above theory.

4.3. For a fixed complex torus $Y = W/\Lambda$ and a Stein group

$$Z = \mathbf{C}^k \times (\mathbf{C}^*)^1 = U' \times U''/\Delta'',$$

we define the rth Torelli map, $r \geqq 1$:

$$T^r_{Z,Y}: \mathrm{Ext}(Y,Z) \to \mathbf{Ext}(H^r_Z, H^r_Y), \quad T_{Z,Y}(X) := H^r_X,$$

where H^r_X is the mixed Hodge structure defined in 4.2.. Furthermore we define the total Torelli map to be the product of the mappings $T^r_{Z,Y}$:

$$T_{Z,Y} := \sum_{r=1}^{\infty} T^r_{Z,Y}: \mathrm{Ext}(Y,Z) \to \prod_{r=1}^{\infty} \mathbf{Ext}(H^r_Z, H^r_Y).$$

Note that $T_{Z,Y}$ is the zero map, if $r > \dim_{\mathbf{C}} Y + \dim_{\mathbf{C}} Z$.

Of course $T^r_{Z,Y}$ is well-defined and in fact determines a morphism of the bifunctors $\mathrm{Ext}(*,*)$ and $\mathbf{Ext}(H^r_*, H^r_*)$.

If $Z = Z' \times Z''$ with $Z' = \mathbf{C}^k$, $Z'' = (\mathbf{C}^{*})^1 = U''/\Delta''$, then by the above considerations the projection $p: Z \to Z''$ induces the following commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}(Y,Z') \times \mathrm{Ext}(Y,Z'') \cong \mathrm{Ext}(Y,Z) & \xrightarrow{\mathrm{Ext}(Y,p)} & \mathrm{Ext}(Y,Z'') \\ \Big\downarrow T^r_{Z,Y} & & \Big\downarrow T^r_{Z'',Y} \\ \mathbf{Ext}(\mathrm{H}^r_Z, H^r_Y) & \xrightarrow{\sim} & \mathbf{Ext}(H^r_{Z''}, H^r_Y) \end{array}$$

where $\mathrm{Ext}(Y,p)$ is the projection to the second factor, and is hence a holomorphic surjective homomorphism of abelian complex Lie groups; the lower arrow is even a holomorphic isomorphism of abelian complex Lie groups.

If we use the identifications of the sections 2 and 3:

$$\mathrm{Ext}(Y,Z'') \cong \mathrm{Hom}_{\mathbf{R}}(W,U'')/(\mathrm{Hom}_{\mathbf{C}}(W,U'') + \mathrm{Hom}_{\mathbf{Z}}(\Lambda,\Delta'')),$$

$$\mathbf{Ext}(H^1_{Z''}, H^1_Y) \cong \mathrm{Hom}_{\mathbf{C}}((U'')^*_{\mathbf{C}}, W^*_{\mathbf{R}})/(\mathrm{Hom}_{\mathbf{C}}((U'')^*_{\mathbf{C}}, W^*_{\mathbf{C}}) + \mathrm{Hom}_{\mathbf{Z}}((\Delta'')^*, \Lambda^*)),$$

then $T^1_{Z'',Y}$ is induces by the (well-defined and) $\mathbf{C}$-linear isomorphism

$$\mathrm{Hom}_{\mathbf{R}}(W,U'') \to \mathrm{Hom}_{\mathbf{C}}((U'')^*_{\mathbf{C}}, W^*_{\mathbf{R}}), \quad \varphi'' \mapsto (-\varphi'')^*.$$

where $(-\varphi'')^*(f) = -f \circ \varphi''$, $f \in (U'')^*_{\mathbf{C}}$. The image of

$$\mathrm{Hom}_{\mathbf{C}}(W,U'') + \mathrm{Hom}_{\mathbf{Z}}(\Lambda,\Delta'') \subset \mathrm{Hom}_{\mathbf{R}}(W,U'')$$

by this mapping is clearly

$$\mathrm{Hom}_{\mathbf{C}}((U'')^*_{\mathbf{C}}, W^*_{\mathbf{C}}) + \mathrm{Hom}_{\mathbf{Z}}((\Delta'')^*, \Lambda^*) \subset \mathrm{Hom}_{\mathbf{C}}((U'')^*_{\mathbf{C}}, W^*_{\mathbf{R}}),$$

hence $T^1_{Z'',Y}$ is a holomorphic isomorphism of abelian complex Lie groups.

Since the higher Torelli maps $T^r_{Z,Y}$ are induced by taking exterior powers of $T^1_{Z,Y}$, we obtain the following result:

THEOREM. *Let Y be a compact complex torus, Z a commutative Stein group.*

a) *The Torelli maps*

$$T^r_{Z,Y}: \mathrm{Ext}(Y,Z) \to \mathbf{Ext}(H^r_Z, H^r_Y), \quad r \geqq 1,$$

and

$$T_{Z,Y}: \mathrm{Ext}(Y,Z) \to \prod_{r=1}^{\infty} \mathbf{Ext}(H^r_Z, H^r_Y)$$

are holomorphic homomorphisms of abelian complex Lie groups.

b) *If $Z \cong Z' \times Z''$ with $Z' = \mathbf{C}^k$, $Z'' = (\mathbf{C}^*)^l$, then the kernel of $T_{Z,Y}$ is exactly the subgroup* $\mathrm{Ext}(Y,Z') \subset \mathrm{Ext}(Y,Z)$. *In particular the Torelli map is injective if and only if Z does not contain a vector subgroup $G_a(\mathbf{C}) = \mathbf{C}$.*

c) *The first Torelli map*

$$T^1_{Z,Y}: \mathrm{Ext}(Y,Z) \to \mathbf{Ext}(H^1_Z, H^1_Y)$$

is surjective and induces an isomorphism of the complex torus

$$\hat{Y}^l \cong \mathrm{Ext}(Y,Z'') \cong \mathrm{Ext}(Y,Z)/\mathrm{Ext}(Y,Z'),$$

$l = \dim_{\mathbf{C}} Z''$, *onto the torus* $\mathbf{Ext}(H^1_Z, H^1_Y)$.

5. Applications and further remarks.

5.1. Let X be an abelian complex Lie group. We denote by $\tilde{H}^{p,q}(X) \subset H^q(X,\Omega^p)$ the finite-dimensional complex vector space, which is determined by (p,q)-forms with constant coefficients. Note that $H^q(X,\Omega^p)$ is in general infinite dimensional (cf. [12], [13]). Now the topological cohomology groups $H^r(X,\mathbf{C})$ can be calculated by means of the groups $\tilde{H}^{p,q}(X)$:

$$H^r(X,\mathbf{C}) \cong \bigoplus_{p+q=r} \tilde{H}^{p,q}(X).$$

For toroidal groups X with finite-dimensional cohomology groups $H^q(X,\Omega^p)$ this was shown by Vogt in [13]. The general case follows very easily from Vogt's considerations.

Now we represent X as quotient V/Γ of the complex vector space V by the discrete subgroup $\Gamma \subset V$. If we set $V' := \Gamma \otimes_{\mathbf{Z}} \mathbf{C}$, $\tilde{V}' := \Gamma \otimes_{\mathbf{Z}} \mathbf{R}$, and if

$\tilde{W} \subset \tilde{V}'$ is the maximal complex vector space contained in the real vector space $\tilde{V}'$, then:

$$H^r(X, \mathbf{C}) \cong \Lambda^r((\tilde{V}')^*_{\mathbf{R}})$$

$$\tilde{H}^{p,q}(X) \cong \Lambda^p((V')^*_{\mathbf{C}}) \otimes_{\mathbf{C}} \Lambda^q(\tilde{W}^*_{\bar{\mathbf{C}}}).$$

REMARKS. a) If $X \cong \mathbf{C}^k \times (\mathbf{C}^*)^1 \times T$, where T is a toroidal group, then V' may be viewed as the complex tangent space at 0 to the subgroup

$$X' := \{0\} \times (\mathbf{C}^*)^1 \times T \subset X,$$

and $\tilde{V}'$ is the tangent space at 0 of the maximal compact real subgroup of X.

b) If $0 \to Z \xrightarrow{\varepsilon} X \xrightarrow{\pi} Y \to 0$ is a pseudo-algebraic structure on X, where $Y = W/\Lambda$ is complex torus, then the universal covering map $\hat{\pi}: V \to W$ projects $\tilde{W}$ isomorphically onto W. Thus

$$\pi^*: H^r(Y, \mathbf{C}) \to H^r(X, \mathbf{C}) \quad \text{induces an isomorphism}$$

$$\pi^*: H^{0,r}(Y) = \Lambda^r(W^*_{\bar{\mathbf{C}}}) \xrightarrow{\sim} \tilde{H}^{0,r}(X) = \Lambda^r(W^*_{\bar{\mathbf{C}}}).$$

In particular $\tilde{H}^{0,r}(X) = (H^r_X)^{0,r}$, where H^r_X is the Hodge structure on $H^r(X, \mathbf{Z})$ induced by the pseudo-algebraic structure on X. Moreover

$$\tilde{H}^{p,r-p}(X) \cong \bigoplus_{s=0}^{p} (H^r_X)^{p,r-s}, \quad \text{and}$$

$$F^p(H^r_X) = \bigoplus_{q=p}^{r} \tilde{H}^{q,r-q}(X).$$

5.2. *Néron-Severi group.*

Again we consider an abelian complex Lie group X.

As usual we denote by $\mathrm{NS}(X) \subset H^2(X, \mathbf{Z})$ the *Néron-Severi group*, i.e. the group of Chern classes of holomorphic line bundles on X.

The exact exponential sequence

$$\begin{array}{c} \cdots \to H^1(X, \mathbf{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}^*_X) \xrightarrow{c} H^2(X, \mathbf{Z}) \xrightarrow{\varphi} H^2(X, \mathcal{O}_X) \to \cdots \\ \| \\ \mathrm{Pic}(X) \end{array}$$

induces the following diagram:

$$\begin{array}{ccccc} H^1(X, \mathcal{O}^*_X) & \xrightarrow{\quad c \quad} & H^2(X, \mathbf{Z}) & \xrightarrow{\varphi} & H^2(X, \mathcal{O}_X) \\ & \searrow & \nearrow & & \\ & \mathrm{NS}(X) = \mathrm{Im}(c) = \mathrm{Ker}(\varphi). & & & \end{array}$$

Here $\operatorname{Im}\varphi \subset H^2(X, \mathcal{O}_X)$ is already contained in the finite dimensional part $\tilde{H}^{0,2}(X) \subset H^2(X, \mathcal{O}_X)$.

Now we provide X with a pseudo-algebraic structure

$$H_X^2 = (H^2(X, \mathbf{Z}), W.(H_X^2), F^{\cdot}(H_X^2)).$$

We consider the commutative diagram:

$$\begin{array}{ccc} H^2(X, \mathbf{Z}) & \xrightarrow{\quad\varphi\quad} & \tilde{H}^{0,2}(X) \\ & {\scriptstyle j}\searrow \quad \nearrow{\scriptstyle \tilde{\varphi}} & \\ & H^2(X, \mathbf{C}) & \end{array}$$

where j is the natural inclusion and $\tilde{\varphi}$ is induced by φ. Furthermore the Hodge structure H_X^2 defines the decomposition:

$$H^2(X, \mathbf{C}) \cong (H_X^2)^{2,0} \oplus (H_X^2)^{2,1} \oplus (H_X^2)^{2,2} \oplus (H_X^2)^{1,1} \oplus (H_X^2)^{1,2} \oplus (H_X^2)^{0,2}.$$

PROPOSITION. *If X is an abelian complex Lie group, provided with an arbitrary pseudo-algebraic structure, then*

$$\mathrm{NS}(X) = F^1(H_X^2) \cap \overline{F^1(H_X^2)} \cap H^2(X, \mathbf{Z}).$$

PROOF. We have

$$F^1(H_X^2) \cap \overline{F^1(H_X^2)} = (H_X^2)^{2,1} \oplus (H_X^2)^{2,2} \oplus (H_X^2)^{1,1} \oplus (H_X^2)^{1,2}.$$

Now

$$\mathrm{NS}(X) = \operatorname{Ker}(\varphi : H^2(X, \mathbf{Z}) \to \tilde{H}^{0,2}(X)),$$

so the Néron-Severi group consists of those integrally defined cohomology classes $\eta \in H^2(X, \mathbf{C})$, such that in the decomposition

$$\eta = \eta_{2,0} + \eta_{2,1} + \eta_{2,2} + \eta_{1.1} + \eta_{1,2} + \eta_{0,2},$$

we have $0 = \varphi(\eta) = \eta_{0,2}$. Since η is real, i.e. $\eta = \bar{\eta}$, and since

$$\overline{(H_X^2)^{0,2}} = (H_X^2)^{2,0},$$

it follows

$$\eta_{2,0} = 0 = \eta_{0,2}, \quad \eta_{2,1} = \overline{\eta_{1,2}}, \quad \eta_{1,1} = \overline{\eta_{1,1}}, \quad \eta_{2,2} = \overline{\eta_{2,2}}.$$

So $\mathrm{NS}(X)$ consists of the integral elements in

$$(H_X^2)^{2,1} \oplus (H_X^2)^{2,2} \oplus (H_X^2)^{1,1} \oplus (H_X^2)^{1,2} = F^1(H_X^2) \cap F^1(H_X^2).$$

REMARK. If $\eta \in \mathrm{NS}(X)$ is a Chern class, such that $\eta_{12} \neq 0$ or $\eta_{22} \neq 0$, then η is not of type $(1, 1)$; a phenomenon which does not occur in the compact case. Note that for pseudo-algebraic groups $0 \to Z \to X \to Y \to 0$, such that $Z = (\mathbf{C}^*)^l$, $l \geqq 1$, and such that $\dim_{\mathbf{C}} Y = 1$, we have $\mathrm{NS}(X) = H^2(X, \mathbf{Z})$ (cf. [10]); hence there are Chern classes of holomorphic line bundles, which are not of type $(1,1)$. For many further examples we refer to [10].

5.3. *Picard group.*

Again let X be an abelian complex Lie group. We denote by

$$\mathrm{Pic}_\theta(X) \subset \mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*)$$

the subgroup given by holomorphic line bundles, which are determined by θ-factors (cf. [12]), which we call simply theta bundles. Furthermore we consider the group

$$\mathrm{Pic}_\theta^0(X) := \mathrm{Pic}_\theta(X) \cap \mathrm{Pic}^0(X)$$

of topologically trivial theta bundles.

From [12], where it is shown, that every topologically trivial theta bundle is given by a representation of the fundamental group $\pi_1(X)$, we get the commutative diagram

$$\begin{array}{ccccccccc}
\cdots \to & H^1(X,\mathbf{Z}) & \xrightarrow{j} & H^1(X,\mathcal{O}_X) & \xrightarrow{\exp} & H^1(X,\mathcal{O}_X^*) & \xrightarrow{c} & H^2(X,\mathbf{Z}) & \to \cdots \\
 & \| & & \uparrow & & \| & & \| & \\
 & & & & & \mathrm{Pic}(X) & & & \\
 & \| & & \cup & & \cup\uparrow & & \| & \\
\cdots \to & H^1(X,\mathbf{Z}) & \xrightarrow{j} & \tilde{H}^{0,1}(X) & \longrightarrow & \mathrm{Pic}_\theta(X) & \xrightarrow{c_\theta} & H^2(X,\mathbf{Z}) & \to \cdots
\end{array}$$

Hence

$$\begin{aligned}
\mathrm{Pic}_\theta^0(X) &= \mathrm{Ker}(c_\theta : \mathrm{Pic}_\theta(X) \to H^2(X,\mathbf{Z})) \\
&= \mathrm{Im}(\exp : \tilde{H}^{0,1}(X) \to \mathrm{Pic}_\theta(X)) = \tilde{H}^{0,1}(X)/j(H^1(X,\mathbf{Z})).
\end{aligned}$$

Now again we endow X with a pseudo-algebraic structure

$$0 \to Z \xrightarrow{\varepsilon} X \xrightarrow{\pi} Y \to 0;$$

then the Hodge structure H_X^1 on $H^1(X,\mathbf{Z})$ by this pseudo-algebraic structure induces the decomposition

$$H^1(X,\mathbf{C}) = (H_X^1)^{1,0} \oplus (H_X^1)^{1,1} \oplus (H_X^1)^{0,1}.$$

In the diagram

$$\begin{array}{ccc}
H^1(X,\mathbf{Z}) & \xrightarrow{\quad j \quad} & \tilde{H}^{0,1}(X) \\
\cap\downarrow & & \uparrow\varphi \\
H^1(X,\mathbf{C}) \cong (H_X^1)^{1,0} \oplus (H_X^1)^{1,1} & \oplus & (H_X^1)^{0,1}
\end{array}$$

the mapping φ projects $(H_X^1)^{0,1}$ isomorphically onto $\tilde{H}^{0,1}(X)$ and the kernel of φ is $(H_X^1)^{1,0} \oplus (H_X^1)^{1,1}$.

So

$$\begin{aligned}\operatorname{Pic}^0_\theta(X) &\cong \tilde{H}^{0,1}(X)/j(H^1(X,\mathbf{Z}))\\ &\cong H^1(X,\mathbf{C})/((H_X^1)^{1,0} \oplus (H_X^1)^{1,1})+j(H^1(X,\mathbf{Z}))\\ &\cong (H_X^1)_{\mathbf{C}}/(F^1(H_X^1)+H^1(X,\mathbf{Z})) = \operatorname{Jac}^1(H_X^1),\end{aligned}$$

where $F^{\cdot}(H_X^1)$ is the Hodge filtration of the Hodge structure H_X^1. Thus we have proved the following

PROPOSITION. *If X is an abelian complex Lie group, endowed with any pseudo-algebraic structure, then*

$$\operatorname{Pic}^0_\theta(X) \cong \operatorname{Jac}^1(H_X^1).$$

REMARK. If X is neither compact nor Stein, the group $H^1(X,\mathbf{Z})$ projects to a subgroup of $(H_X^1)_{\mathbf{C}}/F^1(H_X^1)$, which is not a discrete subgroup of this complex vector space. Hence $\operatorname{Jac}^1(H_X^1)$ does not carry a natural structure of a Lie group.

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