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A NOTE ON THE TRANSFER MAP

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Introduction.

Let X be a space with a base point. We denote by $Q(X)$ the space $\Omega^\infty S^\infty(X)$ with the compact-open topology.

Let G be a compact Lie group, M a compact smooth G-manifold without boundary and $\tilde{\mu}: P \to B$ a principal G-bundle over a finite complex B. We shall denote by μ the associated bundle $P \times_G M \to B$.

In [3] Becker and Gottlieb have defined the transfer map of the bundle μ,

$$\tau(\mu): B_+ \to Q((P \times_G M)_+).$$

Let $p: (P \times_G M)_+ \to S^0$ be the based map which transforms $P \times_G M$ into the non-base point of S^0.

We define a *p-transfer map* of the bundle μ,

$$\bar{\tau}(\mu): B_+ \to Q(S^0)$$

as the composition

$$B_+ \xrightarrow{\tau(\mu)} Q((P \times_G M)_+) \xrightarrow{Q(p)} Q(S^0).$$

Let $Q(S^0)_{(i)}$, $i \in \mathbf{Z}$, be the connected component of maps of degree i in $Q(S^0)$. If $n = \chi(M)$ is the Euler-Poincaré characteristic of M, then $\bar{\tau}(\mu)(B) \subset Q(S^0)_{(n)}$.

The aim of this note is to give some sufficient conditions for the map $\bar{\tau}(\mu): B \to Q(S^0)_{(n)}$ to be contractible over some skeleton of B.

We have a following result in this direction: If the tangent bundle $T(M)$ of M contains $M \times W$, for some representation W of G, as a G-subbundle and $\dim_R W = m$, then $\bar{\tau}(\mu)$ is contractible over the $(m-1)$-skeleton of B (Corollary 1.7).

Suppose now that $\bar{\tau}(\mu)$ is indeed contractible over some $(k-1)$-skeleton

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$B^{(k-1)}$ of B. Then it can be factorized up to homotopy as

$$\begin{array}{ccc} B & \xrightarrow{\bar{\tau}(\mu)} & Q(S^0)_{(n)} \\ & \searrow \quad \nearrow_{F} & \\ & B/B^{(k-1)} & \end{array}$$

for some map $F: B/B^{(k-1)} \to Q(S^0)_{(n)}$. Given any map $g: S^k \to B/B^{(k-1)}$, we can consider the composition $f = F \circ g$,

$$f: S^k \to Q(S^0)_{(n)}$$

and the element it yields in the kth stable stem $[f] \in \pi_k(Q(S^0))$.

It would be interesting, in our opinion, to know what elements in $\pi_*(Q(S^0))$ can be obtained in this way.

We show that for $k = 1, 3, 7$ maps $f: S^k \to Q(S^0)_{(1)}$ with the Hopf invariant one can indeed be obtained in such a way. See Section 3.

The paper is organized as follows: In Section 1 we prove our criterion for the contractibility of the transfer map over the skeleta. Section 2 contains some examples. In Section 3 we conduct some cohomological computations and show how maps with the Hopf invariant one can be obtained from transfer maps. In Section 4 we formulate some problems relevant to the Kervaire invariant one problem.

NOTATIONS AND CONVENTIONS.

Throughout the paper G is a compact Lie group. The term "G-manifold" means a compact smooth Riemannian manifold without boundary equipped with a smooth left action of G such that G acts through Riemannian isometries. We assume that all G-vector bundles which appear are equipped with a G-invariant Riemannian metric and that the Riemannian metric of a Whitney sum of bundles is an orthogonal sum of the metrics of the summands. We assume also that all G-isomorphisms of vector bundles that appear are orthogonal in those metrics.

If M is a G-manifold and $H \subset G$ is a subgroup then

$$M^H = \{x \in M \mid hx = x \quad \text{for} \quad h \in H\}.$$

If X is a topological space, X_+ is a disjoint union of X with a one-point space. The extra point is the base point of X_+. $p_X: X_+ \to S^0$ is a based map such that

$$p_X(X) = \{\text{the non-base point of } S^0\}.$$

If X is a G-space, then X_+ is a G-space as well. G acts trivially on its base point.

$\chi(X)$ is the Euler-Poincaré characteristic of X.

S^m is the unit sphere in $\mathbf{R}^{m+1}$.

1. Properties of the transfer map.

Let G be a compact Lie group and let M be a G-manifold. $\pi: T(M) \to M$ is the tangent bundle of M. π is a G-vector bundle.

Let $\tilde{\mu}: P \to B$ be a principal G-bundle over a finite CW-complex B. Let $\mu: P \times_G M \to B$ be the induced fibration. We denote by η the induced vector bundle over $P \times_G M$

$$P \times_G \pi: P \times_G T(M) \to P \times_G M.$$

Our first aim in this section is to prove

PROPOSITION 1.1. *If η has a nonvanishing cross section over $P \times_G M$, then the transfer map of the fibration μ*

$$\tau(\mu): B_+ \to Q((P \times_G M)_+)$$

is homotopically contractible to the base point.

We start by recalling the Becker-Gottlieb definition of transfer, [3]. If $\xi: D \to X$ is a G-vector bundle over a compact G-space X, then we define the fibrewise one-point compactification X_ξ of ξ to be a quotient space $X_\xi = D/\sim$, where, for $d_1, d_2 \in D$, $d_1 \sim d_2$ if and only if either $d_1 = d_2$ or $\xi(d_1) = \xi(d_2)$, $\|d_1\| \geqq 1$ and $\|d_2\| \geqq 1$. Here $\|\cdot\|$ is the Riemannian norm in ξ.

If the fibre dimension of ξ is 0, we define X_ξ to be the disjoint sum $X \sqcup X$.

There is the projection $\bar{\xi}: X_\xi \to X$ defined by $\bar{\xi}([d]) = \xi(d)$ and there are two cross sections: The zero-section $s_\xi^0: X \to X_\xi$ defined by $s_\xi^0(x) = [d_0]$ with $d_0 \in D$ such that $\xi(d_0) = x$ and $\|d_0\| = 0$, and the section at infinity $s_\xi^\infty: X \to X_\xi$ defined by $s_\xi^\infty(x) = [d_1]$ for any $d_1 \in D$ such that $\xi(d_1) = x$ and $\|d_1\| \geqq 1$.

The triple $(X_\xi, \bar{\xi}, s_\xi^\infty)$ is an ex-space over X, see [11].

We define the Thom space X^ξ of ξ to be the quotient

$$X^\xi = X_\xi / s_\xi^\infty(X).$$

The image of s_ξ^∞ is a base point of X^ξ.

Let $E = P \times_G M$. The transfer map $\tau(\mu): B_+ \to Q(E_+)$ of the bundle $\mu: E \to B$ is defined as follows.

There exists an orthogonal finite-dimensional representation V of G and a G-embedding $i: M \to V$. Let ω be the G-normal bundle of i. Define a G-map

$$\gamma: S^V \to (M_+) \wedge S^V$$

to be the composition

$$S^V \xrightarrow{c} M^{\omega} \xrightarrow{j} M^{\pi \oplus \omega} \xrightarrow{\psi} (M_+) \wedge S^V, \tag{1.2}$$

where S^V is the one-point compactification of V, c is the Pontrjagin-Thom map associated to the embedding i, j is induced by the inclusion $\omega \subset \pi \oplus \omega$ and ψ is induced by a G-trivialization of the bundle $\pi \oplus \omega$.

We have a commutative diagram of maps

$$\begin{array}{ccc} P \times_G S^V & \xrightarrow{\gamma'} & P \times_G ((M_+) \wedge S^V) \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

where $\gamma' = \mathrm{id}_P \times_G \gamma$.

Let ξ denote the vector bundle $P \times_G V \to B$ and let ζ be a vector bundle over B such that $\xi \oplus \zeta$ is trivial with a trivialization $\Phi: \xi \oplus \zeta \to B \times \mathbb{R}^n$.

Let us consider the map

$$\gamma' \wedge_B 1: (P \times_G S^V) \wedge_B B_\zeta \to (P \times_G ((M_+) \wedge S^V)) \wedge_B B_\zeta, \tag{1.3}$$

where "$\wedge_B$" is the fibrewise smash product of the bundles over B, see [3; Section 3]. B is embedded through the sections at ∞ in both the range and the domain of $\gamma' \wedge_B 1$ and $\gamma' \wedge_B 1 | B = 1_B$. There are canonical identifications

$$\begin{cases} (P \times_G S^V) \wedge_B B_\zeta / B \cong B^{\xi \oplus \zeta} \quad \text{and} \\ (P \times_G ((M_+) \wedge S^V)) \wedge_B B_\zeta / B \cong (P \times_G M)^{\mu^*(\xi \oplus \zeta)} = E^{\mu^*(\xi \oplus \zeta)}. \end{cases} \tag{1.4}$$

Therefore, $\gamma' \wedge_B 1$ yields

$$\gamma'': B^{\xi \oplus \zeta} \to E^{\mu^*(\xi \oplus \zeta)}.$$

The trivialization Φ induces isomorphisms $\bar{\Phi}: B^{\xi \oplus \zeta} \to (B_+) \wedge S^n$ and

$$\mu^*(\bar{\Phi}): E^{\mu^*(\xi \oplus \zeta)} \to (E_+) \wedge S^n.$$

The transfer $\tau(\mu): B_+ \to Q(E_+)$ is defined as the adjoint to the composition

$$\mu^*(\bar{\Phi}) \circ \gamma'' \circ \bar{\Phi}^{-1}: (B_+) \wedge S^n \to (E_+) \wedge S^n. \tag{1.5}$$

If $\xi_i: D_i \to X$, $i = 1, 2$, are G-vector bundles over X, then there is a G-embedding

$$f^0_{\xi_1, \xi_2}: X_{\xi_2} \to X_{\xi_1 \oplus \xi_2}$$

defined by $f^0_{\xi_1,\xi_2}([d]) = [s^0_{\xi_1}(\xi_2(d)) \oplus d]$ for $d \in D_2$. There is also a G-map

$$f^{\infty}_{\xi_1,\xi_2}: X_{\xi_2} \to X_{\xi_1 \oplus \xi_2}$$

given by $f^{\infty}_{\xi_1,\xi_2}([d]) = s^{\infty}_{\xi_1 \oplus \xi_2}(\bar{\xi}_2([d]))$. Both $f^0_{\xi_1,\xi_2}$ and $f^{\infty}_{\xi_1,\xi_2}$ are maps of ex-spaces over X.

LEMMA 1.6. *If the G-vector bundle ξ_1 has a nonvanishing G-cross section, then $f^0_{\xi_1,\xi_2}$ and $f^{\infty}_{\xi_1,\xi_2}$ are G-homotopic as ex-maps over X,* i.e. *there exists a G-homotopy h_t from $f^0_{\xi_1,\xi_2}$ to $f^{\infty}_{\xi_1,\xi_2}$ such that*

$$\overline{(\xi_1 \oplus \xi_2)} \circ h_t = \bar{\xi}_2 \quad and \quad h_t \circ s^{\infty}_{\xi_2} = s^{\infty}_{\xi_1 \oplus \xi_2} \quad for \quad t \in I.$$

PROOF. Let $s: X \to D_1$ be the nonvanishing G-cross section of ξ_1. Assume that $\|s(x)\| = 1$ for $x \in X$. We define a G-homotopy

$$h_t: X_{\xi_2} \to X_{\xi_1 \oplus \xi_2}, \quad t \in I = [0, 1],$$

by

$$h_t([d]) = \begin{cases} [(t\sqrt{1-\|d\|^2}s(\xi_2(d))) \oplus d] & \text{if } \|d\| \leqq 1 \\ [s^0_{\xi_1}(\xi_2(d)) \oplus d] & \text{if } \|d\| \geqq 1 \end{cases}$$

for $d \in D_2$.

Then $h_0 = f^0_{\xi_1,\xi_2}$ and $h_1 = f^{\infty}_{\xi_1,\xi_2}$. Observe also that, for every $t \in I$,

$$h_t \circ s^{\infty}_{\xi_2} = s^{\infty}_{\xi_1 \oplus \xi_2} \quad \text{and} \quad \overline{(\xi_1 \oplus \xi_2)} \circ h_t = \bar{\xi}_2.$$

This proves Lemma 1.6.

We shall use (1.6) only in a non-equivariant form.

PROOF OF PROPOSITION 1.1. Let $\{pt\}$ be a one point G-space. The G-map $f: M^{\omega} \to \{pt\}$ induces $\bar{f}: P \times_G M^{\omega} \to P \times_G \{pt\} = B$. Let $g: \{pt\} \to M^{\pi \oplus \omega}$ be the embedding of the base point. g is a G-map and it induces an embedding

$$\bar{g}: B = P \times_G \{pt\} \to P \times_G M^{\pi \oplus \omega}$$

Let $l: P \times_G M^{\omega} \to P \times_G M^{\pi \oplus \omega}$ be the composition $l = \bar{g} \circ \bar{f}$. l is an ex-map over B.

If η has a nonvanishing cross section over $E = P \times_G M$, then the ex-map over B (see (1.2))

$$\mathrm{id}_P \times_G j: P \times_G M^{\omega} \to P \times_G M^{\pi \oplus \omega}$$

is ex-homotopic to l. Indeed, let λ be the vector bundle $P \times_G \omega$ over $E = P \times_G M$. Observe that $P \times_G M^{\omega}$ is a quotient space of E_{λ} and $P \times_G M^{\tau \oplus \omega}$ is a quotient space of $E_{\eta \oplus \lambda}$. In fact, $P \times_G M^{\omega} = E_{\lambda}/\sim$, where, for $x,y \in E_{\lambda}$, we have $x \sim y$ if and only if either $x = y$ or $x = s^{\infty}_{\lambda}(a)$,

$y = s_\lambda^\infty(b)$, $a, b \in E$ and $\mu(a) = \mu(b)$. Similarly, for $P \times_G M^{\pi \oplus \omega}$. Moreover, under those identifications the map l is a quotient of $f_{\eta,\lambda}^\infty$ and the map $\mathrm{id}_P \times_G j$ is a quotient of $f_{\eta,\lambda}^0$. According to (1.6) there is an ex-homotopy h_t over E from $f_{\eta,\lambda}^0$ to $f_{\eta,\lambda}^\infty$. h_t induces an ex-homotopy over B from $\mathrm{id}_P \times_G j$ to l on the quotient spaces.

Define now a G-map

$$\gamma_\infty : S^V \to (M_+) \wedge S^V$$

to be a composition

$$S^V \xrightarrow{c} M^\omega \xrightarrow{g \circ f} M^{\pi \oplus \omega} \xrightarrow{\psi} (M_+) \wedge S^V.$$

Since $g \circ f$ maps the whole M^ω into the base point of $M^{\pi \oplus \omega}$, γ_∞ maps S^V into the base point of $(M_+) \wedge S^V$.

Define

$$\gamma'_\infty : P \times_G S^V \to P \times_G ((M_+) \wedge S^V)$$

as $\gamma'_\infty = \mathrm{id}_P \times_G \gamma_\infty$. Since $\mathrm{id}_P \times_G (g \circ f) = l$ is ex-homotopic over B to $\mathrm{id}_P \times_G j$, we get that γ' and γ'_∞ are ex-homotopic over B as well. Moreover

$$\gamma'_\infty(P \times_G S^V) \subset B \subset P \times_G ((M_+) \wedge S^V).$$

Consequently,

$$\gamma'_\infty \wedge_B 1 : (P \times_G S^V) \wedge_B B_\zeta \to (P \times_G ((M_+) \wedge S^V)) \wedge_B B_\zeta$$

is ex-homotopic over B to the map $\gamma' \wedge_B 1$ of (1.3). When we pass to the quotient spaces of (1.4), $\gamma'_\infty \wedge_B 1$ and $\gamma' \wedge_B 1$ induce homotopic maps. On the other hand $\gamma'_\infty \wedge_B 1$ induces trivial map while $\gamma' \wedge_B 1$ induces the transfer $\tau(\mu)$.

COROLLARY 1.7. *If there exists a representation W of the group G such that the equivariant tangent bundle $T(M)$ contains $M \times W$ as a G-subbundle and* $\dim_{\mathbb{R}} W > \dim B$, *then the transfer*

$$\tau(\mu) : B_+ \to Q(E_+)$$

is homotopically trivial.

PROOF. Let $\lambda : P \times_G W \to B$ be the vector bundle associated to the principal G-bundle $\tilde{\mu} : P \to B$ and the representation W. Since $M \times W \subset T(M)$ equivariantly, the vector bundle η over $E = P \times_G M$ contains a subbundle

$$\bar{\lambda} : P \times_G (M \times W) \to P \times_G M = E.$$

We have $\bar{\lambda} = \mu^*(\lambda)$, $\mu : P \times_G M \to B$. Since $\dim B < \dim_{\mathbb{R}} W$, λ has a non-vanishing cross section (see [10; Section 8.1]). Therefore $\bar{\lambda} = \mu^*(\lambda)$ also has a nonvanishing cross section and, consequently, the bundle η has one such as well. Corollary 1.7 follows now from (1.1).

Lest us recall that the p-transfer of the bundle $\mu: P \times_G M \to B$,

$$\bar{\tau}(\mu): B_+ \to Q(S^0)$$

is the composition

$$B_+ \xrightarrow{\tau(\mu)} Q((P \times_G M)_+) \xrightarrow{Q(p)} Q(S^0),$$

where $p = p_{P \times_G M}: (P \times_G M)_+ \to S^0$. (See Introduction.)

We shall now recall some well-known facts about the p-transfer. Let $A(G)$ be the Burnside ring of the group G, (see [7], [8; Chapter 5]). The G-manifold M represents an element $[M]$ in $A(G)$.

For a (finite dimensional) representation V of G let $[S^V, S^V]_G$ be the set of G-homotopy classes of G-maps from the one point compactification S^V of V into itself. If W is another representation of G, there is a suspension map

$$\sigma_{V,W}: [S^V, S^V]_G \to [S^{V \oplus W}, S^{V \oplus W}]_G.$$

Let

$$\omega_G^0 = \varinjlim_V [S^V, S^V]_G,$$

where V runs over the set of isomorphism classes of real representations of G and $\sigma_{V,W}$ are the transformations in the direct system. ω_G^0 is a commutative ring with unit.

There is a ring isomorphism $I_G: A(G) \to \omega_G^0$, see [9], [15], [8; Theorem 8.5.1]. I_G may be described as follows: let $\alpha \in A(G)$ be represented by a compact G-manifold M. We choose a representation V of G and a G-embedding $i: M \hookrightarrow V$. Let $\mathrm{pr}_2: (M_+) \wedge S^V \to S^V$ be the projection on the second factor. Then $I_G(\alpha) \in \omega_G^0$ is represented by the composition

$$S^V \xrightarrow{\gamma} (M_+) \wedge S^V \xrightarrow{\mathrm{pr}_2} S^V,$$

where γ is the map defined in (1.2). In particular, the stable G-homotopy class of $\mathrm{pr}_2 \circ \gamma$ depends only on the element in the Burnside ring $A(G)$ represented by M.

Let M_i, $i = 1, 2$, be two G-manifolds. We have two fibre bundles

$$\mu_i: P \times_G M_i \to B, \quad i = 1, 2,$$

and two transfer maps

$$\tau(\mu_i): B_+ \to Q((P \times_G M_i)_+), \quad i = 1, 2.$$

Furthermore, we have two maps

$$Q(\mu_i): Q((P \times_G M_i)_+) \to Q(B_+).$$

Let us consider compositions

$$Q(\mu_i) \circ \tau(\mu_i): B_+ \to Q(B_+).$$

PROPOSITION 1.8. *If* $[M_1] = [M_2]$ *in the Burnside ring* $A(G)$, *then* $Q(\mu_1) \circ \tau(\mu_1)$ *is homotopic to* $Q(\mu_2) \circ \tau(\mu_2)$.

PROOF. Since $[M_1] = [M_2]$ in $A(G)$, it follows from the construction of I_G described above that there exists a representation V of G and G-embeddings $i_1 : M_1 \to V$, $i_2 : M_2 \to V$ such that the compositions

$$S^V \xrightarrow{\gamma_1} (M_{1+}) \wedge S^V \xrightarrow{\mathrm{pr}_2} S^V \quad \text{and} \quad S^V \xrightarrow{\gamma_2} (M_{2+}) \wedge S^V \xrightarrow{\mathrm{pr}_2} S^V$$

are G-homotopic. Here γ_1 is constructed as in (1.2) from i_1 and γ_2 from i_2.
Let us consider the maps

$$\bar{\gamma}_i': P \times_G S^V \to P \times_G S^V, \quad \bar{\gamma}_i' = \mathrm{id}_P \times_G (\mathrm{pr}_2 \circ \gamma_i), \quad i = 1, 2,$$

and then the maps

$$\bar{\gamma}_i' \wedge_B 1 : (P \times_G S^V) \wedge_B B_\zeta \to (P \times_G S^V) \wedge_B B_\zeta, \quad i = 1, 2$$

(compare (1.3)). As in (1.4), the maps $\bar{\gamma}_i' \wedge_B 1$ yield

$$\gamma_i'': B^{\xi \oplus \zeta} \to B^{\xi \oplus \zeta}, \quad i = 1, 2.$$

Finally, under the identification $\bar{\Phi}: B^{\xi \oplus \zeta} \to (B_+) \wedge S^n$ we get

$$\tilde{\gamma}_i = \bar{\Phi} \gamma_i'' \bar{\Phi}^{-1} : (B_+) \wedge S^n \to (B_+) \wedge S^n, \quad i = 1, 2.$$

Since $\mathrm{pr}_2 \circ \gamma_1$ and $\mathrm{pr}_2 \circ \gamma_2$ were G-homotopic, $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are homotopic. Observe now that $Q(\mu_i) \circ \tau(\mu_i)$ is represented by $\tilde{\gamma}_i$, $i = 1, 2$.

COROLLARY 1.9. *If* $[M_1] = [M_2]$ *in the Burnside ring* $A(G)$, *then the p-transfers*

$$\bar{\tau}(\mu_1), \bar{\tau}(\mu_2): B_+ \to Q(S^0)$$

are homotopic.

PROOF. $\bar{\tau}(\mu_i) = Q(p_i) \circ \tau(\mu_i)$, where

$$p_i: (P \times_G M_i)_+ \to S^0, \quad p_i = p_{P \times_G M_i}.$$

Let us consider $p_B: B_+ \to S^0$. Then

$$p_i = p_B \circ \mu_i \quad \text{and} \quad \bar{\tau}(\mu_i) = Q(p_B) \circ Q(\mu_i) \circ \tau(\mu_i).$$

Corollary 1.9 follows now from (1.8).

Let $\mu_G: EG \to BG$ be the universal principal G-bundle over the classifying space BG of the group G. Let M be a G-manifold. We denote by $\mu_G(M)$ the associated bundle $\mu_G(M): EG \times_G M \to BG$.

BG is not a finite complex. It follows, however, from the Sullivan theory of compact Brownian functors [17; Section 3] that there is a well-defined homotopy class of a map

$$(1.10) \qquad \bar{\tau}(\mu_G(M)): BG_+ \to Q(S^0)$$

characterized by the property:

> if $B \subset BG$ is a finite subcomplex, $P = \mu_G^{-1}(B)$, $\tilde{\mu}: P \to B$, $\tilde{\mu} = \mu_G|P$ and $\mu: P \times_G M \to B$ is the associated bundle, then the p-transfer $\bar{\tau}(\mu): B_+ \to Q(S^0)$ of the bundle μ is homotopic to $\bar{\tau}(\mu_G(M))|B_+$.

It follows from Corollary 1.9 that the homotopy class of $\bar{\tau}(\mu_G(M))$ depends only on the element $[M]$ in the Burnside ring $A(G)$. Thus we get a transformation

$$(1.11) \qquad \bar{\tau}: A(G) \to [BG_+, Q(S^0)]_*$$

defined by $\bar{\tau}([M]) = \bar{\tau}(\mu_G(M))$. Here $[\cdot,\cdot]_*$ denotes the set of based homotopy classes of maps.

The loop-sum and the composition product on $Q(S^0)$ induce in $[BG_+, Q(S^0)]_*$ a structure of a commutative ring with unit. $\bar{\tau}$ is a homomorphism of rings.

There are other ways to describe the transformation $\bar{\tau}$, see for example [12].

2. Examples.

Let R^n be the space of n-tuples of real numbers and let Q_n be the negative definite form $Q_n(x_1, \ldots, x_n) = -\sum x_i^2$ on R^n. C_n is the Clifford algebra of the form Q_n, $i_{Q_n}: \mathsf{R}^n \to C_n$, see [2], [5], [10]. We shall follow the notations and definitions of [2]. C_n^* is the group of invertible elements of C_n, $\Gamma_n \subset C_n^*$ is the Clifford group. $\mathrm{Pin}(n) \subset \Gamma_n$ is the subgroup of elements of norm 1.

Let $\varrho_n: \mathrm{Pin}(n) \to \mathrm{O}(\mathsf{R}^n)$ be the *twisted adjoint representation* of $\mathrm{Pin}(n)$ (see [2; p. 7–8]), $\mathrm{Spin}(n) = \varrho_n^{-1}(\mathrm{SO}(\mathsf{R}^n))$. Let $\varphi: \mathrm{Pin}(n) \to \mathrm{O}(\mathsf{R}^1)$ be the nontrivial representation with $\mathrm{Ker}\,\varphi = \mathrm{Spin}(n)$.

We define $\tilde{\varrho}_n: \mathrm{Pin}(n) \to \mathrm{O}(\mathsf{R}^n)$ as the tensor product of representations $\tilde{\varrho}_n = \varrho_n \otimes \varphi$ and we refer to $\tilde{\varrho}_n$ as the *untwisted adjoint representation* of $\mathrm{Pin}(n)$ on R^n. Compare [5; Section 2.3, p. 49]. Let us identify R^n with its image by the embedding $i_{Q_n}: \mathsf{R}^n \to C_n$. Then, in C_n, we have $sxs^{-1} = \tilde{\varrho}_n(s)(x)$ for $x \in \mathsf{R}^n$ and $s \in \mathrm{Pin}(n)$.

Let N be a (finite-dimensional) module over C_n. We may assume that N is equipped with an inner product $\langle \cdot, \cdot \rangle$ such that $\mathrm{Pin}(n)$ acts on N as a group of isometries. Let $S(N)$ be the unit sphere in N. $S(N)$ is a $\mathrm{Pin}(n)$-manifold, and we have:

PROPOSITION 2.1. *The* $\mathrm{Pin}(n)$*-equivariant tangent bundle* $T(S(N))$ *contains a* $\mathrm{Pin}(n)$*-vector subbundle* $S(N) \times \tilde{\varrho}_n$ *of dimension* n.

PROOF. If we identify

$$T(S(N)) = \{(x, v) \in S(N) \times N \mid \langle x, v \rangle = 0\},$$

then the action of $\mathrm{Pin}(n)$ on $T(S(N))$ is the restriction of the diagonal action on $S(N) \times N$.

The map $\mu: S(N) \times \mathbf{R}^n \to S(N) \times N$, $\mu(x, w) = (x, i_{Q_n}(w)(x))$ gives an embedding of the trivial bundle $S(N) \times \mathbf{R}^n$ on a subbundle of $T(S(N))$, see [10; Section 11.2].

If $\mathrm{Pin}(n)$ acts on $S(N) \times \mathbf{R}^n$ through $g(x, w) = (gx, \tilde{\varrho}_n(g)(w))$ for $g \in \mathrm{Pin}(n)$, $x \in S(N)$ and $w \in \mathbf{R}^n$, then μ is a $\mathrm{Pin}(n)$-map. Indeed,

$$\begin{aligned}
\mu(g(x, w)) &= \mu(gx, \tilde{\varrho}_n(g)(w)) = (gx, i_{Q_n}(\tilde{\varrho}_n(g)(w))(gx)) \\
&= (gx, (g i_{Q_n}(w) g^{-1})(gx)) = (gx, g i_{Q_n}(w)(x)) \\
&= g(x, i_{Q_n}(w)(x)) = g\mu(x, w),
\end{aligned}$$

since $\tilde{\varrho}_n(g)(w) = g i_{Q_n}(w) g^{-1}$ in C_n.

Thus μ gives a Pin-embedding of the subbundle $S(N) \times \tilde{\varrho}_n$ in $T(S(N))$.

COROLLARY 2.2. *If* N *is a* C_n*-module and* $a(N) \in A(\mathrm{Pin}(n))$ *is the element in the Burnside ring of* $\mathrm{Pin}(n)$ *represented by the sphere* $S(N)$, *then the transfer map*

$$\bar{\tau}(a(N)): B\mathrm{Pin}(n)_+ \to Q(S^0)$$

is contractible on the $(n-1)$*-skeleton of* $B\mathrm{Pin}(n)_+$.

For further examples see [16; Section 5].

3. Some cohomology computations. The Hopf invariant.

All homology and cohomology groups which appear in this and the next Section are with $\mathbf{Z}/2$-coefficients. We write $H^*(X)$ for the cohomology ring $H^*(X; \mathbf{Z}/2)$ of a space X.

Let us consider the elementary abelian 2-group $G_k = \mathbf{Z}/2 \times \ldots \times \mathbf{Z}/2$ (k factors) and let V_k be the real regular representation of G_k. We assume that

V_k is equipped with a G_k-invariant scalar product. $S(V_k)$ is the unit sphere in V_k.

Let α_k be the element of the Burnside ring $A(G_k)$ of the group G_k represented by the G_k-manifold $S(V_k)$. For a subgroup $H \subset G_k$, we denote by $\chi_H : A(G_k) \to \mathbf{Z}$ the homomorphism given by $\chi_H([M]) = \chi(M^H)$. Here M is a G_k-manifold and $[M]$ is the class of M in $A(G_k)$. According to [16; (5.2)] the element α_k is characterized by

$$\chi_H(\alpha_k) = \begin{cases} 2 & \text{if } H = G_k \\ 0 & \text{if } H \subsetneqq G_k. \end{cases}$$

Let $\eta : G_k \to \mathbf{Z}/2$ be a group homomorphism, $\eta \in \mathrm{Hom}(G_k, \mathbf{Z}/2)$. We define an element $\tilde{\eta} \in A(G_k)$ as $\tilde{\eta} = [G_k/\mathrm{Ker}\,\eta]$.

PROPOSITION 3.1. (E. Laitinen). *In* $A(G_k)$

$$1 - \alpha_k = \prod_{\substack{\eta \in \mathrm{Hom}(G_k, \mathbf{Z}/2) \\ \eta \neq 0}} (\tilde{\eta} - 1).$$

PROOF. See [12; p. 68].

Let us consider the p-transfer map of the element $1 - \alpha_k$

$$\bar{\tau}(1 - \alpha_k) : BG_k \to Q(S^0)_{(1)}.$$

Let

$$\sigma(w_n) \in H^{n-1}(Q(S^0)_{(1)})$$

be the suspension of the nth Stiefel-Whitney class $w_n \in H^n(B(Q(S^0)_{(1)}))$. We denote $\sigma(W) = \sum_{i \geq 2} \sigma(w_i)$.

We shall compute $(\bar{\tau}(1 - \alpha_k))^*(\sigma(W)) \in H^*(BG_k)$.

Let us consider first the case $G_1 = \mathbf{Z}/2$. Then $BG_1 = \mathbf{RP}^\infty$. The p-transfer of the element $[G_1] - 1 \in A(G_1)$,

$$\bar{\tau}([G_1] - 1) : \mathbf{RP}^\infty \to Q(S^0)_{(1)}$$

is homotopic to the James map

$$\mathbf{RP}^\infty \to \mathrm{SO} \xrightarrow{J} Q(S^0)_{(1)},$$

(see [6; p. 120]). Thus

$$(\bar{\tau}([G_1] - 1))^*(\sigma(w_n)) = u^{n-1} \tag{3.2}$$

in $H^*(\mathbf{RP}^\infty)$, where $u \in H^1(\mathbf{RP}^\infty)$ is the generator. See [4; Lemma 3.5].

Let

$$\varphi : \mathrm{Hom}(G_k, \mathbf{Z}/2) \to H^1(BG_k)$$

be the isomorphism given by $\varphi(\eta) = B(\eta)^*(u)$ for $\eta \in \operatorname{Hom}(G_k, \mathbf{Z}/2)$. We identify $\operatorname{Hom}(G_k, \mathbf{Z}/2)$ with $H^1(BG_k)$ through φ.

Let $\eta \in \operatorname{Hom}(G_k, \mathbf{Z}/2)$. Then the p-transfer

$$\bar{\tau}(\tilde{\eta}-1)\colon BG_k \to Q(S^0)_{(1)}$$

of the element $\tilde{\eta}-1 \in A(G_k)$ satisfies

$$(3.3) \qquad (\bar{\tau}(\tilde{\eta}-1))^*(\sigma(w_n)) = \eta^{n-1}$$

in $H^{n-1}(BG_k)$. Indeed, let $\eta^*\colon A(\mathbf{Z}/2) \to A(G_k)$ be the homomorphism induced by η. Then $\tilde{\eta}-1 = \eta^*([G_1]-1)$ in $A(G_k)$. It follows that $\bar{\tau}(\tilde{\eta}-1)$ is equal to the composition

$$BG_k \xrightarrow{B(\eta)} BG_1 \xrightarrow{\bar{\tau}([G_1]-1)} Q(S^0)_{(1)}.$$

Consequently,

$$(\bar{\tau}(\tilde{\eta}-1))^*(\sigma(w_n)) = B(\eta)^*(u^{n-1}) = (B(\eta)^*(u))^{n-1} = \eta^{n-1}.$$

COROLLARY 3.4.

$$(\bar{\tau}(1-\alpha_k))^*(\sigma(w_{n+1})) = \sum_{x \in H^1(BG_k)} x^n$$

in $H^n(BG_k)$ *for* $n \geqq 1$.

PROOF. It follows from (3.1) that

$$\bar{\tau}(1-\alpha_k) = \prod_{\substack{\eta \in \operatorname{Hom}(G_k,\, \mathbf{Z}/2) \\ \eta \neq 0}} (\bar{\tau}(\tilde{\eta}-1))$$

in $[BG_k, Q(S^0)_{(1)}]$. In this formula the product on the right hand side is the multiplication in $[BG_k, Q(S^0)_{(1)}]$ induced by the composition product in $Q(S^0)_{(1)}$. The classes $\sigma(w_{n+1}) \in H^n(Q(S^0)_{(1)})$ are primitive with respect to the composition product in $Q(S^0)_{(1)}$, (see [4; Lemma 3.5]). Thus, Corollary 3.4 follows from (3.3).

We identify $H^*(BG_k)$ with the graded polynomial ring $\mathbf{Z}/2[x_1, \ldots, x_k]$ in k independent variables x_i, $\deg x_i = 1$.

LEMMA 3.5. *In* $H^*(BG_k) = \mathbf{Z}/2[x_1, \ldots, x_k]$ *we have*

(i) *if* $1 \leqq n < 2^k-1$, *then* $\sum_{x \in H^1(BG_k)} x^n = 0$,

(ii) *if* $n = 2^k-1$, *then* $\sum_{x \in H^1(BG_k)} x^n \neq 0$.

PROOF. Let $V \subset Z/2[x_1, \ldots, x_k]$ be the vector subspace spanned by $x_1, \ldots, x_k$. Let

$$T_n(x_1, \ldots, x_k) = \sum_{x \in V} x^n \in Z/2[x_1, \ldots, x_k].$$

We denote by $W(i_1, \ldots, i_k)$ the monomial $x_1^{i_1} \ldots x_k^{i_k}$. We shall determine which monomials $W(i_1, \ldots, i_k)$ can appear in $T_n(x_1, \ldots, x_k)$ with nontrivial coefficients.

STEP 1. Let us first consider those monomials $W(i_1, \ldots, i_k)$ for which at least one $i_j = 0$. We may assume that

$$W(i_1, \ldots, i_k) = W(i_1, \ldots, i_s, 0, \ldots, 0), \quad s < k \text{ and } i_1 \neq 0, \ldots, i_s \neq 0.$$

If such a $W(i_1 \ldots, i_k)$ appears nontrivially in $T_n(x_1, \ldots, x_k)$, then it must already appear with the same coefficient in the polynomial $(x_1 + \ldots + x_s)^n$. Indeed, this follows directly from the equality

$$T_n(x_1, \ldots, x_k) = \sum_{A \subset \{1, \ldots, k\}} \left(\sum_{a \in A} x_a \right)^n.$$

Furthermore, such a $W(i_1, \ldots, i_k)$ cannot appear nontrivially in $(x_{l_1} + \ldots + x_{l_t})^n$ if

$$\{1, \ldots, s\} \not\subset \{l_1, \ldots, l_t\}.$$

$W(i_1, \ldots, i_s, 0, \ldots, 0)$ appears with the same coefficient in all polynomials $((x_1 + \ldots + x_s) + y)^n$, where $y \in Y = \operatorname{span}\{x_{s+1}, \ldots, x_k\}$. Thus

$$T_n(x_1, \ldots, x_k) = \sum_{\substack{A \subset \{1, \ldots, k\} \\ \{1, \ldots, s\} \not\subset A}} \left(\sum_{a \in A} x_a \right)^n + \sum_{y \in Y} ((x_1 + \ldots + x_s) + y)^n.$$

Since cardinality of Y is 2^{k-s} and $k - s > 0$, it follows that $W(i_1, \ldots, i_s, 0, \ldots, 0)$ has multiplicity 2^{k-s} in $T_n(x_1, \ldots, x_k)$, i.e. multiplicity 0.

Thus we have proved that *no* monomial $W(i_1, \ldots, i_k)$ with at least one $i_j = 0$ can appear with a nontrivial coefficient in $T_n(x_1, \ldots, x_k)$.

STEP 2. We shall now consider monomials $W(i_1, \ldots, i_k)$ with all $i_j > 0$. The coefficient of such a $W(i_1, \ldots, i_k)$ in $T_n(x_1, \ldots, x_k)$ is the same as its coefficient in $(x_1 + \ldots + x_k)^n$.

Let $n = \sum_{j=0}^{m} a_j 2^j$, $a_j = 0$ or 1, be the 2-adic expansion of n. Then

$$(x_1 + \ldots + x_k)^n = \prod_{j=0}^{m} (x_1^{2^j} + \ldots + x_k^{2^j})^{a_j}.$$

If $n \leqq 2^k - 1$, then $n = \sum_{j=0}^{k-1} a_j 2^j$. For $n < 2^k - 1$ the number of these a_j's for which $a_j \neq 0$ is less than k. It follows that *no* $W(i_1, \ldots, i_k)$ with $i_j > 0$ for $j = 1, \ldots, k$, appears with a nontrivial coefficient in $(x_1 + \ldots + x_k)^n$.

Hence $T_n(x_1, \ldots, x_k) = 0$ if $n < 2^k - 1$. If $n = 2^k - 1$, then $n = \sum_{j=0}^{k-1} 2^j$ and

$$(x_1 + \ldots + x_k)^n = \prod_{j=0}^{k-1} (x_1^{2^j} + \ldots + x_k^{2^j})$$

$$= \mathrm{Symm}\left(\prod_{i=1}^{k} x_i^{2^{k-i}}\right) + \begin{cases}\text{monomials which do not include} \\ \text{all variabels at the same time.}\end{cases}$$

Here Symm($\cdot$) is the symmetrization operator with respect to the group of permutations of all variables x_i.

Thus

$$T_n(x_1, \ldots, x_k) = \mathrm{Symm}\left(\prod_{i=1}^{k} x_i^{2^{k-i}}\right) \neq 0 \quad \text{if } n = 2^k - 1.$$

REMARK 3.6. The element $T_{2^k-1}(x_1, \ldots, x_k) \in H^{2^k-1}(BG_k)$ is *not* detected by any proper subgroup of G_k. It is of the lowest possible dimension among all the elements of $H^*(BG_k)$ with this property and the only one in this dimension. Furthermore,

$$T_{2^k-1}(x_1, \ldots, x_k) = \prod_{\substack{y \in H^1(BG_k) \\ y \neq 0}} y.$$

Compare also [13; Lemma 3.25].

COROLLARY 3.7. *$(\bar{\tau}(1-\alpha_k))^*(\sigma(w_{2^k})) \neq 0$ in $H^{2^k-1}(BG_k)$.*

It follows from (1.7) and [16; (5.4)] that for $k = 1, 2, 3$ the p-transfer of $\alpha_k \in A(G_k)$

$$\bar{\tau}(\alpha_k): BG_k \to Q(S^0)_{(0)}$$

is contractible over (2^k-2)-skeleton of BG_k. Let $I: BG_k \to Q(S^0)_{(1)}$ be a map which transforms all BG_k into one point of $Q(S^0)_{(1)}$. Then

$$\bar{\tau}(1-\alpha_k) = I - \bar{\tau}(\alpha_k)$$

in the ring of (non-based) homotopy classes $[BG_k, Q(S^0)]$.

It follows that for $k = 1, 2, 3$ the p-transfer

$$\bar{\tau}(1-\alpha_k): BG_k \to Q(S^0)_{(1)}$$

is contractible over (2^k-2)-skeleton $BG_k^{(2^k-2)}$ of BG_k.

Let

$$p_k: BG_k \to BG_k/BG_k^{(2^k-2)}$$

be the contraction map and let

$$F_k : BG/BG_k^{(2^k-2)} \to Q(S^0)_{(1)}$$

be a map such that the diagram

$$\begin{array}{ccc} BG_k & \xrightarrow{\bar{\tau}(1-\alpha_k)} & Q(S^0)_{(1)} \\ & {\scriptstyle p_k}\searrow \quad \nearrow{\scriptstyle F_k} & \\ & BG_k/BG_k^{(2^k-2)} & \end{array}$$

is homotopy commutative. Such F_k exists for $k = 1, 2, 3$. We are going to show that it does not exist for any other value of k. It follows from (3.7) that $F_k^*(\sigma(w_{2^k})) \neq 0$ in $H^{2^k-1}(BG_k/BG_k^{(2^k-2)})$. Since $BG_k/BG_k^{(2^k-2)}$ is (2^k-2)-connected, there exists a map

$$f_k : S^{2^k-1} \to BG_k/BG_k^{(2^k-2)}$$

such that the composition

$$S^{2^k-1} \xrightarrow{f_k} BG_k/BG_k^{(2^k-2)} \xrightarrow{F_k} Q(S^0)_{(1)}$$

satisfies $(F_k \circ f_k)^*(\sigma(w_{2^k})) \neq 0$ in $H^{2^k-1}(S^{2^k-1})$, i.e. the composition $F_k \circ f_k$ is a map with the Hopf invariant 1.

COROLLARY 3.8. *Let* $\bar{\tau}(1-\alpha_k): BG_k \to Q(S^0)_{(1)}$ *be the p-transfer of* $1-\alpha_k \in A(G_k)$.

(i) *If* $k > 3$, *then* $\bar{\tau}(1-\alpha_k)$ *is not homotopically trivial over the* (2^k-2)-*skeleton of* BG_k.

(ii) *If* $k = 1, 2, 3$, *then* $\bar{\tau}(1-\alpha_k)$ *is homotopically trivial over the* (2^k-2)-*skeleton* $BG_k^{(2^k-2)}$ *of* BG_k. *There exist maps*

$$F_k : BG_k/BG_k^{(2^k-2)} \to Q(S^0)_{(1)} \quad \text{and} \quad f_k : S^{2^k-1} \to BG_k/BG_k^{(2^k-2)}$$

such that $F_k \circ p_k : BG_k \to Q(S^0)_{(1)}$ *is homotopic to* $\bar{\tau}(1-\alpha_k)$ *and* $F_k \circ f_k : S^{2^k-1} \to Q(S^0)_{(1)}$ *is a map with the Hopf invariant one.*

PROOF. (i) follows directly from the non-existence of maps with the Hopf invariant one in dimensions greater than 7, (see [1]). (ii) has been shown above.

4. Concluding remarks. The Kervaire invariant.

The notations of Section 3 are preserved.

We shall now consider the elements $1-\alpha_n^2 \in A(G_n)$.

Let $k_{2^{n+1}-2} \in H^{2^{n+1}-2}(G/PL)$ be the Kervaire class, (see [14]), and let

$$i^*(k_{2^{n+1}-2}) \in H^{2^{n+1}-2}(Q(S^0)_{(1)})$$

be its image through the map $i: Q(S^0)_{(1)} \to G/PL$, see [4; Section 3].

We are going to show that the p-transfer $\bar{\tau}(1-\alpha_n^2): BG_n \to Q(S^0)_{(1)}$ satisfies

$$\bar{\tau}(1-\alpha_n^2)^*(k_{2^{n+1}-2}) \neq 0$$

in $H^{2^{n+1}-2}(BG_n)$.

Let us first consider the p-transfer of the element $(1-\alpha_n)^2 \in A(G_n)$:

$$\bar{\tau}((1-\alpha_n)^2): BG_n \to Q(S^0)_{(1)}.$$

Lemma 4.1.

$$\bar{\tau}((1-\alpha_n)^2)^*(\sigma(W)) = 0$$

$$\bar{\tau}((1-\alpha_n)^2)^*(i^*(k_{2^m-2})) = 0 \quad \textit{for every } m \geqq 2.$$

Proof. We have

$$\bar{\tau}((1-\alpha_n)^2) = (\bar{\tau}((1-\alpha_n)))^2$$

in $[BG_n, Q(S^0)]$. Lemma 4.1 follows now from the fact that both $\sigma(W)$ and $i^*(k_{2^m-2})$ are primitive with respect to the composition product in $Q(S^0)_{(1)}$, (see [14] and [4; Lemma 3.5]).

We recall now a theorem of Brumfiel, Madsen and Milgram. We quote from [4; Section 3, p. 94]. Let

$$\Delta_*: H^*(Q(S^0)_{(1)}) \to H^*(Q(S^0)_{(1)}) \otimes H^*(Q(S^0)_{(1)})$$

be the coproduct induced by the $*$-structure on $Q(S^0)_{(1)}$, i.e. the loop sum structure adjusted with a component shift. Let $\bar{\Delta}_*(x) = \Delta_*(x) - x \otimes 1 - 1 \otimes x$.

Theorem (Brumfiel-Madsen-Milgram).

$$\bar{\Delta}_*(i^*(k_{2^j-2})) = \sum_{\substack{s+t=2^j \\ s,t \geq 2}} \sigma(w_s) \otimes \sigma(w_t).$$

Proposition 4.2. *The p-transfer*

$$\bar{\tau}(1-\alpha_n^2): BG_n \to Q(S^0)_{(1)}$$

satisfies

$$(\bar{\tau}(1-\alpha_n^2))^*(i^*(k_{2^{n+1}-2})) \neq 0$$

in $H^*(BG_n)$.

Proof. We have $1-\alpha_n^2 = 2(1-\alpha_n) - (1-\alpha_n)^2$. It follows from Lemma 4.1 and the Theorem of Brumfiel-Madsen-Milgram that

$$
\begin{aligned}
(\bar{\tau}(1-\alpha_n^2))^*(i^*(k_{2^{n+1}-2})) &= (\bar{\tau}(2(1-\alpha_n)-1))^*(i^*(k_{2^{n+1}-2})) \\
&= [(\bar{\tau}(1-\alpha_n))^*(\sigma(w_{2^n}))]^2 \in H^*(BG_n).
\end{aligned}
$$

According to (3.7), $(\bar{\tau}(1-\alpha_n))^*(\sigma(w_{2^n})) \neq 0$. Since $H^*(BG_n)$ has no zero-divisors, we get the conclusion of Proposition 4.2. As a matter of fact we have

$$
(\bar{\tau}(1-\alpha_n^2))^*(i^*(k_{2^{n+1}-2})) = \prod_{\substack{x \in H^1(BG_n) \\ x \neq 0}} x^2.
$$

Thus, for every positive integer n, we have the map

$$
f_n = \bar{\tau}(1-\alpha_n^2) : BG_n \to Q(S^0)_{(1)}
$$

such that

$$
f_n^*(i^*(k_{2^{n+1}-2})) \neq 0.
$$

In this way we are led to

PROBLEM 1. Is the p-transfer $f_n = \bar{\tau}(1-\alpha_n^2)$ contractible on the $(2^{n+1}-3)$-skeleton of BG_n?

The positive answer to Problem 1 implies the existence of a framed manifold of dimension $2^{n+1}-2$ with the Kervaire invariant one. Indeed, if the answer is positive, then there exists a map

$$
F_n : BG_n/BG_n^{(2^{n+1}-3)} \to Q(S^0)_{(1)}
$$

such that the diagram

$$
\begin{array}{ccc}
BG_n & \xrightarrow{f_n} & Q(S^0)_{(1)} \\
{\scriptstyle p}\searrow & & \nearrow{\scriptstyle F_n} \\
& BG_n/BG_n^{(2^{n+1}-3)} &
\end{array}
$$

is homotopy commutative. Furthermore,

$$
F_n^*(i^*(k_{2^{n+1}-2})) \neq 0 \quad \text{in} \quad H^{2^{n+1}-2}(BG_n/BG_n^{(2^{n+1}-3)}).
$$

Since $BG_n/BG_n^{(2^{n+1}-3)}$ is $(2^{n+1}-3)$-connected, there exists also a map

$$
h_n : S^{2^{n+1}-2} \to BG_n/BG_n^{(2^{n+1}-3)}
$$

such that $h_n^*(F_n^*(k_{2^{n+1}-2}))) \neq 0$. Thus the composition

$$
S^{2^{n+1}-2} \xrightarrow{h_n} BG_n/BG_n^{(2^{n+1}-3)} \xrightarrow{F_n} Q(S^0)_{(1)}
$$

would represent a framed manifold of dimension $2^{n+1}-2$ with the Kervaire invariant one.

PROBLEM 2. Do there exist a G_n-manifold M and an orthogonal representation W of G_n such that

(i) $[M] = \alpha_n^2$ in the Burnside ring $A(G_n)$,

(ii) $\dim_{\mathbb{R}} W = 2^{n+1} - 2$, and

(iii) $M \times W$ is a G_n-vector subbundle of the tangent bundle $T(M)$ of M?

According to Corollary 1.7, the positive answer to Problem 2 implies the positive answer to Problem 1.

Let us consider the G_n-manifolds $M_n = S(V_n) \times S(\mathring{V}_n)$. M_n is of dimension $2^{n+1} - 2$ and it represents the element α_n^2 in $A(G_n)$. According to [16; (5.4) and (5.14)], M_n is G_n-parallelizable if and only if $n = 1, 2, 3$. It follows that for $n = 1, 2, 3$ the answer to Problem 2 and, consequently, to Problem 1 is positive.

Originally it was this connection of manifolds M_n with Problems 1 and 2 that was the motivation behind [16].

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