MATH. SCAND. 62 (1988), 217-245

EQUIVARIANTLY PARALLELIZABLE MANIFOLDS

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1. Introduction.

In [10], M. Kervaire has proved that a stably parallelizable closed smooth connected manifold M of dimension $2n$, $n > 0$, is parallelizable if and only if its Euler-Poincaré characteristic $\chi(M) = 0$.

The aim of this paper is to extend Kervaire's result to the case with a finite group action.

Let G be a finite group and let M be a closed G-manifold. $T(M)$ is the tangent bundle of M. We assume that there exists a representation V of G such that

$$T(M) \oplus \mathsf{R}^s \cong M \times (V \oplus \mathsf{R}^s)$$

equivariantly, for some integer s.

Assume furthermore that for all subgroups $H \subset G$ all connected components of M^H are even-dimensional.

We construct an elementary abelian 2-group $C(M, V)$ and an element $\zeta(M, V)$ in $C(M, V)$, both depending only on M and V, and we prove:

THEOREM 2.8. *$T(M) \cong M \times V$ equivariantly if and only if*

(1) $\zeta(M, V) = 0$ *in* $C(M, V)$, *and*

(2) $\chi(L) = 0$ *for any subgroup* $H \subset G$ *and any connected component* L *of* M^H *with* $\dim L > 0$.

Then we examine a series of examples. Let G_k be the elementary abelian 2-group $\mathsf{Z}/2 \times \ldots \times \mathsf{Z}/2$ (k factors). V_k is the real regular representation of G_k. $\tilde{V}_k$ is the orthogonal complement of the trivial summand in V_k. $S(V_k)$ is the unit sphere in V_k and $M_k = S(V_k) \times S(V_k)$. Then

$$T(S(V_k)) \oplus \mathsf{R}^1 \cong S(V_k) \times (\tilde{V}_k \oplus \mathsf{R}^1)$$

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* The author was supported in part by the Swedish Natural Sciences Research Council grant No KTO 511964101-6.

Received July 24, 1985; in revised form April 28, 1987.

and

$$T(M_k) \oplus \mathrm{R}^2 \cong M_k \times (\tilde{V}_k \oplus \tilde{V}_k \oplus \mathrm{R}^2)$$

equivariantly and, moreover, M_k satisfy the condition (2) of Theorem 2.8. We show, however, that M_k and $S(V_k)$ are G_k-parallelizable if and only if $k = 1, 2, 3$. This is a consequence of the fact that $\zeta(M_k, \tilde{V}_k \oplus \tilde{V}_k) \neq 0$ for $k > 3$.

The G_k-manifolds $S(V_k)$ and M_k have some connection with the Hopf invariant one problem and the Kervaire invariant one problem. This is explained in [13].

The paper is organized as follows: in Section 2 we define the group $C(M, V)$ and the element $\zeta(M, V)$ and we formulate the main Theorem 2.8. Section 3 is purely technical. Section 4 contains a proof of the main theorem. In Section 5 we investigate the manifolds M_k.

Throughout the paper G is a finite group. The term "G-manifold" means a compact smooth Riemannian manifold without boundary equipped with a smooth left action of G such that G acts through Riemannian isometries. All G-vector bundles which appear are assumed to be equipped with G-invariant Riemannian metrics. All G-isomorphisms of vector bundles are orthogonal in those metrics. If M is a G-manifold and $H \subset G$ is a subgroup, then

$$M^H = \{x \in M \mid hx = x \quad \text{for} \quad h \in H\}.$$

2. The main theorem.

G is a finite group.

Let V be an orthogonal (finite dimensional) representation of G and let M be a G-manifold.

DEFINITION 2.1. (i) M is *V-parallelizable* if its tangent bundle $T(M)$ and the product bundle $M \times V$ are G-isomorphic as G-vector bundles.

(ii) M is *V-specially stably parallelizable* (we shall abbreviate this to: M is V-ssp) if, for some nonnegative integer s, $T(M) \oplus \mathrm{R}^s$ and $M \times (V \oplus \mathrm{R}^s)$ are G-isomorphic G-vector bundles. Here R^s is the trivial s-dimensional representation of G.

(iii) M is *G-parallelizable* if it is W-parallelizable for some representation W of G.

For the rest of this section M is a V-specially stably parallelizable G-manifold. We define an abelian group $B(M)$ as follows:

Let

$$M_0 = \{x \in M \mid \dim M^{G_x} = 0\}.$$

Here G_x is the isotropy group of the point x. (Observe that, for any subgroup $H \subset G$, all connected components of M^H have the same dimension equal to $\dim V^H$ Hence $M_0 = \{x \in M \mid \dim V^{G_x} = 0\}$). M_0 is an invariant, finite subset of M. We define $B(M)$ as

$$B(M) = \mathrm{Map}(M_0/G, \mathsf{Z}/2),$$

the group of all mappings from the set M_0/G to the group $\mathsf{Z}/2 = \{\pm 1\}$.

We define now a group $A(M, V)$. Let $\mathrm{O}(V \oplus \mathsf{R}^n)$, $n = 0, 1, \dots$ be the orthogonal group of $V \oplus \mathsf{R}^n$. G acts on $\mathrm{O}(V \oplus \mathsf{R}^n)$ by conjugation: $g(f) = gfg^{-1}$ for $f \in \mathrm{O}(V \oplus \mathsf{R}^n)$ and $g \in G$. The embedding of R^n into R^{n+1} on the first n coordinates induces G-embeddings

$$\mathrm{O}(V \oplus \mathsf{R}^n) \subset \mathrm{O}(V \oplus \mathsf{R}^{n+1}).$$

Let

$$\mathrm{O}(V, \infty) = \varinjlim_n \mathrm{O}(V \oplus \mathsf{R}^n)$$

be the limit space with the inductive limit topology. We define $A(M, V)$ as the group of all G-homotopy classes of G-maps from M to $\mathrm{O}(V, \infty)$,

$$A(M, V) = [M, \mathrm{O}(V, \infty)]_G.$$

The group structure of $A(M, V)$ is induced by that of $\mathrm{O}(V, \infty)$.

We define also a group homomorphism $h: A(M, V) \to B(M)$ as follows: Let

$$f: M \to \mathrm{O}(V \oplus \mathsf{R}^n)$$

be a G-map and $x \in M_0$. Then $\dim V^{G_x} = 0$ and $(V \oplus \mathsf{R}^n)^{G_x} = \mathsf{R}^n$. Furthermore, $f(x) \in \mathrm{O}(V \oplus \mathsf{R}^n)^{G_x}$. It follows from the Schur lemma that $f(x)(\mathsf{R}^n) = \mathsf{R}^n$. Observe that the restricted isometry $f(x)|\mathsf{R}^n$ is independent of the choice of x within its orbit, i.e. $f(x)|\mathsf{R}^n = f(gx)|\mathsf{R}^n$ for any $g \in G$. We define $h: A(M, V) \to B(M)$ by

$$h([f])(x) = \det(f(x)|\mathsf{R}^n) \in \{\pm 1\}.$$

Let $\widetilde{\mathrm{KO}}_G^{-1}(\cdot)$ be the (-1)st functor of the reduced equivariant real KO-theory. Let W_{reg} be the real regular representation of G,

$$nW_{\mathrm{reg}} = W_{\mathrm{reg}} \oplus \dots \oplus W_{\mathrm{reg}}$$

(n summands). G acts on the orthogonal group $\mathrm{O}(nW_{\mathrm{reg}})$ by conjugation, $g(f) = gfg^{-1}$ for $f \in \mathrm{O}(nW_{\mathrm{reg}})$, $g \in G$, and in the similar way on

$$\mathrm{O}_G(\infty) = \varinjlim_n \mathrm{O}(nW_{\mathrm{reg}}).$$

Then $\widetilde{\mathrm{KO}}_G^{-1}(M_+) = [M, \mathrm{O}_G(\infty)]_G$.

We choose an orthogonal G-embedding $i: V \hookrightarrow n_0 W_{\mathrm{reg}}$, for some integer n_0, and an orthogonal G-embedding $j: \mathrm{R}^1 \hookrightarrow W_{\mathrm{reg}}$. i and j yield G-embeddings

$$i_n: V \oplus \mathrm{R}^n \hookrightarrow (n_0+n)W_{\mathrm{reg}}, \quad i_n = i \oplus \underbrace{j \oplus \ldots \oplus j}_{n}.$$

i_n induce G-maps

$$\tilde{i}_n: \mathrm{O}(V + \mathrm{R}^n) \to \mathrm{O}((n_0+n)W_{\mathrm{reg}}),$$

$\tilde{i}_n(f)|\mathrm{Im}\, i_n = f$, $\tilde{i}_n(f)|(\mathrm{Im}\, i_n)^{\perp} = \mathrm{id}$ for $f \in \mathrm{O}(V \oplus \mathrm{R}^n)$. The sequence $\{\tilde{i}_n\}$ gives a G-map $i_\infty: \mathrm{O}(V, \infty) \to \mathrm{O}_G(\infty)$, which in turn induces a group homomorphism

$$J: A(M, V) \to \widetilde{\mathrm{KO}}_G^{-1}(M_+).$$

We have also a group homomorphism $H: \widetilde{\mathrm{KO}}_G^{-1}(M_+) \to B(M)$. It is defined as follows: let $k: M \to \mathrm{O}(nW_{\mathrm{reg}})$ be a G-map and $x \in M_0$. Then $k(x) \in \mathrm{O}(nW_{\mathrm{reg}})^{G_x}$ and it follows from the Schur lemma that

$$k(x)((nW_{\mathrm{reg}})^{G_x}) = (nW_{\mathrm{reg}})^{G_x}.$$

If $y \in M_0$, $y = gx$ for some $g \in G$, then

$$g((nW_{\mathrm{reg}})^{G_x}) = (nW_{\mathrm{reg}})^{G_y}$$

and the diagram

$$\begin{array}{ccc} (nW_{\mathrm{reg}})^{G_x} & \xrightarrow{k(x)} & (nW_{\mathrm{reg}})^{G_x} \\ g \downarrow & & g \downarrow \\ (nW_{\mathrm{reg}})^{G_y} & \xrightarrow{k(y)} & (nW_{\mathrm{reg}})^{G_y} \end{array}$$

commutes. It follows that $\det(k(x)|(nW_{\mathrm{reg}})^{G_x}) = \det(k(y)|(nW_{\mathrm{reg}})^{G_y})$. We define the homomorphism

$$H: \widetilde{\mathrm{KO}}_G^{-1}(M_+) \to B(M)$$

by

$$H([k])(x) = \det(k(x)|(nW_{\mathrm{reg}})^{G_x}) \in \{\pm 1\}.$$

H is well defined since $H([k])(x) = H([k])(gx)$ for any $g \in G$.

It follows directly from the definition of the homomorphisms h, J and H that

$$h = H \cdot J. \tag{2.2}$$

Let $\Phi\colon T(M)\oplus \mathsf{R}^s \to M\times(V\oplus \mathsf{R}^s)$ be a G-isomorphism. We shall assign to Φ an element $\alpha(\Phi)\in B(M)$. It is constructed as follows: Let $x\in M_0$. The tangent space of M at x, $T_x(M)$ is a G_x-space and $(T_x(M))^{G_x}=0$.

$$\Phi_x\colon T_x(M)\oplus \mathsf{R}^s \to V\oplus \mathsf{R}^s$$

is a G_x-isomorphism, $V^{G_x}=0$. It follows that $\Phi_x(\mathsf{R}^s)=\mathsf{R}^s$. We define a map $\bar\alpha(\Phi)\colon M_0\to O(\mathsf{R}^s)$ by $\bar\alpha(\Phi)(x)=(\Phi_x|\mathsf{R}^s)$. Observe that $\bar\alpha(\Phi)(x)=\bar\alpha(\Phi)(gx)$ for any $g\in G$. Finally we define $\alpha(\Phi)\colon M_0/G\to \mathsf{Z}/2$ by

$$\alpha(\Phi)[x]=\det(\bar\alpha(\Phi)(x)). \tag{2.3}$$

Of course, $\alpha(\Phi)$ depends only on the stable class of Φ, that is $\alpha(\Phi)=\alpha(\Phi\oplus \mathrm{id}_{\mathsf{R}^n})$, and if

$$\Phi_1,\Phi_2\colon T(M)\oplus \mathsf{R}^s\to M\times(V\oplus \mathsf{R}^s)$$

are G-homotopic, then $\alpha(\Phi_1)=\alpha(\Phi_2)$.

Every pair of G-isomorphisms $\Phi_1,\Phi_2\colon T(M)\oplus \mathsf{R}^s\to M\times(V\oplus \mathsf{R}^s)$ determines an element $\gamma(\Phi_1,\Phi_2)\in A(M,V)$. $\gamma(\Phi_1,\Phi_2)$ is the G-homotopy class of the map

$$\bar\gamma(\Phi_1,\Phi_2)\colon M\to O(V\oplus \mathsf{R}^s),\quad \bar\gamma(\Phi_1,\Phi_2)(x)=\Phi_{2,x}\circ\Phi_{1,x}^{-1}.$$

Lemma 2.4. *If $\Phi_1,\Phi_2\colon T(M)\oplus \mathsf{R}^s\to M\times(V\oplus \mathsf{R}^s)$ are two G-isomorphisms of vector bundles, then*

$$\alpha(\Phi_2)=\alpha(\Phi_1)+h(\gamma(\Phi_1,\Phi_2))$$

in $B(M)$.

Proof. $\alpha(\Phi_1)+h(\gamma(\Phi_1,\Phi_2))$ is represented by a map $f\colon M_0/G\to \mathsf{Z}/2$,

$$\begin{aligned} f([x]) &= \det(\bar\alpha(\Phi_1)(x))\cdot\det((\Phi_{2,x}\circ\Phi_{1,x}^{-1})|\mathsf{R}^s)\\ &= \det(((\Phi_{2,x}\circ\Phi_{1,x}^{-1})|\mathsf{R}^s)\circ(\bar\alpha(\Phi_1)(x)))\\ &= \det(((\Phi_{2,x}\circ\Phi_{1,x}^{-1})|\mathsf{R}^s)\circ(\Phi_{1,x}|\mathsf{R}^s)) = \det(\Phi_{2,x}|\mathsf{R}^s)=\det(\bar\alpha(\Phi_2)(x))\\ &= \alpha(\Phi_2)[x]\quad\text{for}\quad x\in M_0. \end{aligned}$$

Thus $\alpha(\Phi_2)=\alpha(\Phi_1)+h(\gamma(\Phi_1,\Phi_2))$.

Lemma 2.5. *Let $\Phi_1\colon T(M)\oplus \mathsf{R}^s\to M\times(V\oplus \mathsf{R}^s)$ be a G-isomorphism and $\omega\in A(M,V)$. There exists an integer n and a G-isomorphism*

$$\Phi_2\colon T(M)\oplus \mathsf{R}^n\to M\times(V\oplus \mathsf{R}^n)$$

such that $\alpha(\Phi_2)=\alpha(\Phi_1)+h(\omega)$ in $B(M)$.

Proof. Let ω be represented by a G-map $f\colon M\to O(V\oplus \mathsf{R}^t)$. We set

$n = \max(s, t)$. We may assume that $s = t = n$. Let

$$\Phi_2 : T(M) \oplus \mathsf{R}^n \to M \times (V \oplus \mathsf{R}^n)$$

be defined by $\Phi_{2,x} = f(x) \circ \Phi_{1,x}$ for $x \in M$. Then $\gamma(\Phi_1, \Phi_2) = \omega$. Lemma 2.5 follows now from (2.4).

DEFINITION 2.6. If M is a V-specially stably parallelizable manifold, we define an abelian group $C(M, V)$ and an element $\zeta(M, V) \in C(M, V)$. $C(M, V)$ is the quotient Coker(h). $\zeta(M, V)$ is the image of $\alpha(\Phi) \in B(M)$, where

$$\Phi : T(M) \oplus \mathsf{R}^s \to M \times (V \oplus \mathsf{R}^s)$$

is a G-isomorphism of vector bundles, in Coker (h).

REMARK 2.7. (i) It follows from (2.4) that $\zeta(M, V)$ does not depend on the choice of Φ.

(ii) If $\zeta(M, V) = 0$, then there exists a G-isomorphism

$$\Phi : T(M) \oplus \mathsf{R}^s \to M \times (V \oplus \mathsf{R}^s)$$

with $\alpha(\Phi) = 0$. Indeed, if $\zeta(M, V) = 0$, then there exist $\beta \in A(M, V)$ and

$$\Phi_1 : T(M) \oplus \mathsf{R}^s \to M \times (V \oplus \mathsf{R}^s)$$

such that $\alpha(\Phi_1) = h(\beta)$. Apply now (2.5) with $\omega = -\beta$.

(iii) If M is V-parallelizable, then $\zeta(M, V) = 0$. Indeed, let $\Psi : T(M) \to M \times V$ be a G-isomorphism of vector bundles. By the definition, $\alpha(\Psi) = 0$. Consequently, $\zeta(M, V) = 0$ as well.

Let us observe that if M is a V-ssp manifold then, for every subgroup $H \subset G$, M^H is a stably parallelizable manifold.

The following is our equivariant generalization of the Kervaire theorem ([10], [7; Theorem 2], [15; Section 3]):

THEOREM 2.8. *Let M be a V-specially stably parallelizable manifold such that*

(i) *for every subgroup $H \subset G$, $\dim M^H$ is even or M^H is empty,*

(ii) *if $H \subset G$ is such a subgroup that $\dim M^H > 0$ and L is a connected component of M^H, then $\chi(L) = 0$,*

(iii) *$\zeta(M, V) = 0$ in $C(M, V)$.*

Then M is V-parallelizable.

Theorem 2.8 will be proved in Section 4. (2.8) (ii) and (2.8) (iii) are also necessary conditions for M to be V-parallelizable.

3. *G*-fibrations.

G is a finite group.

DEFINITION 3.1. Let E and B be G-spaces and let $p: E \to B$ be a G-map. p is said to be a G-fibration if

(i) for any G-CW-pair (X, A), G-homotopy $F_t: X \to B$, G-map $H: X \to E$, and G-homotopy $h_t: A \to E$ such that $p \circ h_t = F_t|_A$ for $t \in I$, $h_0 = H|_A$ and $p \circ H = F_0$, there exists a G-homotopy $H_t: X \to E$ such that $h_t = H_t|_A$, $p \circ H_t = F_t$ for $t \in I$ and $H_0 = H$,

(ii) for every subgroup $H \subset G$, $p^H: E^H \to B^H$ is surjective.

If p satisfies the property (3.1) (i) for a fixed G-CW-pair (X, A), we say that p has the G-covering homotopy extension property with respect to (X, A).

REMARK 3.2. If Z is any space, Y is a G-space and $H \subset G$ is a subgroup, then there is a bijective correspondence between maps $f: Z \to Y^H$ and G-maps $\bar{f}$: $(G/H) \times Z \to Y$, where G acts on $(G/H) \times Z$ via trivial action on Z and the left multiplication on G/H. Indeed, the correspondence is given by $\bar{f}(gH, z) = gf(z)$. Compare [12; (2.1)].

LEMMA 3.3. *$p: E \to B$ is a G-fibration if and only if $p^H: E^H \to B^H$ is a Serre fibration for every subgroup $H \subset G$.*

PROOF. Let $p: E \to B$ be a G-fibration, $H \subset G$ be a subgroup and (Z, Y) be a CW-pair. The covering homotopy extensions property of $p^H: E^H \to B^H$ with respect to (Z, Y) follows from the G-covering homotopy extension property of p with respect to the G-CW-complex $((G/H) \times Z, (G/H) \times Y)$. See (3.2).

Suppose now that, for all subgroups $H \subset G$, $p^H: E^H \to B^H$ are Serre fibrations. In order to prove that p satisfies the requirements of (3.1) it is enough to show that p satisfies them in the case when $X = A \cup_f ((G/H) \times D^n)$ and $f: (G/H) \times S^{n-1} \to A$ is a G-map. This in turn would follow from (3.1) being satisfied for $X = (G/H) \times D^n$, $A = (G/H) \times S^{n-1}$. But the G-covering homotopy extension property of p with respect to the pair $((G/H) \times D^n, (G/H) \times S^{n-1})$ follows from the covering homotopy extension property of p^H with respect to the pair (D^n, S^{n-1}). See (3.2).

If $p: E \to B$ is a G-fibration and $b \in B$, then the fibre of p over b, $p^{-1}(b)$ is a G_b-space. If $H \subset G$ is a subgroup and $b \in B^H$, then $(p^{-1}(b))^H$ is the fibre over b of the fibration $p^H: E^H \to B^H$.

Let $\xi_i: E_i \to X$, $i = 1, 2$, be G-vector bundles over a G-CW-complex X. Suppose that there exists a G-isomorphism $\Phi: \xi_1 \to \xi_2$.

We have a G-vector bundle $\operatorname{Hom}(\xi_1, \xi_2)$ over X, see [2; Sections 1.2 and 1.6]. If $h \in \operatorname{Hom}(\xi_1, \xi_2)$, $h: \xi_1^{-1}(x) \to \xi_2^{-1}(x)$, $x \in X$ and $g \in G$, then $g(h) \in \operatorname{Hom}(\xi_1^{-1}(gx), \xi_2^{-1}(gx))$ is given by $g(h) = g \circ h \circ g^{-1}$.

The G-vector bundle $\operatorname{Hom}(\xi_1, \xi_2)$ contains a nonlinear G-subbundle

$$\operatorname{Iso}(\xi_1, \xi_2) = \{h \in \operatorname{Hom}(\xi_1, \xi_2) \mid h \text{ is an orthogonal isomorphism}\}.$$

We denote by $I: \operatorname{Iso}(\xi_1, \xi_2) \to X$ the bundle projection.

LEMMA 3.4. *$I: \operatorname{Iso}(\xi_1, \xi_2) \to X$ is a G-fibration.*

PROOF. Since there exists a G-isomorphism $\Phi: \xi_1 \to \xi_2$, (3.1) (ii) is satisfied. Let $H \subset G$ be a subgroup. According to (3.3) it is enough to prove that

$$I^H: \operatorname{Iso}(\xi_1, \xi_2)^H \to X^H$$

is a Serre fibration. We have

$$\operatorname{Iso}(\xi_1, \xi_2)^H = \operatorname{Iso}(\xi_1|_{X^H}, \xi_2|_{X^H})^H.$$

For every $x \in X^H$ there is a neighbourhood $U \subset X^H$ such that $\xi_i|_U$ is H-isomorphic to $U \times (\xi_i^{-1}(x))$. Indeed, for complex H-vector bundles this fact is a consequence of [2; Proposition 1.6.2]. If ζ is a real H-vector bundle over X^H, we choose a neighbourhood $x \in U_1 \subset X^H$ and an H-isomorphism

$$\varphi: \zeta \otimes \mathbf{C}|_{U_1} \to U_1 \times (\zeta^{-1}(x) \otimes \mathbf{C})$$

such that $\varphi|_{\zeta^{-1}(x) \otimes \mathbf{C}}$ is the identity. Let $i: \zeta \to \zeta \otimes \mathbf{C}$ be the embedding corresponding to the embedding $\mathbf{R} \hookrightarrow \mathbf{C}$ and let

$$\operatorname{pr}: U_1 \times (\zeta^{-1}(x) \otimes \mathbf{C}) \to U_1 \times \zeta^{-1}(x)$$

be the projection corresponding to the projection $\alpha: \mathbf{C} \to \mathbf{R}$, $\alpha(z) = \frac{1}{2}(z + \bar{z})$. The composition $\psi = \operatorname{pr} \circ \varphi \circ (i|_{U_1})$ is an H-vector bundle homomorphism and it is the identity on the fibre $\zeta^{-1}(x)$. Hence there exists a neighbourhood U of x in X^H such that $\psi|_U: \zeta|_U \to U \times \zeta^{-1}(x)$ is an H-isomorphism. (Compare [6; p. 15].) Observe that we may assume that $\psi|_U$ is orthogonal on fibres. If not, we can apply the Gramm-Schmidt orthogonalization procedure.

It follows that

$$\operatorname{Iso}(\xi_1|_U, \xi_2|_U)^H \cong U \times (\operatorname{Iso}(\xi_1^{-1}(x), \xi_2^{-1}(x))^H).$$

Consequently $I^H: \operatorname{Iso}(\xi_1, \xi_2)^H \to X^H$ is locally a product bundle and, therefore, a Serre fibration.

Let V_1 and V_2 be two orthogonal representations of G. We shall denote by $\operatorname{St}(V_1, V_2)$ the space of all linear isometric embeddings of V_1 into V_2 (that is $\operatorname{St}(V_1, V_2)$ is homeomorphic to the Stiefel variety of orthonormal n-frames in $\mathbf{R}^m$, where $n = \dim V_1$ and $m = \dim V_2$). $\mathrm{St}(V_1, V_2)$ is a G-space with the action of G given by $g(h) = ghg^{-1}$ for $h \in \mathrm{St}(V_1, V_2)$. If $V_1 = V_2$, then $\mathrm{St}(V_1, V_1)$ is the group $\mathrm{O}(V_1)$, i.e. the group of all orthogonal automorphisms of V_1 with the usual action of G.

Let $i_2 : V_2 \to V_1 \oplus V_2$ be the embedding of the second summand. There are two G-maps

$$\mathrm{O}(V_1) \xrightarrow{j} \mathrm{O}(V_1 \oplus V_2) \xrightarrow{\pi} \mathrm{St}(V_2, V_1 \oplus V_2), \tag{3.5}$$

j is a G-embedding given by $j(f) = f \oplus \mathrm{id}_{V_2}$ for $f \in \mathrm{O}(V_1)$. The projection π is defined by $\pi(h) = h \circ i_2$ for $h \in \mathrm{O}(V_1 \oplus V_2)$.

LEMMA 3.6. *For every subgroup $H \subset G$*

$$\mathrm{O}(V_1)^H \xrightarrow{j^H} \mathrm{O}(V_1 \oplus V_2)^H \xrightarrow{\pi^H} \mathrm{St}(V_2, V_1 \oplus V_2)^H$$

is a Serre fibration.

PROOF. $\mathrm{O}(V_1 \oplus V_2)^H$ is a closed subgroup of the compact Lie group $\mathrm{O}(V_1 \oplus V_2)$. Hence $\mathrm{O}(V_1 \oplus V_2)^H$ is itself a compact Lie group. Similarly $\mathrm{O}(V_1)^H$ is a closed subgroup of the compact Lie group $\mathrm{O}(V_1)$ and, consequently, itself a compact Lie group.

$$j^H : \mathrm{O}(V_1)^H \to \mathrm{O}(V_1 \oplus V_2)^H$$

is an embedding onto a closed Lie subgroup. Therefore the sequence

$$\mathrm{O}(V_1)^H \xrightarrow{j^H} \mathrm{O}(V_1 \oplus V_2)^H \to \mathrm{O}(V_1 \oplus V_2)^H/\mathrm{O}(V_1)^H \tag{3.7}$$

is a Serre fibration. (3.7) is homeomorphic to the sequence

$$\mathrm{O}(V_1)^H \xrightarrow{j^H} \mathrm{O}(V_1 \oplus V_2)^H \xrightarrow{\pi^H} \mathrm{St}(V_2, V_1 \oplus V_2)^H. \tag{3.8}$$

Consequently, (3.8) is a Serre fibration.

COROLLARY 3.9. *The sequence of G-maps*

$$\mathrm{O}(V_1) \xrightarrow{j} \mathrm{O}(V_1 \oplus V_2) \xrightarrow{\pi} \mathrm{St}(V_2, V_1 \oplus V_2)$$

is a G-fibration.

Let us now consider a case when $V_2 = \mathbf{R}$ is the 1-dimensional trivial representation of G. Let $S(V_1 \oplus \mathbf{R})$ be the unit sphere in $V_1 \oplus \mathbf{R}$. Then there is a G-homeomorphism

$$\varphi : \mathrm{St}(\mathbf{R}, V_1 \oplus \mathbf{R}) \xrightarrow{\cong} S(V_1 \oplus \mathbf{R}), \quad \varphi(h) = h(1) \quad \text{for} \quad h \in \mathrm{St}(\mathbf{R}, V_1 \oplus \mathbf{R}),$$

and upon this identification (3.5) takes form

$$\text{(3.10)} \qquad O(V_1) \xrightarrow{j} O(V_1 \oplus \mathsf{R}) \xrightarrow{\pi} S(V_1 \oplus \mathsf{R}),$$

with $\pi(f) = f(0 \oplus 1)$.

Setting $V = V_1$ we obtain

COROLLARY 3.11. *For any orthogonal representation V of G and any subgroup $H \subset G$*

(i) $O(V) \xrightarrow{j} O(V \oplus \mathsf{R}) \xrightarrow{\pi} S(V \oplus \mathsf{R})$ *is a G-fibration,*

(ii) $$\pi_n(O(V \oplus \mathsf{R})^H, O(V)^H) = \begin{cases} 0 & \text{if } n < \dim_{\mathsf{R}} V^H \\ \mathsf{Z} & \text{if } n = \dim_{\mathsf{R}} V^H. \end{cases}$$

PROOF. (i) is a special case of Corollary (3.9).

(ii) According to Lemma 3.6

$$O(V)^H \xrightarrow{j^H} O(V \oplus \mathsf{R})^H \xrightarrow{\pi^H} S(V \oplus \mathsf{R})^H$$

is a Serre fibration. Thus

$$\pi_n(O(V \oplus \mathsf{R})^H, O(V)^H \cong \pi_n(S(V \oplus \mathsf{R})^H, 0 \oplus 1) = \pi_n(S(V^H \oplus \mathsf{R}), 0 \oplus 1)$$

and (3.11) (ii) follows.

4. Proof of Theorem 2.8.

If (X, A) is a CW-pair, then $\dim(X, A)$ is the supremum of the dimensions of cells in $X \setminus A$. Let (X, A) be a G-CW-pair and let $\xi : E \to X$ be a G-vector bundle over X.

DEFINITION 4.1. ξ satisfies the (GD)-condition over (X, A) if for every subgroup $H \subset G$ and every connected component $(X^H)_i$ of the subcomplex X^H,

$$\dim(\xi^H|_{(X^H)_i}) > \dim((X^H)_i, \quad (X^H)_i \cap A).$$

Here $\dim(\xi^H|_{(X^H)_i})$ is the dimension of the fibre of ξ^H over any of the points of $(X^H)_i$.

PROPOSITION 4.2. *Let $\xi_i : E_i \to X$, $i = 1, 2$, be two G-vector bundles over a G-CW-pair (X, A), both satisfying the (GD)-condition over (X, A). Let $\bar{\Psi} : \xi_1|_A \to \xi_2|_A$ be a G-isomorphism of vector bundles over A and $\Phi : \xi_1 \oplus \mathsf{R} \to \xi_2 \oplus \mathsf{R}$ a G-isomorphism of vector bundles over X such that $\Phi|_A = \bar{\Psi} \oplus \mathrm{id}_{\mathsf{R}}$. Then there exists a G-isomorphism $\Psi : \xi_1 \to \xi_2$ of vector bundles over X and a G-homotopy $F_t : \xi_1 \oplus \mathsf{R} \to \xi_2 \oplus \mathsf{R}$, $t \in I$, such that*

(i) $F_t: \xi_1 \oplus \mathsf{R} \to \xi_2 \oplus \mathsf{R}$ *is a G-isomorphism of vector bundles over X for every* $t \in I$,

(ii) $F_0 = \Phi$ *and* $F_1 = \Psi \oplus \mathrm{id}_{\mathsf{R}}$,

(iii) $F_t|_A = \bar{\Psi} \oplus \mathrm{id}_{\mathsf{R}}$ *for every* $t \in I$.

In particular $\Psi|_A = \bar{\Psi}$.

PROOF. We prove (4.2) by induction on $\dim(X, A)$.

If $\dim(X, A) = 0$, it is enough to prove the absolute version of (4.2) for every orbit of G in $X \setminus A$ separately. We can hence assume that $X = G/H$ for some subgroup $H \subset G$ and that A is empty. In such a case, the bundles ξ_i correspond to two representations V_i, $i = 1, 2$, of the subgroup H. Existence of Φ implies that $V_1 \oplus \mathsf{R}$ and $V_2 \oplus \mathsf{R}$ are isomorphic representations of H. Consequently, V_1 and V_2 are isomorphic representations and we can identify them with a representation V of H, $V_1 \cong V \cong V_2$. Given the identification, Φ corresponds to an element $\tilde{\Phi} \in \mathrm{O}(V \oplus \mathsf{R})^H$. Since ξ_i satisfy the (GD)-condition over $G/H, \dim_{\mathsf{R}} V^H > 0$. The unit sphere $S(V \oplus \mathsf{R})^H$ is connected. It follows from (3.11) (i) that

$$j^H_\# : \pi_0(\mathrm{O}(V)^H) \to \pi_0(\mathrm{O}(V \oplus \mathsf{R})^H)$$

is surjective. Let $\tilde{\Psi} \in \mathrm{O}(V)^H$ be such that $j^H_\#([\tilde{\Psi}]) = [\tilde{\Phi}]$. Here $[\tilde{\Phi}]$ and $[\tilde{\Psi}]$ denote the connected components of $\tilde{\Phi}$ in $\mathrm{O}(V \oplus \mathsf{R})^H$ and of $\tilde{\Psi}$ in $\mathrm{O}(V)^H$, respectively. $\tilde{\Psi}$ corresponds to a G-isomorphism $\Psi: \xi_1 \to \xi_2$ over G/H and it follows from our choice of $\tilde{\Psi}$ that

$$\Psi \oplus \mathrm{id}_{\mathsf{R}} : \xi_1 \oplus \mathsf{R} \to \xi_2 \oplus \mathsf{R}$$

is G-homotopic to Φ. This proves (4.2) if $\dim(X, A) = 0$.

We assume now that (4.2) holds for G-CW-pairs (X, A) with $\dim(X, A) < k$. Let (X, A) be a G-CW-pair, $\dim(X, A) = k$, and let $X^{(k-1)}$ be the $(k-1)$st skeleton of X. Denote $Y = X^{(k-1)} \cup A$. (4.2) holds for the G-pair (Y, A). We apply (4.2) to the G-isomorphism

$$\Phi' = \Phi|_Y : \xi_1 \oplus \mathsf{R}|_Y \to \xi_2 \oplus \mathsf{R}|_Y.$$

Let $\Psi' : \xi_1|_Y \to \xi_2|_Y$ be a G-isomorphism and

$$F'_t : \xi_1 \oplus \mathsf{R}|_Y \to \xi_2 \oplus \mathsf{R}|_Y, \quad t \in I,$$

a G-homotopy such that $F'_0 = \Phi'$, $F'_1 = \Psi' \oplus \mathrm{id}_{\mathsf{R}}$ and $F'_t|_A = \Phi'|_A$ for $t \in I$. It follows from (3.4) that the G-homotopy F'_t can be extended over X to a G-homotopy $K_t : \xi_1 \oplus \mathsf{R} \to \xi_2 \oplus \mathsf{R}$ such that $K_t|_Y = F'_t$ for $t \in I$ and $K_0 = \Phi$. It is now enough to prove (4.2) for the G-isomorphism $K_1 : \xi_1 \oplus \mathsf{R} \to \xi_2 \oplus \mathsf{R}$ instead of Φ.

We have $K_1|_Y = F'_1 = \Psi' \oplus \mathrm{id}_{\mathsf{R}}$. It is therefore necessary to deform K_1

only over the interior of top dimensional cells of (X, A). That can be done at each G-cell separately. We can thus assume that $X = (G/H) \times D^k$ and $A = (G/H) \times S^{k-1}$, where $H \subset G$ is a subgroup, D^k is a k-dimensional disc and $\partial D^k = S^{k-1}$. In this case $K_1 = \Phi$. Let $p_0 \in S^{k-1}$ and let

$$x_0 = (eH, p_0) \in (G/H) \times S^{k-1} = A.$$

x_0 is a fixed point of H. $\xi_i^{-1}(x_0)$, $i = 1, 2$, are orthogonal representations of H. Existence of Φ implies that they are isomorphic. We denote one of them, say $\xi_1^{-1}(x_0)$, by V. Let H act trivially on D^k. Then $(G/H) \times D^k$ can be identified with $G \times_H D^k$. Furthermore, both bundles ξ_1 and ξ_2 can be identified with the G-vector bundle

$$\xi : G \times_H (D^k \times V) \to G \times_H D^k.$$

Here H acts on $D^k \times V$ via the second factor.

Let $\tilde{\xi} : D^k \times V \to D^k$ be the product H-vector bundle.

Given all these identifications, the G-isomorphism Φ corresponds to an H-isomorphism of H-vector bundles $\tilde{\Phi} : \tilde{\xi} \oplus \mathsf{R} \to \tilde{\xi} \oplus \mathsf{R}$ over D^k. $\tilde{\Phi}$ is given by a map $\hat{\Phi} : D^k \to \mathrm{O}(V \oplus \mathsf{R})^H$. Since $\Phi|_A = \Psi \oplus \mathrm{id}_{\mathsf{R}}$, we have that

$$\hat{\Phi}(S^{k-1}) \subset j^H(\mathrm{O}(V)^H).$$

Let us consider the homotopy class

$$[\hat{\Phi}] \in \pi_k(\mathrm{O}(V \oplus \mathsf{R})^H, \mathrm{O}(V)^H).$$

The bundles ξ_i satisfy the (GD)-condition over (X, A) and, consequently, $\dim_{\mathsf{R}} V^H > k$. Therefore, according to (3.11) (ii),

$$\pi_k(\mathrm{O}(V \oplus \mathsf{R})^H, \mathrm{O}(V)^H) = 0.$$

It follows that $\hat{\Phi}$ is homotopic rel. S^{k-1} to a map $\hat{\Psi} : D^k \to \mathrm{O}(V)^H$. This in turn implies that there exists an H-homotopy

$$\tilde{F}_t : \tilde{\xi} \oplus \mathsf{R} \to \tilde{\xi} \oplus \mathsf{R}, \quad t \in I,$$

over D^k such that $\tilde{F}_t$ are H-isomorphisms of H-vectors bundles and

$$\tilde{F}_t|_{S^{k-1}} = \tilde{\Phi}|_{S^{k-1}} \quad \text{for all } t \in I,$$

$\tilde{F}_0 = \tilde{\Phi}$ and $\tilde{F}_1 = \tilde{\Psi} \oplus \mathrm{id}_{\mathsf{R}}$ for some H-isomorphism $\tilde{\Psi} : \tilde{\xi} \to \tilde{\xi}$.

It is now enough to define the G-homotopy F_t as

$$F_t = G \times_H (\tilde{F}_t) : \xi \oplus \mathsf{R} \to \xi \oplus \mathsf{R}$$

and the G-isomorphism Ψ as $\Psi = G \times_H (\tilde{\Psi}) : \xi \to \xi$. This completes the proof of Proposition 4.2.

Let M be a G-manifold and V be an orthogonal representation of G.

COROLLARY 4.3. *If $\Phi: T(M) \oplus \mathsf{R}^s \to M \times (V \oplus \mathsf{R}^s)$, $s \geqq 1$, is a G-isomorphism of G-vector bundles over M, then there exists a G-isomorphism*

$$\Psi: T(M) \oplus \mathsf{R}^1 \to M \times (V \oplus \mathsf{R}^1)$$

such that Φ and $\Psi \oplus \mathrm{id}_{\mathsf{R}^{s-1}}$ are G-homotopic.

PROOF. If $s > 1$ then $T(M) \oplus \mathsf{R}^{s-1}$ (and, consequently, $M \times (V \oplus \mathsf{R}^{s-1})$) satisfies the (GD)-condition over (M, ϕ). It follows from (4.2) that there exists a G-isomorphism

$$\Phi': T(M) \oplus \mathsf{R}^{s-1} \to M \times (V \oplus \mathsf{R}^{s-1})$$

such that Φ and $\Phi' \oplus \mathrm{id}_{\mathsf{R}^1}$ are G-homotopic. In the next step we apply this reduction process to Φ' and continue until $s = 1$.

The proof of Theorem 2.8 will go along the lines of the Bredon-Kosiński proof, [7], of the corresponding case (without any group action) of the Kervaire theorem.

Let M be a G-specially stably parallelizable manifold. Let V be an orthogonal representation of G and $\Phi: T(M) \oplus \mathsf{R}^1 \to M \times (V \oplus \mathsf{R}^1)$ a G-isomorphism of vector bundles over M. We denote by $\varepsilon: M \to T(M) \oplus \mathsf{R}^1$ the cross section of $T(M) \oplus \mathsf{R}^1$ given by $1 \in \mathsf{R}^1$. Let

$$p_2: M \times (V \oplus \mathsf{R}^1) \to V \oplus \mathsf{R}^1$$

be the projection on the second factor. The Gauss map of the trivialization Φ is the G-map

$$v_\Phi: M \to S(V \oplus \mathsf{R}^1)$$

defined by $v_\Phi(x) = p_2 \Phi \varepsilon(x)$, compare [7; p. 86].

Observe that if $H \subset G$ is a subgroup, then $T(M)^H = T(M^H)$,

$$\Phi^H: T(M^H) \oplus \mathsf{R}^1 \to M^H \times (V^H \oplus \mathsf{R}^1)$$

is a stable trivialization of $T(M^H)$ and $(v_\Phi)^H: M^H \to S(V^H \oplus \mathsf{R}^1)$ is the Gauss map of Φ^H,

$$(v_\Phi)^H = v_{\Phi^H}. \tag{4.4}$$

We choose $z_0 \in S(V \oplus \mathsf{R}^1)^G$, $z_0 = 0 \oplus 1$ as a base point of $S(V \oplus \mathsf{R}^1)$. Just as in the non-equivariant case we have

LEMMA 4.5. *Φ is G-homotopic to $\Psi \oplus \mathrm{id}_{\mathsf{R}}: T(M) \oplus \mathsf{R}^1 \to M \times (V \oplus \mathsf{R}^1)$ for some G-isomorphism $\Psi: T(M) \to M \times V$ if and only if v_Φ is G-homotopically contractible to the base point $z_0 \in S(V \oplus \mathsf{R}^1)$.*

Compare [7; 4.1].

PROOF. If Φ is G-homotopic to $\Psi \oplus \mathrm{id}_{\mathsf{R}}$ then ν_Φ is G-homotopic to $\nu_{\Psi \oplus \mathrm{id}}$. But $\nu_{\Psi \oplus \mathrm{id}}$ maps whole M onto the point $z_0 \in S(V \oplus \mathsf{R}^1)$.

Let us assume now that ν_Φ is G-contractible to z_0. Let

$$f_t : M \to S(V \oplus \mathsf{R}^1), \quad t \in I,$$

be a G-homotopy such that $f_0 = \nu_\Phi$, $f_1(M) = \{z_0\}$. Let

$$\pi : \mathrm{O}(V \oplus \mathsf{R}^1) \to S(V \oplus \mathsf{R}^1)$$

be the map from (3.5) and (3.11), i.e. $\pi(h) = h(z_0)$ for $h \in \mathrm{O}(V \oplus \mathsf{R}^1)$. Let

$$y_0 = \mathrm{id}_{V \oplus \mathsf{R}^1} \in \mathrm{O}(V \oplus \mathsf{R}^1)^G.$$

We have $\pi(y_0) = z_0$ and we define $g : M \to \mathrm{O}(V \oplus \mathsf{R}^1)$ by $g(x) = y_0$ for all $x \in M$. Since, according to (3.11), π is a G-fibration, there exists a G-homotopy $g_t : M \to \mathrm{O}(V \oplus \mathsf{R}^1)$, $t \in I$, such that $g_1 = g$ and $f_t = \pi \circ g_t$. This homotopy yields a G-homotopy of G-vector bundles isomorphisms

$$G_t : M \times (V \oplus \mathsf{R}^1) \to M \times (V \oplus \mathsf{R}^1), \quad t \in I,$$

given by $G_t(x, w) = (x, g_t(x)(w))$ for $x \in M, w \in V \oplus \mathsf{R}^1$. Let

$$F_t : T(M) \oplus \mathsf{R}^1 \to M \times (V \oplus \mathsf{R}^1), \quad t \in I,$$

be defined by $F_t = G_t^{-1} \circ \Phi$. F_t is a G-homotopy of G-vector bundle isomorphisms over M. Since

$$G_1 = \mathrm{id}_{M \times (V \oplus \mathsf{R}^1)},$$

$F_1 = \Phi$. Let us consider F_0. The Gauss map ν_{F_0} of F_0 satisfies

$$\begin{aligned} \nu_{F_0}(x) &= p_2 F_0 \Phi \varepsilon(x) = p_2 G_0^{-1} \Phi \varepsilon(x) = (g_0(x))^{-1}(p_2 \Phi \varepsilon(x)) \\ &= (g_0(x))^{-1}(\nu_\Phi(x)) \quad \text{for } x \in M. \end{aligned}$$

We have also

$$g_0(x)(z_0) = \pi g_0(x) = f_0(x) = \nu_\Phi(x).$$

Consequently, $\nu_{F_0}(x) = z_0$ for all $x \in M$. It follows that there exists a G-isomorphism $\Psi : T(M) \to M \times V$ such that $F_0 = \Psi \oplus \mathrm{id}_{\mathsf{R}}$. This proves (4.5).

We shall now recall some results which allow us to determine the G-homotopy class of the Gauss map ν_Φ.

Let M be a G-manifold, $m = \dim M$. We assume that every connected component of M is orientable (in the ordinary sense) and that M is G-connected, i.e. M/G is connected.

We shall define an orientation homomorphism of the action of G on M. Let $\{M_j\}_{j\in J}$ be the set of all connected components M_j of M. G acts transitively on J. We denote by G_j the subgroup $\{g\in G|g(M_j)=M_j\}$ of G. Let $[M_j]\in H_m(M_j, \mathbf{Z})$ be an orientation class of M_j. We choose $j_0\in J$ and define the orientation homomorphism of the action of G on M,

$$\omega_{M,G}: G_{j_0}\to \mathbf{Z}/2\mathbf{Z} \quad \text{by} \quad g_*[M_{j_0}] = (-1)^{\omega_{M,G}(g)}[M_{j_0}] \quad \text{for} \quad g\in G_{j_0}.$$

The orientation homomorphism $\omega_{M,G}$ depends on the choice of $j_0\in J$. However, if we choose another $j_1\in J$ and define an orientation homomorphism $\omega'_{M,G}: G_{j_1}\to \mathbf{Z}/2\mathbf{Z}$ for this choice of component, then there exists an isomorphism of groups $\alpha: G_{j_1}\to G_{j_0}$ such that $\omega_{M,G}\cdot\alpha = \omega'_{M,G}$.

Let $\varrho: G\to \mathrm{O}(V)$ be an orthogonal representation of G. We define an orientation homomorphism of ϱ,

$$\omega_{\varrho,G}: G\to \mathbf{Z}/2\mathbf{Z} \quad \text{by} \quad \det(\varrho(g)) = (-1)^{\omega_{\varrho,G}(g)} \quad \text{for} \quad g\in G.$$

We are going to write $\omega_{V,G}$ instead of $\omega_{\varrho,G}$ if it is clear which representation ϱ on the space V we mean.

Observe that if $\dim V > 1$, then $\omega_{V,G} = \omega_{S(V),G}$.

LEMMA 4.6. *If V is an orthogonal representation of G and*

$$\Phi: T(M)\oplus \mathbf{R}^s \to M\times (V\oplus \mathbf{R}^s)$$

is a G-isomorphism of vector bundles over M, then $\omega_{M,G} = \omega_{V,G}|_{G_{j_0}}$.

PROOF. Let D_1 be the unit disc bundle of $T(M_{j_0})\oplus \mathbf{R}^s$ and let D_2 be the unit disc bundle of $M_{j_0}\times (V\oplus \mathbf{R}^s)$. Then $\Phi(D_1) = D_2$. D_1 is orientable as a manifold with boundary. If $f: M_{j_0}\to M_{j_0}$ is an isometry, then the differential

$$T(f)\oplus \mathrm{id}_{\mathbf{R}^s}: T(M_{j_0})\oplus \mathbf{R}^s \to T(M_{j_0})\oplus \mathbf{R}^s$$

satisfies

$$(T(f)\oplus \mathrm{id}_{\mathbf{R}^s})(D_1) = D_1$$

and $T(f)\oplus \mathrm{id}_{\mathbf{R}^s}$ preserves an orientation of D_1, see [1; 14.15 and 14.16]. In particular, the induced action of G_{j_0} on D_1 preserves orientation. It follows that the action of G_{j_0} on D_2 also preserves orientation. However, if $[D_2]\in H_{2m+s}(D_2, \partial D_2, \mathbf{Z})$ is an orientation class of D_2 and $g\in G_{j_0}$ then

$$g_*[D_2] = (-1)^{\omega_{M,G}(g)+\omega_{V,G}(g)}[D_2].$$

We have therefore that $\omega_{M,G}(g)+\omega_{V,G}(g) = 0$ in $\mathbf{Z}/2\mathbf{Z}$ for every $g\in G_{j_0}$ and, consequently,

$$\omega_{M,G} = \omega_{V,G}|_{G_{j_0}}.$$

Observe that if M is a G-specially stably parallelizable manifold and V a representation such that $T(M) \oplus \mathrm{R}^1$ and $M \times (V \oplus \mathrm{R}^1)$ are G-isomorphic, then for every subgroup $H \subset G$ and every connected component $(M^H)_i$ of M^H we have

$$\dim (M^H)_i = \dim S(V \oplus \mathrm{R}^1)^H.$$

Let $N(H) \subset G$ be the normalizer of H in G and let $W(H) = N(H)/H$ be the Weyl group of H in G. M^H is a $W(H)$-manifold. If

$$\Phi : T(M) \oplus \mathrm{R}^s \to M \times (V \oplus \mathrm{R}^s)$$

is a G-isomorphism of vector bundles over M, then V^H is a representation of $W(H)$ and

$$\Phi^H : T(M^H) \oplus \mathrm{R}^s \to M^H \times (V^H \oplus \mathrm{R}^s)$$

is a $W(H)$-isomorphism of vector bundles over M^H.

COROLLARY 4.7. *For every subgroup $H \subset G$ and every $W(H)$-connected component N of M^H, the orientation homomorphism $\omega_{N,W(H)}$ is a restriction of the orientation homomorphism $\omega_{V^H,W(H)}$ of the representation V^H of the Weyl group $W(H)$.*

We shall now recall some results from the equivariant homotopy theory. Let M be a compact smooth G-manifold and U be an orthogonal representation of G. Assume that for every subgroup $H \subset G$ and every non-empty connected component L of M^H

(4.8)
- (i) L is an orientable manifold,
- (ii) $\dim L = \dim U^H - 1$ and $\dim L$ is even,
- (iii) if $\tilde{G}(L) = \{g \in G \mid g(L) = L\}$, $G(L) = \tilde{G}(L)/H \subset W(H)$, $\omega_{L,G(L)}$ is the orientation homomorphism of the action of $G(L)$ on L and $\omega_{U^H,W(H)}$ is the orientation homomorphism of the representation U^H of the Weyl group $W(H)$, then $\omega_{L,G(L)} = \omega_{U^H,W(H)} | G(L)$.

PROPOSITION 4.9. *Under the assumptions* (4.8), *two G-maps $f_1, f_2 : M \to S(U)$ are G-homotopic, if and only if for every subgroup $H \subset G$ the restriction maps $f_1^H, f_2^H : M^H \to S(U^H)$ are homotopic.*

PROOF. This proposition is essentially proved, for example, in [8; Theorem 8.4.1]. The only difference is that we do not assume that M^H are connected as well as we allow $\dim M^{H_1} = \dim M^{H_2}$ for two different isotropy types (H_1), (H_2) on M with $(H_1) \subsetneq (H_2)$. It follows, however, from our assumption (4.8) (ii) that if (H) is an isotropy type on M, L is a connected component of M^H and $\tilde{L} = \{x \in L \mid H \subsetneq G_x\}$, then either $L = \tilde{L}$ or $\dim \tilde{L} \leqq \dim L - 2$.

Consequently, if $L \setminus \tilde{L} \neq \phi$, then $L - \tilde{L}$ is connected and

$$H^{\dim L}(L/G(L), \tilde{L}/G(L); \omega_{L,G(L)}) \cong \mathbf{Z}.$$

The rest of the proof proceeds just as in [8; Theorem 8.4.1].

Proof of Theorem 2.8. Let $\Phi: T(M) \oplus \mathbf{R}^s \to M \times (V \oplus \mathbf{R}^s)$ be a G-isomorphism over M. It follows from the assumption (2.8) (iii) as well as from the remark (2.7) (ii) that we can choose Φ in such a way that $\alpha(\Phi) = 0$. Moreover, according to (4.3) we may assume that $s = 1$.

Let $v_\Phi: M \to S(V \oplus \mathbf{R}^1)$ be the Gauss map of Φ. It follows from (4.5) that Theorem 2.8 will be proved as soon as we show that v_Φ is G-homotopically contractible to the base point $z_0 \in S(V \oplus \mathbf{R}^1)$.

Let $U = V \oplus \mathbf{R}^1$. For every subgroup $H \subset G$

$$\Phi^H: T(M^H) \oplus \mathbf{R}^1 \to M^H \times U^H$$

is an isomorphism. Consequently, M^H is a stably parallelizable manifold and, in particular, it is orientable. Furthermore, for every connected component L of M^H, we have $\dim L = \dim U^H - 1$. According to (2.8) (i), $\dim L$ is even. Thus, assumptions (4.8) (i) and (ii) are satisfied. Corollary 4.7 implies that (4.8) (iii) is satisfied as well. We can now apply Proposition 4.9 in order to determine G-homotopy class of v_Φ.

Let $\gamma: M \to S(U)$, $\gamma(M) = \{z_0\}$. We are going to show that for every subgroup $H \subset G$, $(v_\Phi)^H$ and γ^H are homotopic maps from M^H to $S(U^H)$.

If $\dim M^H = 0$, then $M^H \subset M_0$. In such a case $U^H = \mathbf{R}^1$ and

$$(v_\Phi)^H = \varphi \circ (\bar{\alpha}(\Phi)|M^H): M^H \to S(\mathbf{R}^1),$$

where $\bar{\alpha}(\Phi): M_0 \to \mathrm{O}(\mathbf{R}^1)$ is defined as in (2.3) and $\varphi: \mathrm{O}(\mathbf{R}^1) \to S(\mathbf{R}^1)$ is the evaluation map $\varphi(f) = f(z_0)$. Since $\alpha(\Phi)$ is the homotopy class of $\bar{\alpha}(\Phi)$ and $\alpha(\Phi) = 0$, it follows that $(v_\Phi)^H = \gamma^H$.

If $\dim M^H > 0$ then, according to (4.4), $(v_\Phi)^H = v_{\Phi^H}$, where

$$v_{\Phi^H}: M^H \to S(V^H \oplus \mathbf{R}^1)$$

is the Gauss map of the stable trivialization

$$\Phi^H: T(M^H) \oplus \mathbf{R}^1 \to M^H \times (V^H \oplus \mathbf{R}^1)$$

of the tangent bundle of the manifold M^H. If L is a connected component of M^H, then

$$v_{\Phi^H}|L: L \to S(V^H \oplus \mathbf{R}^1)$$

is the Gauss map of the stable trivialization

$$\Phi^H|_L: T(L) \oplus \mathbf{R}^1 \to L \times (V^H \oplus \mathbf{R}^1).$$

According to [7; p. 89], $\chi(L) = \pm 2\deg(\nu_{\Phi^H}|_L)$. (Observe that we can claim the equality only up to the sign since we have not followed the Bredon-Kosiński orientation convention on p. 86 of [7].) Let us recall that this result holds only for connected even-dimensional manifolds L with $\dim L > 0$.

However, according to the assumption (2.8) (ii), $\chi(L) = 0$. It follows that $\deg(\nu_{\Phi^H}|_L) = 0$ and $\nu_{\Phi^H}|_L$ is contractible. Consequently, ν_{Φ^H} is contractible, i.e. $(\nu_\Phi)^H \sim \gamma^H$.

It follows from Proposition 4.9 that ν_Φ and γ are G-homotopic, i.e. ν_Φ is G-contractible to the base point $z_0 \in S(V \oplus \mathsf{R}^1)$. This proves Theorem 2.8.

5. Examples.

Let $G_k = \mathsf{Z}/2 \oplus \ldots \oplus \mathsf{Z}/2$ (k summands), $k = 1, 2, \ldots$, be the elementary abelian 2-group. We denote by V_k the real regular representation of G_k, $\dim V_k = 2^k$. As before, we assume that V_k is equipped with an invariant inner product. $\tilde{V}_k \subset V_k$ is the orthogonal complement of the trivial subrepresentation $V_k^{G_k} \subset V_k$.

Let $S(V_k)$ be the unit sphere in V_k. $S(V_k)$ is a G_k-manifold and it is $\tilde{V}_k$-specially stably parallelizable.

Let $M_k = S(V_k) \times S(V_k)$ with the diagonal action of G_k. M_k is $(\tilde{V} \oplus \tilde{V}_k)$-specially stable parallelizable. Since $M_k^{G_k} \neq \emptyset$, $W = \tilde{V}_k \oplus \tilde{V}_k$ is the only representation of G_k for which M_k can be W-ssp.

In this section we shall show that

(i) if $k = 1, 2, 3$, then $S(V_k)$ is a $\tilde{V}_k$-parallelizable (and, consequently, M_k is $(\tilde{V}_k \oplus \tilde{V}_k)$-parallelizable for $k \leqq 3$),

(ii) if $k > 3$, then M_k is *not* $(\tilde{V}_k \oplus \tilde{V}_k)$-parallelizable.

REMARK 5.1. Since $\dim S(V_k) = 2^k - 1$, $S(V_k)$ is not parallelizable for $k > 3$ even without any group action. On the other hand, manifolds M_k, when considered as H-manifolds for any *proper* subgroup $H \subset G_k$, are $(\tilde{V}_k \oplus \tilde{V}_k|H)$-parallelizable. This follows either from Theorem 2.8 and Lemma 5.2 below or can be shown directly.

LEMMA 5.2.
(i) $\dim S(V_k)^{G_k} = 0$ *and* $\chi(S(V_k)^{G_k}) = 2$,
(ii) *if* $H \subset G_k$, $H \neq G_k$, *then* $\dim(S(V_k)^H) > 0$ *and* $\chi(S(V_k)^H) = 0$.

PROOF. V_k is the representation of G_k induced from the 1-dimensional representation of the trivial subgroup $\{e\}$, $V_k = \mathrm{Ind}_{\{e\}}^{G_k}(1)$. Let $H \subset G_k$ be a subgroup. Then

$$V_k = \mathrm{Ind}_{\{e\}}^{G_k}(1) = \mathrm{Ind}_H^{G_k}\mathrm{Ind}_{\{e\}}^H(1)$$

and, since G_k is abelian, the restriction of V_k to the subgroup H satisfies

$$\mathrm{Res}_H(V_k) = \mathrm{Res}_H \mathrm{Ind}_H^{G_k} \mathrm{Ind}_{\{e\}}^H(1) = |G_k/H| \mathrm{Ind}_{\{e\}}^H(1), \tag{5.3}$$

where $|G_k/H|$ is the cardinality of G_k/H, see [14; Section 7.3, Proposition 22]. It follows that

$$\dim V_k^H = \dim(\mathrm{Res}_H(V_k))^H = |G_k/H|.$$

Hence, if $H = G_k$ then $\dim V_k^H = 1$, and if $H \neq G_k$ then $\dim V_k^H$ is even and greater than 0. Since $S(V_k)^H$ is the unit sphere in V_k^H, Lemma 5.2 follows.

The next three examples show that the usual trivializations of the tangent bundles of S^1, S^3, and S^7 are, respectively, G_1-, G_2-, and G_3-equivariant.

EXAMPLES 5.4. (i) Let $k = 1$ and let ξ_1 be a generator of $G_1 = \mathbf{Z}/2$. We identify V_1 with the complex plane $\mathbf{C}$, ξ_1 acts on $\mathbf{C}$ by the complex conjugation. Thus G_1 acts on $\mathbf{C}$ through ring isomorphisms. The norm $\|z\| = z\bar{z} \in \mathbf{R}$ is a real quadratic form on $\mathbf{C}$. Let $\langle \cdot, \cdot \rangle$ be the associated symmetric bilinear form. $\langle \cdot, \cdot \rangle$ is a G_1-invariant inner product on $\mathbf{C}$. $\tilde{V}_1$ is then identified with the imaginary axis $\mathbf{R}i \subset \mathbf{C}$.

$S(V_1)$ is the unit circle S^1 in $\mathbf{C}$. The tangent bundle $T(S(V_1)) \subset S^1 \times \mathbf{C}$,

$$T(S(V_1)) = \{(z, z_1) \in S^1 \times \mathbf{C} \mid \langle z, z_1 \rangle = 0\}.$$

G_1 acts on $T(S(V_1))$ by restriction of the diagonal action on $S^1 \times \mathbf{C}$, $\xi_1(z, z_1) = (\bar{z}, \bar{z}_1)$.

The map $\varphi: S^1 \times \mathbf{C} \to \mathbf{C}$, $\varphi(zz_1) = z_1 z^{-1}$ is G_1-equivariant and

$$\varphi(T(S(V_1))) = \mathbf{R}i = \tilde{V}_1.$$

Thus φ yields a G_1-isomorphism of G_1-vector bundles

$$\Phi: T(S(V_1)) \to S(V_1) \times \tilde{V}_1, \quad \Phi(zz_1) = (z, \varphi(z, z_1)).$$

$S(V_1)$ is $\tilde{V}_1$-parallelizable.

(ii) Let $k = 2$ and let ξ_1, ξ_2 be generators of $G_2 = \mathbf{Z}/2 \oplus \mathbf{Z}/2$. We shall identify V_2 with the algebra of quaternions $\mathbf{H}$. Thus

$$\mathbf{H} = \mathbf{R} \cdot 1 \oplus \mathbf{R} \cdot i \oplus \mathbf{R} \cdot j \oplus \mathbf{R} \cdot ij$$

and $i^2 = j^2 = -1$, $ij = -ji$, $1 \cdot i = i$, $1 \cdot j = j$. G_2 acts on $\mathbf{H}$ through $\xi_1(h) = ihi^{-1}$, $\xi_2(h) = jhj^{-1}$ for $h \in \mathbf{H}$. With this action of G_2, $\mathbf{H}$ is indeed isomorphic to V_2.

Let $(\bar{\cdot})$ be the quaternion conjugation, i.e.

$$\overline{a + bi + cj + dij} = a - bi - cj - dij.$$

Observe that G_2 acts on H through ring isomorphisms and that this action commutes with $(\bar{\cdot})$.

Let $\|h\| = h\bar{h} \in \mathsf{R}$ for $h \in \mathsf{H}$. $\|\cdot\|$ is a real quadratic form on H. Let $\langle \cdot,\cdot \rangle$ be the associated symmetric bilinear form. Since the form $\|\cdot\|$ is G_2-invariant, $\langle \cdot,\cdot \rangle$ is a G_2-invariant inner product on H. $\mathsf{H}^{G_2} = \mathsf{R} \cdot 1$ and $\tilde{V}_2$ is identified with the orthogonal complement of $\mathsf{R} \cdot 1$.

$S(V_2) = S^3 = \{h \in \mathsf{H} \mid \|h\| = 1\}$. Again $T(S^3) \subset S^3 \times \mathsf{H}$,

$$T(S^3) = \{(h,v) \in S^3 \times \mathsf{H} \mid \langle h, v \rangle = 0\},$$

and the action of G_2 on $T(S^3)$ is restriction of the diagonal action of G_2 on $S^3 \times \mathsf{H}$. Let $\varphi: S^3 \times \mathsf{H} \to \mathsf{H}$, $\varphi(h,v) = vh^{-1}$. φ is a G_2-map and $\varphi(T(S^3)) = \tilde{V}_2$. φ yields a G_2-isomorphism

$$\Phi: T(S^3) \to S^3 \times \tilde{V}_2, \quad \Phi(h,v) = (h, \varphi(h,v)).$$

Thus $S(V_2)$ is $\tilde{V}_2$-parallelizable.

(iii) Let $k = 3$ and let ξ_1, ξ_2, ξ_3 be generators of $G_3 = \mathsf{Z}/2 \oplus \mathsf{Z}/2 \oplus \mathsf{Z}/2$. We shall identify V_3 with the algebra of Cayley numbers $\mathscr{C}$. Thus, see [9; p. 17], $\mathscr{C} = \mathsf{H} \oplus \mathsf{H}$ with the multiplication

$$(a,b)(c,d) = (ac - \bar{d}b, da + b\bar{c}).$$

Let G_3 act on $\mathscr{C}$ by

$$\xi_1(a,b) = (\xi_1(a), \xi_1(b))$$

$$\xi_2(a,b) = (\xi_2(a), \xi_2(b)),$$

where the action of ξ_1 and ξ_2 on H is that defined in (ii), and

$$\xi_3(a,b) = (a, -b).$$

Observe that G_3 acts on $\mathscr{C}$ through algebra isomorphisms.

The action of G_3 on $\mathscr{C}$ commutes also with the involution $(\bar{\cdot})$, where $(\bar{\cdot}): \mathscr{C} \to \mathscr{C}$ is given by $\overline{(a,b)} = (\bar{a}, -b)$. This representation of G_3 on $\mathscr{C}$ is isomorphic to V_3.

The element $(1,0)$ is the unit of $\mathscr{C}$. For $x \in \mathscr{C}$ the norm $\|x\| \in \mathsf{R}$ is defined by

$$x\bar{x} = \|x\|(1,0).$$

The norm $\|\cdot\|$ is preserved by the action of G_3. Since $\overline{xy} = \bar{y}\bar{x}$ for $x, y \in \mathscr{C}$, we have $\|xy\| = \|x\| \cdot \|y\|$. The norm $\|\cdot\|$ is a real quadratic form on $\mathscr{C}$. Let $\langle \cdot,\cdot \rangle$ be the associated symmetric bilinear form. $\langle \cdot,\cdot \rangle$ is a G_3-invariant inner product on $\mathscr{C}$. Let $\mathsf{R} \subset \mathscr{C}$ be the real subspace of $\mathscr{C}$ spanned by $(1,0)$. $\mathsf{R} = \mathscr{C}^{G_3}$. Thus $\tilde{V}_3$ is the orthogonal complement of R.

$$S(V_3) = S^7 = \{x \in \mathscr{C} \mid \|x\| = 1\}.$$

Again $T(S^7) \subset S^7 \times \mathscr{C}$,

$$T(S^7) = \{(c, w) \in S^7 \times \mathscr{C} \mid \langle c, w \rangle = 0\}$$

and the action of G_3 on $T(S^7)$ is the restriction of the diagonal action of G_3 on $S^7 \times \mathscr{C}$. The map $\varphi: S^7 \times \mathscr{C} \to \mathscr{C}$, $\varphi(c, w) = wc^{-1}$ is G_3-equivariant and $\varphi(T(S^7)) = \tilde{V}_3$. φ yields a G_3-isomorphism

$$\Phi: T(S^7) \to S^7 \times \to \tilde{V}_3, \quad \Phi(c, w) = (c, \varphi(c, w)).$$

Thus $S(V_3)$ is $\tilde{V}_3$-parallelizable.

REMARK 5.5. (i) If $k \geqq 2$, then G_k acts on V_k through oriented orthogonal transformations. This follows directly from (5.3).

(ii) The Stiefel-Whitney classes $w_i(V_k) \in H^i(BG_k, \mathsf{Z}/2)$ of the representations V_k were computed in [11; Lemma 3.26, p. 59]. In particular, the second Stiefel-Whitney class $w_2(V_2) \neq 0$ and $w_2(V_k) = 0$ for $k \neq 2$.

(iii) Let $\varrho_n: \mathrm{Spin}(n) \to \mathrm{SO}(\mathsf{R}^n)$ be the twisted adjoint representation of $\mathrm{Spin}(n)$, [5; p. 7–8]. It follows from (i) and (ii) that the representation V_k of G_k can be lifted to a homomorphism $\gamma_k: G_k \to \mathrm{Spin}(2^k)$ such that $\varrho_{2^k} \circ \gamma_k = V_k$ if and only if $k \geqq 3$.

(iv) For the same reason as in (iii), the representation $\tilde{V}_k$ of G_k can be lifted to a homomorphism $\tilde{\gamma}_k: G_k \to \mathrm{Spin}(2^k - 1)$ such that $\varrho_{2^k-1} \circ \tilde{\gamma}_k = \tilde{V}_k$ provided $k \geqq 3$. This shows that the $\tilde{V}_3$-parallelizability of $S(V_3)$ follows also from [13; (2.1)].

The rest of this section is devoted to proving that the G_k-manifolds $M_k = S(V_k) \times S(V_k)$ are *not* $(\tilde{V}_k \oplus \tilde{V}_k)$-parallelizable (and, consequently, are not W-parallelizable for any representation W of G_k) provided that $k > 3$.

We assume that $k \geqq 3$. Let $\gamma_k: G_k \to \mathrm{Spin}(2^k)$ be a lifting of the representation V_k, see (5.5), (iii).

We start by considering the $\tilde{V}_k$-ssp manifold $S(V_k)$. Let

$$H: \widetilde{\mathrm{KO}}_{G_k}^{-1}(S(V_k)_+) \to B(S(V_k))$$

be the homomorphism defined in Section 2. Let $A \in \mathrm{O}(V_k)$ be the reflection in the hyperplane $\tilde{V}_k \subset V_k$. Then

$$A \in \mathrm{O}(V_k)^{G_k} \subset \mathrm{O}_{G_k}(\infty)^{G_k}.$$

Let

$$f_A: S(V_k) \to \mathrm{O}_{G_k}(\infty)$$

be the map which takes all $S(V_k)$ into the point A,

$$f_A(S(V_k)) = \{A\}.$$

f_A is a G_k-map. $[f_A] \in \widetilde{\mathrm{KO}}_{G_k}{}^{-1}(S(V_k)_+)$ is the G_k-homotopy class of f_A.

PROPOSITION 5.6. *If $k \geqq 4$, then the image of*

$$H : \widetilde{KO}_{G_k}^{-1}(S(V_k)_+) \to B(S(V_k))$$

is generated by $H([f_A])$.

PROOF. Let $y_0 \in S(V_k)$ be a fixed point of G_k. We shall consider $S(V_k)$ as a G_k-space with the base point y_0. Let S^0 be the 0-dimensional sphere with the trivial action of G_k. Then

$$\widetilde{KO}_{G_k}{}^{-1}(S(V_k)_+) = \widetilde{KO}_{G_k}^{-1}(S(V_k)) \oplus \widetilde{KO}_{G_k}^{-1}(S^0).$$

The subgroup $H(\widetilde{KO}_{G_k}^{-1}(S^0)) \subset B(S(V_k))$ is generated by $H([f_A])$.

We shall show that $H(\widetilde{KO}_{G_k}^{-1}(S(V_k))) = 0$, provided that $k \geqq 4$.

Let V_k^c be the one-point compactification of V_k. Then

$$\widetilde{KO}_{G_k}^{-1}(S(V_k)) = \widetilde{KO}_{G_k}^{0}(V_k^c).$$

The structure of the $\mathrm{RO}(G_k)$-module $\widetilde{KO}_{G_k}^{0}(V_k^c)$ has been determined in [3; Theorem 6.1]. We assume that $k \geqq 3$. Let $\lambda_k \in \widetilde{KO}_{G_k}^{0}(V_k^c)$ be the Bott class of γ_k. Then $\widetilde{KO}_{G_k}^{0}(V_k^c)$ is a free $\mathrm{RO}(G_k)$-module on one generator λ_k.

Let $j : \mathbf{R}^1 \hookrightarrow V_k$ be an isometric embedding of $\mathbf{R}^1$ on the trivial subrepresentation of V_k. After compactification we obtain a G_k-embedding $j : S^1 \to V_k^c$, G_k acts trivially on S^1. j induces an $\mathrm{RO}(G_k)$-module homomorphism

$$j^* : \mathrm{KO}_{G_k}^{0}(V_k^c) \to \widetilde{KO}_{G_k}^{0}(S^1).$$

Since all real irreducible representations of G_k are of real dimension 1, we have

$$\widetilde{KO}_{G_k}^{0}(S^1) \cong \widetilde{KO}^0(S^1) \otimes \mathrm{RO}(G_k),$$

see [6; Proposition 8.1], and

$$j^* : \widetilde{KO}_{G_k}^{0}(V_k^c) \to \widetilde{KO}^0(S^1) \otimes \mathrm{RO}(G_k).$$

Let $\varepsilon : \mathrm{RO}(G_k) \to \mathbf{Z}$ be a group homomorphism defined by $\varepsilon(W) = \dim_{\mathbf{R}}(W^{G_k})$ for a representation W of G_k.

We have

$$S(V_k)_0 = S(V_k)^{G_k} = \{y_0, -y_0\},$$

see Section 2 and Lemma 5.2. Let $\zeta \in B(S(V_k))$ be given by

$$\zeta : S(V_k)_0 \to \mathbf{Z}/2, \mathbf{Z}/2 = \{\pm 1\}, \zeta(y_0) = 1, \zeta(-y_0) = -1.$$

We define a group homomorphism

$$F : \widetilde{KO}^0(S^1) \otimes \mathrm{RO}(G_k) \to B(S(V_k))$$

by

$$F(\xi \otimes W) = \varepsilon(W)\zeta \quad \text{for} \quad W \in \mathrm{RO}(G_k).$$

It follows from the definition of H that

$$H|\widetilde{\mathrm{KO}}_{G_k}^{-1}(S(V_k)) = F \circ j^*. \tag{5.7}$$

Let $\xi \in \widetilde{\mathrm{KO}}^0(S^1) \approx \mathbf{Z}/2$ be the generator. Then

$$j^*(\lambda_k) = \begin{cases} \xi \otimes V_k, & \text{if } k = 3 \\ 0, & \text{if } k > 3. \end{cases} \tag{5.8}$$

This equality is proved in Corollary 5.12 below. Assuming it for a moment we shall complete the proof of Proposition 5.6.

Since λ_k generates $\mathrm{RO}(G_k)$-module $\widetilde{\mathrm{KO}}_{G_k}^0(V_k^c)$ and j^* is a $\mathrm{RO}(G_k)$-module homomorphism, it follows from (5.8) that $j^* = 0$ provided that $k > 3$. Consequently, (5.7) implies that $H(\widetilde{\mathrm{KO}}_{G_k}^{-1}(S(V_k))) = 0$ if $k > 3$. This proves Proposition 5.6.

We shall now proceed to prove Corollary 5.12 which we have used in the proof of Proposition 5.6. To this end we need the next two lemmas.

Let Δ^+, Δ^- be the two real $\frac{1}{2}$-Spin representations of $\mathrm{Spin}(2^k)$, see [4; p. 483] and [5; Proposition (5.5)].

LEMMA 5.9. *$j^*(\lambda_k) = \xi \otimes \gamma_k^*(\Delta^+)$ in $\widetilde{\mathrm{KO}}^0(S^1) \otimes \mathrm{RO}(G_k)$.*

PROOF. $\Delta = \Delta^+ \oplus \Delta^-$ is a graded module over the Clifford algebra C_{2^k}, [5; Proposition (5.5)]. Let $\langle \cdot, \cdot \rangle$ be the G_k-invariant inner product on V_k and let $q: V_k \to \mathbf{R}$ be the quadratic form $q(x) = -\langle x, x \rangle$. We identify C_{2^k} with the Clifford algebra $C(q)$ of the form q.

Let $D(V_k)$ be the unit disc in $V_k, \partial D(V_k) = S(V_k)$. Thus $D(V_k) \subset C_{2^k}$. The Bott class

$$\lambda_k \in \widetilde{\mathrm{KO}}_{G_k}^0(V_k^c) = \mathrm{KO}_{G_k}(D(V_k), S(V_k))$$

is represented by the G_k-equivariant sequence

$$0 \to D(V_k) \times \Delta^- \xrightarrow{\sigma} D(V_k) \times \Delta^+ \to 0,$$

$$\sigma(v, x) = (v, -vx),$$

for $v \in D(V_k)$, $x \in \Delta^-$. See [5: Section 9 and Section 11]. Here G_k acts on Δ^+, Δ^- via lifting $\gamma_k: G_k \to \mathrm{Spin}(2^k)$.

Let $D^1 = D(V_k)^{G_k}$, $S^0 = \partial D^1$. It follows that

$$j^*(\lambda_k) \in \widetilde{\mathrm{KO}}_{G_k}^0(S^1) = \mathrm{KO}_{G_k}(D^1, S^0)$$

is represented by the sequence

$$0 \to D^1 \times \Delta^- \xrightarrow{\omega} D^1 \times \Delta^+ \to 0$$

$$\omega = \sigma | D^1 \times \Delta^-.$$

We have $S^0 = \{y_0, -y_0\}$ and $(\omega_{-y_0}) \circ (\omega_{y_0})^{-1} = -\mathrm{Id}_{\Delta^+}$. It follows that

$$j^*(\lambda_k) = \xi \otimes \gamma_k^*(\Delta^+),$$

where $\gamma_k : G_k \to \mathrm{Spin}(2^k)$ is the lifting of the representation V_k and $\gamma_k^*(\Delta^*) \in \mathrm{RO}(G_k)$.

REMARK. Since y_0 is fixed by G_k, the left multiplication by y_0 establishes an isomorphism between representations $\gamma_k^*(\Delta^+)$ and $\gamma_k^*(\Delta^-)$.

LEMMA 5.10. *$\gamma_k^*(\Delta^+) = 2^{2^{k-1}-k-1} V_k$ in* $\mathrm{RO}(G_k)$.

PROOF. The representation V_k of G_k splits into an orthogonal sum of 1-dimensional real subrepresentations. We may therefore assume that the vector space V_k has an orthonormal basis $e_1, \ldots, e_{2^k}$ such that for every $i = 1, \ldots, 2^k$ and every $g \in G_k$, $ge_i = \pm e_i$. Consequently, G_k acts on V_k through compositions of reflections in hyperplanes orthogonal to e_i's. Let $E \subset \mathrm{Pin}(2^k)$ be the subgroup generated by $e_1, \ldots, e_{2^k}$. It follows that $\mathrm{Im}\,\gamma_k \subset \mathrm{Spin}(2^k) \cap E$.

We shall now compute the character of the representation $\Delta = \Delta^+ \oplus \Delta^-$ restricted to the subgroup $E^0 = \mathrm{Spin}(2^k) \cap E$.

Let $c_+ : \mathrm{Spin}(2^k) \to \mathrm{Aut}(C^0_{2^k})$ be the natural representation given by the left multiplication by elements of $\mathrm{Spin}(2^k) \subset (C^0_{2^k})^*$. After complexification we have

$$c_+ \otimes_{\mathbf{R}} \mathbf{C} = 2^{2^{k-1}-1}(\Delta \otimes_{\mathbf{R}} \mathbf{C}),$$

see [4; (8.25)].

Let $A \subset \{1, 2, 3, \ldots, 2^k\}$, $A = \{a_1, \ldots, a_s\}$, $a_i < a_{i+1}$ for $1 \leqq i \leqq s-1$. Let $e_A \in C_{2^k}$ be the element $e_A = e_{a_1} e_{a_2} \cdots e_{a_s}$. Then $\{e_A\}_{A \subset \{1, \ldots 2^k\}}$ is a basis of the vector space C_{2^k}.

For $A_1, A_2 \subset \{1, \ldots, 2^k\}$ we define $B \subset \{1, \ldots, 2^k\}$ as

$$B = (A_1 \setminus A_2) \cup (A_2 \setminus A_1).$$

Then

$$e_{A_1} \cdot e_{A_2} = \pm e_B \tag{5.11}$$

in C_{2^k}. It follows that $e_{A_1} \cdot e_{A_2} = \pm e_{A_2}$ if and only if $A_1 = \phi$.

Since $\dim_{\mathbf{R}} C^0_{2^k} = 2^{2^k - 1}$, we obtain for $h \in E^0$

$$\mathrm{Tr}((c_+ \dot{\otimes}_{\mathbf{R}} \mathbf{C})(h) = \begin{cases} 0, & \text{if } h \neq \pm 1 \\ 2^{2^k-1}, & \text{if } h = 1 \\ -2^{2^k-1}, & \text{if } h = -1. \end{cases}$$

V_k is a faithful representation of G_k and γ_k is a lifting of V_k. Hence $-1 \notin \operatorname{Im} \gamma_k$. It follows that the character $\chi(\gamma_k^*(c_+ \otimes_{\mathbf{R}} \mathbf{C}))$ of the representation $\gamma_k^*(c_+ \otimes_{\mathbf{R}} \mathbf{C})$ of the group G_k is

$$\chi(\gamma_k^*(c_+ \otimes_{\mathbf{R}} \mathbf{C}))(g) = \begin{cases} 0, & \text{if } g \neq 1 \\ 2^{2^k-1}, & \text{if } g = 1 \end{cases}$$

for $g \in G_k$.

Since

$$\gamma_k^*(c_+ \otimes_{\mathbf{R}} \mathbf{C}) = 2^{2^{k-1}-1}(\gamma_k^*(\Delta) \otimes_{\mathbf{R}} \mathbf{C}) \quad \text{and} \quad \gamma_k^*(\Delta) = 2\gamma_k^*(\Delta^+),$$

we obtain the character of $\gamma_k^*(\Delta^+)$:

$$\chi(\gamma_k^*(\Delta^+))(g) = \begin{cases} 0, & \text{if } g \neq 1 \\ 2^{2^{k-1}-1}, & \text{if } g = 1 \end{cases}$$

for $g \in G_k$. The character of V_k is

$$\chi(V_k)(g) = \begin{cases} 0, & \text{if } g \neq 1 \\ 2^k, & \text{if } g = 1 \end{cases}$$

for $g \in G_k$.

Consequently $\gamma_k^*(\Delta^+) = 2^{2^{k-1}-k-1} V_k$ in $\mathrm{RO}(G_k)$.

COROLLARY 5.12.

$$j^*(\lambda_k) = \begin{cases} \xi \otimes V_k, & \text{if } k = 3 \\ 0, & \text{if } k > 3 \end{cases}$$

in $\widetilde{\mathrm{KO}}^0(S^1) \otimes \mathrm{RO}(G_k)$.

PROOF. If $k = 3$, then $2^{k-1} - k - 1 = 0$. If $k > 3$, then $2^{k-1} - k - 1 > 0$. Corollary 5.12 follows, since $\xi \in \widetilde{\mathrm{KO}}^0(S^1)$ has order 2.

Let us recall that $M_k = S(V_k) \times S(V_k)$. M_k is $(\tilde{V}_k \oplus \tilde{V}_k)$-specially stably parallelizable. Let

$$g_A : M_k \to \mathrm{O}_{G_k}(\infty)$$

be the G_k-map which takes all M_k into the point $A \in \mathrm{O}_{G_k}(\infty)$. $[g_A] \in \widetilde{\mathrm{KO}}_{G_k}^{-1}((M_k)_+)$ is the G_k-homotopy class of g_A. Let

$$H : \widetilde{\mathrm{KO}}_{G_k}^{-1}((M_k)_+) \to B(M_k)$$

be the homomorphism defined in Section 2. The element $H[g_A] \in B(M_k)$ is represented by the map $\tilde{g}:(M_k)_0/G_k \to Z/2$, where $\tilde{g}(x) = -1$ for every $x \in (M_k)_0$.

PROPOSITION 5.13. *If $k \geqq 4$, then the image of $H: \widetilde{KO}_{G_k}^{-1}((M_k)_+) \to B(M_k)$ is generated by $H([g_A])$.*

PROOF. Let $f \in B(M_k)$, $f:(M_k)_0/G_k \to Z/2$ be in the image of H. It is enough to show that $f(x) = f(y)$ for all $x,y \in (M_k)_0$. Indeed, in such a case if $f(x) \equiv 1$, then $f = H(0)$ and if $f(x) \equiv -1$, then $f = H([g_A])$.

$(M_k)_0 = (M_k)^{G_k}$ consists of four points. For every pair $x,y \in (M_k)_0$ there exists a G_k-embedding $i_{x,y}: S(V_k) \to M_k$ such that

$$i_{x,y}(S(V_k)^{G^k}) = \{x, y\}.$$

Indeed, let $S(V_k)^{G_k} = \{y_0, -y_0\}$. If

$$x = (y_0, y_0) \quad \text{and} \quad y = (-y_0, -y_0),$$

we set

$$i_{x,y}(z) = (z, z) \quad \text{for} \quad z \in S(V_k).$$

If

$$x = (y_0, -y_0) \quad \text{and} \quad y = (-y_0, y_0),$$

we set

$$i_{x,y}(z) = (z, -z) \quad \text{for} \quad z \in S(V_k).$$

If

$$x = (y_0, y_0) \quad \text{and} \quad y = (y_0, -y_0),$$

we set

$$i_{x,y}(z) = (y_0, z) \quad \text{for} \quad z \in S(V_k),$$

etc.

Let $x,y \in (M_k)_0$ and let $i_{x,y}: S(V_k) \to M_k$ be a G_k-embedding such that

$$i_{x,y}(S(V_k)^{G_k}) = \{x, y\}.$$

$i_{x,y}$ induces homomorphisms

$$i_{x,y}^{\#}: B(M_k) \to B(S(V_k)) \quad \text{and} \quad i_{x,y}^{*}: \widetilde{KO}_{G_k}^{-1}((M_k)_+) \to \widetilde{KO}_{G_k}^{-1}(S(V_k)_+)$$

such that the diagram

$$\begin{array}{ccc} \widetilde{KO}_{G_k}^{-1}((M_k)_+) & \xrightarrow{i_{x,y}^{*}} & \widetilde{KO}_{G_k}^{-1}(S(V_k)_+) \\ H\downarrow & & \downarrow H \\ B(M_k) & \xrightarrow{i_{x,y}^{\#}} & B(S(V_k)) \end{array}$$

commutes.

Let $a \in \widetilde{KO}_{G_k}^{-1}((M_k)_+)$, $H(a) = f$. Then $i^{\#}_{x,y}(f) = H(i^*_{x,y}(a))$. It follows from Proposition 5.6 that either $i^{\#}_{x,y}(f) = 0$ or $i^{\#}_{x,y}(f) = H([f_A])$. In both cases $f(x) = f(y)$. This proves Proposition 5.13.

Let $h: A(M_k, \tilde{V}_k \oplus \tilde{V}_k) \to B(M_k)$ be the homomorphism defined in Section 2. It is clear that $H([g_A]) \in \operatorname{Im} h$. Let $C(M_k, \tilde{V}_k \oplus \tilde{V}_k)$ be the group and let

$$\zeta(M_k, \tilde{V}_k \oplus \tilde{V}_k) \in C(M_k, \tilde{V}_k \oplus \tilde{V}_k)$$

be the element defined in (2.6).

THEOREM 5.14. *If $k \geqq 4$ then*

(i) $C(M_k, \tilde{V}_k \oplus \tilde{V}_k) \cong Z/2 \oplus Z/2 \oplus Z/2$,

(ii) $\zeta(M_k, \tilde{V}_k \oplus \tilde{V}_k) \neq 0$,

(iii) *M_k is not G_k-parallelizable* (i.e. *M_k is not W-parallelizable for any representation W of G_k*).

PROOF. (i) $C(M_k, \tilde{V}_k \oplus \tilde{V}_k) = \operatorname{Coker}(h)$. According to (2.2) the homomorphism h can be factored through

$$H: \widetilde{KO}_{G_k}^{-1}((M_k)_+) \to B(M_k).$$

Since $H([g_A]) \in \operatorname{Im}(h)$ and $H([g_A]) \neq 0$, we have $\operatorname{Im}(h) = \operatorname{Im}(H) \approx Z/2$.

$$(M_k)_0 = (M_k)^{G_k} = (M_k)_0/G_k$$

consists of four points. Thus

$$\dim_{Z/2} B(M_k) = 4 \quad \text{and} \quad \dim_{Z/2} C(M_k, \tilde{V}_k \oplus \tilde{V}_k) = 3.$$

(ii) We choose an isomorphism of representations $V_k \cong \tilde{V}_k \oplus R^1$. Let n be the standard normal field to the embedding $S(V_k) \subset V_k$, that is

$$n: S(V_k) \to T(V_k) = V_k \times V_k, \quad n(x) = (x, x).$$

The field n gives us a G_k-isomorphism of G_k-vector bundles over $S(V_k)$,

$$\psi: T(S(V_k)) \oplus R^1 \to S(V_k) \times (\tilde{V} \oplus R^1).$$

Let $y_0 \in S(V_k)^{G_k}$. Thus

$$S(V_k)_0/G_k = \{y_0, -y_0\}.$$

Let $\alpha(\psi): S(V_k)_0/G_k \to Z/2$ be the map associated to ψ according to the definition (2.3). Then $\alpha(\psi)(y_0) \neq \alpha(\psi)(-y_0)$.

We identify $T(M_k)$ with $T(S(V_k)) \times T(S(V_k))$ and $M_k \times (\tilde{V}_k \oplus \tilde{V}_k \oplus R^2)$ with

$$(S(V_k) \times (\tilde{V}_k \oplus R^1)) \times (S(V_k) \times (\tilde{V}_k \oplus R^1)).$$

Let us consider the G_k-isomorphism

$$\Phi: T(M_k) \oplus \mathbf{R}^2 \to M_k \times (\tilde{V}_k \oplus \tilde{V}_k \oplus \mathbf{R}^2), \quad \Phi = \psi \times \psi.$$

Let $\alpha(\Phi): (M_k)_0/G_k \to \mathbf{Z}/2$ be the map associated to Φ according to (2.3). If $z_1, z_2 \in S(V_k)^{G^k}$, then $(z_1, z_2) \in (M_k)_0$ and

$$\alpha(\Phi)(z_1, z_2) = (\alpha(\psi)(z_1)) \cdot (\alpha(\psi)(z_2))$$

(in the multiplicative group $\mathbf{Z}/2 = \{\pm 1\}$). Thus $\alpha(\Phi)(y_0, y_0) = 1$ and $\alpha(\Phi)(y_0, -y_0) = -1$.

According to the proof of (i) above, $\mathrm{Im}(h)$ consists of those maps

$$f: (M_k)_0 \to \{\pm 1\}$$

which are constant on $(M_k)_0$. Thus $\alpha(\Phi) \notin \mathrm{Im}(h)$ and, consequently,

$$\zeta(M_k, \tilde{V}_k \oplus \tilde{V}_k) \neq 0 \quad \text{in} \quad C(M_k, \tilde{V}_k \oplus \tilde{V}_k) = \mathrm{Coker}(h).$$

(iii) $(M_k)^{G_k} \neq \phi$ and for any $x \in (M_k)^{G_k}$, the tangent representation of G_k in $T_x(M_k)$ is isomorphic to $\tilde{V}_k \oplus \tilde{V}_k$. It follows that if M_k is W-parallelizable for a representation W of G_k, then $W \approx \tilde{V}_k \oplus \tilde{V}_k$. However, since $\zeta(M_k, \tilde{V}_k \oplus \tilde{V}_k) \neq 0$, M_k is *not* $(\tilde{V}_k \oplus \tilde{V}_k)$-parallelizable (see Remark 2.7 (iii)).

Let us recall that, as a consequence of (5.4) (i)–(iii), M_k *is* $(\tilde{V}_k \oplus \tilde{V}_k)$-parallelizable for $k = 1, 2, 3$.

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