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BILINEAR REPRESENTATION FORMULAS FOR POLYNOMIALS

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1. Introduction.

Recently A. Ramm showed that the set of finite sums $\sum u_i v_i$ where each u_i and v_i is a polynomial solution of the three-dimensional Laplace equation, is dense in $L^2(\boldsymbol{D})$ where $\boldsymbol{D}$ is any bounded domain in $\mathbf{R}^3$. He presented this result at a seminar in Stockholm in the spring of 1985, after which I pointed out that an analogous result is true for arbitrarily many variables and, moreover, with the Laplace equation replaced by any one of a large class of partial differential equations with constant coefficients. These results appear in a paper of Ramm [5].

The purpose of this note is to extend Ramm's contribution (i.e. Theorem 1 of [5]) in another direction. As A. Atzmon remarked to the author, the sums figuring in Ramm's theorem in fact comprise *all* polynomials in the two-dimensional case (i.e. *every* polynomial in x, y is representable as $\sum c_{mn} z^m \bar{z}^n$, where $z = x + iy$). We show here that this phenomenon persists in more variables, and for a large class of constant-coefficient partial differential operators. Since the argument builds on our earlier theorem, a version of which was incorporated as Theorem 2 of [5], our proof of this is also included in the present paper, as Theorem 2.1.

NOTATIONS. We use usual multi-variable notations, as in [3]. For each polynomial $P \in \mathbf{C}[\xi_1, \ldots, \xi_n]$ we denote by $P(\partial)$ the differential operator that arises upon replacing each ξ_j by $\partial_j = (\partial/\partial x_j)$. $V(P)$ denotes the set of $z = (z_1, \ldots, z_n)$ in $\mathbf{C}^n$ where $P(z) = 0$. For $x \in \mathbf{R}^n$ and $z \in \mathbf{C}^n$ we denote $z_1 x_1 + \cdots + z_n x_n$ by $z \cdot x$. *We tacitly assume $n \geqq 2$ always.* We use often the familiar fact that $P(\partial)(e^{z \cdot x}) = P(z) e^{z \cdot x}$ and so, in particular, $P(\partial)(e^{z \cdot x}) = 0$ for $z \in V(P)$.

For any $t = (t_1, \ldots, t_n)$ with all $t_i > 0$, the *t-degree* of a monomial

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$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ (where α_i are integers $\geqq 0$) is $\sum t_i \alpha_i$, and $P(\xi) = \sum c_\alpha x^\alpha$ is *t-homogeneous* if all x^α with $c_\alpha \neq 0$ have the same t-degree.

(In case $t = (1, 1, \dots 1)$ the t-degree of x^α is its degree, and t-homogeneity coincides with the usual notion of homogeneity). Until further notice all polynomials may have complex coefficients. We can now state our main result, to be proved in Section 3:

THEOREM 1.1. *Let P, Q be nonconstant polynomials, t-homogeneous (for the same t) and satisfying:*

(1.2). *P, Q are not constant multiples of powers of one and the same linear polynomial.*

Then every polynomial f is representable as a finite sum $\sum u_j v_j$, where u_j, v_j are polynomials, and

$$P(\partial)u_j = Q(\partial)v_j = 0. \tag{1.3}$$

REMARKS. For $P = Q = \sum_{i=1}^n \xi_i^2$ we get the refinement of Ramm's theorem alluded to above. We do not know whether the requirement of t-homogeneity is essential for the conclusion in Theorem 1.1. (Of course, if P has a non-vanishing constant term, $P(\partial)u = 0$ has no polynomial solutions. Observe that the hypothesis of t-homogeneity implies $P(0) = Q(0) = 0$.) As an illustration, take $P = Q = \xi_1^2 - \xi_2$ which is t-homogeneous with $t = (1, 2)$. Thus, the conclusion of the theorem holds, when u_j, v_j are polynomials solutions of the heat equation $(\partial_1^2 - \partial_2)u = 0$.

2. Exponential solutions.

THEOREM 2.1. *If P, Q are any polynomials satisfying* (1.2) *then*

(a) *$V(P) + V(Q)$ contains a ball in $\mathbf{C}^n$,*

(b) *the set of finite sums $\sum c_j u_j v_j$, where $c_j \in \mathbf{C}$, $u_j = e^{a_j \cdot x}$, $v_j = e^{b_j \cdot x}$, and a_j, b_j are in $V(P)$, $V(Q)$ respectively, is dense in $C(K)$, for every compact $K \subset \mathbf{R}^n$. (Note that u_j and v_j satisfy* (1.3).)

PROOF. We show first (a) $\Rightarrow$ (b). Assuming (a), if $\mu \in M(K)$ (complex bounded measures on K) satisfies

$$\int uv\,d\mu = 0$$

whenever $u = e^{a \cdot x}$, $v = e^{b \cdot x}$ and a, b are in $V(P)$, $V(Q)$ respectively, then

$$F(\gamma) := \int e^{\gamma \cdot x} d\mu(x) = 0$$

for all γ in an open subset of $\mathbf{C}^n$ and hence, since F is entire, for all $\gamma \in \mathbf{C}^n$. It follows from the Hahn-Banach theorem that the closure in $C(K)$ of the above sums $\sum c_j u_j v_j$ contains all $e^{\gamma \cdot x}$ with $\gamma \in \mathbf{C}^n$ and hence the algebra of all finite sums $\sum d_j e^{\gamma_j \cdot x}$ with arbitrary d_j, γ_j in $\mathbf{C}$. By the Stone-Weierstrass theorem these are dense in $C(K)$, proving (b).

Proof of (a). If P_0, Q_0 are nonconstant irreducible (over $\mathbf{C}$) factors of P, Q respectively, then $V(P) \supset V(P_0)$ and $V(Q) \supset V(Q_0)$. Moreover if (1.2) holds we may choose P_0, Q_0 so that they are not both constant multiples of the same linear polynomial. Hence, there is no loss of generality if we assume henceforth (dropping the subscripts):

(2.2). *P and Q are irreducible over* $\mathbf{C}$, *and not constant multiples of the same linear polynomial.*

If P, Q are both linear then $V(P)$ and $V(Q)$ are both hyperplanes and, because of (2.2), $V(P) + V(Q) = \mathbf{C}^n$. So assume, say, $\deg P \geqq 2$. Then the span of the vectors

$$\operatorname{grad} P(z) := (P_1(z), \ldots, P_n(z)), \quad z \in V(P),$$

where P_j denotes $\partial P/\partial z_j$, is of dimension at least 2. For otherwise, there exist complex $w_1, \ldots, w_n$ not all 0 such that $\sum w_j P_j(z) = 0$ for every $z \in V(P)$ and, since P is irreducible, $\sum w_j P_j$ is divisible by P in $\mathbf{C}[\xi_1, \ldots, \xi_n]$. However its degree is $< \deg P$ so $\sum w_j P_j = 0$. Writing $f(s) = P(w_1 s, \ldots, w_n s)$ for $s \in \mathbf{C}$ we have $df/ds = 0$ so f is constant. Letting $(a_1, \ldots, a_n)$ be complex numbers, not all zero, with $\sum a_j w_j = 0$, we see that $P(z) - P(0)$ and $L(z) := \sum a_j z_j$ have infinitely many common zeroes (namely $\{sw : s \in \mathbf{C}\}$). Since $P - P(0)$ is irreducible it divides L, a contradiction since $\deg P \geqq 2$.

Hence we can find $z^1 \in V(P)$ and $z^2 \in V(Q)$ such that $\operatorname{grad} P(z^1)$ and $\operatorname{grad} Q(z^2)$ are linearly independent (over $\mathbf{C}$). To complete the proof we shall assume (purely for notational reasons, as will be evident) $n = 3$. Let $\zeta \in \mathbf{C}^3$ be such that $\operatorname{grad} P(z^1), \operatorname{grad} Q(z^2)$, and ζ are linearly independent over $\mathbf{C}$ and consider the map

$$\varphi : q \mapsto (P(z^1 + w), Q(z^2 - w), \textstyle\sum \zeta_j w_j)$$

from $\mathbf{C}^3 \to \mathbf{C}^3$. Note that $\varphi(0) = 0$ and the Jacobian of the map at 0 does not vanish. Hence by the implicit function theorem, there exists $\varrho > 0$ such that φ maps the ball

$$B_\varrho = \{w \in \mathbf{C}^3 : \|w\| < \varrho\}$$

homeomorphically (even, biholomorphically) onto a neighborhood of 0 in $\mathbf{C}^3$. Moreover the Brouwer degree of the image point 0, written $\deg(\varphi, B_\varrho, 0)$ in the notation of [6, p.7] is $+1$. By homotopy invariance of the degree, each map

$$w \mapsto (P(\lambda^1+w), Q(\lambda^2-w), \sum \zeta_j w_j)$$

has the same property so long as λ^1, λ^2 lie sufficiently close to z^1, z^2 respectively. Hence each of these maps takes the value $(0,0,0)$ for some $w \in B_\varrho$ and, a fortiori, the equations

$$P(\lambda^1+w) = 0, \ Q(\lambda^2-w) = 0$$

are solvable for w. In other words, for all λ^i in suitable neighborhoods U_i of z^i in C^3 ($i = 1, 2$), $\lambda^1 + w \in V(P)$ and $\lambda^2 - w \in V(Q)$ hold for some w, that is $\lambda^1 + \lambda^2 \in V(P) + V(Q)$. This completes the proof.

Remark. In the last part of the proof the fact that P, Q are polynomials played no role. This allows (b) to be extended to convolution equations, however, we will not pursue this here.

3. Fischer spaces and Theorem 1.1.

In this section we make use of the Fischer space F_n of holomorphic functions f on C^n such that

$$\|f\|^2 := \sum \alpha! |c_\alpha|^2 = \pi^{-n} \int |f(z)|^2 e^{-|z|^2} dz_1 \wedge \cdots \wedge dz_n$$

is finite. Here $\sum c_\alpha z^\alpha$ is the Taylor expansion of f. For basic properties of this space see [1] or [4]. Especially important for us is the identity

$$f(\zeta) = \langle f, K_\zeta \rangle$$

where $K_\zeta(z) = \exp(\bar{\zeta}_1 z_1 + \cdots \bar{\zeta}_n z_n)$ is the reproducing kernel (r.k.) of F_n.

Proof of Theorem 1.1. Let H_ϱ, for each $\varrho > 0$, be the set of polynomials which are t-homogeneous, of "degree" ϱ (i.e. linear combinations of monomials of t-degree ϱ). Then H_ϱ, augmented by 0, is a finite-dimensional subspace of F_n whose r.k. $J_{\varrho,\zeta}$ is the orthogonal projection of K_ζ into H_ϱ. Since the set $\{J_{\varrho,\zeta} : \zeta \in \Omega\}$ spans H_ϱ if Ω contains a ball of C^n and every polynomial has a (unique) decomposition as a sum of elements of different H_ϱ, it is enough to show that each $J_{\varrho,\zeta}$ with ζ in Ω admits a representation $\sum u_i v_i$ where (1.3) holds. Now, for each $\zeta \in \mathrm{C}^n$ we have

$$\exp\left(\sum \bar{\zeta}_j z_j\right) = \sum_\varrho J_{\zeta,\varrho}(z) \tag{3.1}$$

the summation being over that discrete set R of $\varrho \in [0, \infty)$ which are of the form $\varrho = \sum_{i=1}^{n} t_i \alpha_i$ for some integers $\alpha_i \geqq 0$. In view of Theorem 2.1, the set of $\zeta = a+b$ with $a \in V(P)$ and $b \in V(Q)$ contains a ball of $\mathbf{C}^n$. Now, if $\exp(\sum a_j z_j)$ is expanded in a series of t-homogeneous polynomials:

$$\exp\left(\sum a_j z_j\right) = \sum u_\varrho(z, a) \tag{3.2}$$

where $u_\varrho \in H_\varrho$, each u_ϱ satisfies $P(\partial)u_\varrho = 0$. To see this, observe that

$$P(\partial)x^\beta = \sum c_\alpha(\partial^\alpha x^\beta) = \sum{}' c_\alpha x^{\beta-\alpha},$$

where $\sum'$ means that only terms with $\alpha \leqq \beta$ in the lexicographic order of $\mathbf{Z}_+^n$ are counted. This shows that $P(\partial)$ maps H_ϱ to $H_{\varrho-\delta}$ if $\varrho \geqq \delta =$ t-degree of P, otherwise to 0. So, if u is any entire solution of $P(\partial)u = 0$ we have

$$u = \sum u_\varrho$$
$$0 = P(\partial)u = \sum P(\partial)u_\varrho$$

and hence the $P(\partial)u_\varrho$ are all 0. The formal steps are all justified because the series converge in the space F_n, indeed the spaces H_ϱ for different ϱ are mutually orthogonal.

Hence, coming back to (3.2) and the analogous decomposition

$$\exp\left(\sum b_j z_j\right) = \sum v_\sigma(z, b)$$

where $v_\sigma \in H_\sigma$ and $Q(\partial)v_\sigma = 0$, we have for all ζ in an open set of $\mathbf{C}^n$

$$\exp\left(\sum \zeta_j z_j\right) = \sum u_\varrho(z, a) \sum v_\sigma(z, b). \tag{3.3}$$

The series on the right converge absolutely and uniformly on compact sets of $\mathbf{C}^n$ so we may write the expression on the right as

$$\sum_{\tau \in \mathrm{R}} \sum_{\varrho+\sigma=\tau} u_\varrho(z, a) v_\sigma(z, b)$$

(note that the product of a polynomial in H_ϱ with one in H_σ is in $H_{\varrho+\sigma}$). Combining this with (3.3) and (3.1) gives, since the expansion into elements of distinct H_ϱ is unique,

$$J_{\zeta,\tau}(z) = \sum_{\varrho+\sigma=\tau} u_\varrho(z, a) v_\sigma(z, b),$$

a representation of the form required, and the proof is complete.

4. Concluding remarks.

(4.1). Hitherto we have allowed all functions to be complex-valued. If however P, Q have real coefficients, then in Theorem 1.1, if f has real coefficients, u_j and v_j can also be chosen to have real coefficients. Indeed, writing $f = \sum u_j v_j$, where $u_j = u'_j + iu''_j$, $v_j = v'_j + iv''_j$ and u'_j etc. have real coefficients, we have

$$f = \sum_j (u'_j v'_j - u''_j v''_j)$$

and $P(\partial)u'_j = \cdots = Q(\partial)v''_j = 0$. A similar remark applies to Theorem 2.1.

(4.2). It is possible to generalize the above theorems in (at least) two ways:
(a) use a *multilinear* representation, e.g.

$$f = \sum u_i v_i w_i$$

trilinear), where $P(\partial)u_i = Q(\partial)v_i = R(\partial)w_i = 0$ for suitable polynomials P, Q, R. Provided that $V(P) + V(Q) + V(R)$ has interior points one can formulate similar results to (1.1) and (2.1).

(b) Replace $P, Q, \ldots$ by polynomial ideals. To take a simple illustration when $n = 3$ we can consider the ideal generated by (ξ_2, ξ_3), which we write $I_1 = (\xi_2, \xi_3)$, and the ideals $I_2 = (\xi_1, \xi_3)$, $I_3 = (\xi_1, \xi_2)$. This gives us three corresponding sets of polynomials, namely

$$U_1 = \{u : P(\partial)u = 0, \forall P \in I_1\}$$

and correspondingly U_2 and U_3. Here U_j consists precisely of polynomials in the variable x_j. Since the vector sum of the varieties (i.e. zero-sets) of these ideals has interior in $\mathbf{C}^3$ (here of course it is all of $\mathbf{C}^3$) we have the corresponding algebraic result that every polynomial in (x_1, x_2, x_3) is a finite sum of terms of the type $a(x_1)b(x_2)c(x_3)$ with a, b, c polynomials. (Nontrivial examples can easily be supplied.) Since the formulation of general theorems along these lines is somewhat unwieldy we content ourselves with the above indications.

(4.3). Bilinear representations of another kind, for polynomials, are also known. For example, it is well-known from the theory of spherical harmonics that every polynomial is a finite sum $\sum u_i v_i$, where u_i satisfies the Laplace equation and v_i is of the form $(\sum_{j=1}^{n} x_j^2)^r$ $(r = 0, 1, \ldots)$. Using Fischer's results in [2] one can extend this as follows: if $P_1, \ldots, P_r$ are arbitrary homogeneous polynomials, every polynomial is representable as a finite sum $\sum u_i v_i$ where

$$P_j(\partial)u_i = 0, \quad j = 1, 2, \ldots, r$$

and v_i is in the algebra generated by $P_1, \ldots, P_r$. These representations seem essentially different that those yielded by (1.1).

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