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A BOCHNER-HERZ PROPERTY IN BOUNDED SYNTHESIS

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Abstract.

A Bochner-Herz property for closed sets in R that is related to both Bochner's Property, as described by Meyer, and to the Herz criterion, is introduced. We study its connection with bounded synthesis and give several results. Our methods lead us to some results concerning the union problem for sets of spectral synthesis. We also note that if E_1 and E_2 are S-sets whose intersection is a U-set, then every pseudofunction on $E_1 \cup E_2$ is synthesizable on $E_1 \cup E_2$.

In ([8, pp. 226–232]), Y. Meyer describes the notion of Bochner's Property holding for a closed set E when proving that certain perfect symmetric sets are sets of bounded synthesis (see property (P) below). In our paper, we introduce a modified version of this property (property (BH)) and study its connections with bounded synthesis. In this setting, we show that certain M_1-sets possess some favorable synthesis properties (Theorem 1). Our methods also lead us to results concerning the question of when the union of S-sets is again an S-set (Corollaries 2, 3, and Proposition 2). Our hope is that this study will help in the understanding of basic spectral synthesis properties of pseudomeasures.

We refer to [2] for notation and basic facts. A closed set E in the real line R is a set of spectral synthesis, or an S-set, if every pseudomeasure supported by E is the limit of a net of measures $\{\mu_\alpha\} \subseteqq M(E)$ in the w^*-topology $\sigma(PM, A)$. Whenever possible, we define the constant B_E to be the infimum of all numbers B such that for every S in $N(E)$, the w^*-closure of $M(E)$, there exists a sequence $\{\mu_n\} \subseteqq M(E)$ satisfying

$$\sup_n \|\mu_n\|_{PM} \leqq B\|S\|_{PM} \quad \text{and} \quad \mu_n \xrightarrow{w^*} S.$$

In the case that there exists no such B, we set $B_E = \infty$. A set E is then a set of bounded synthesis, or bounded-S-set., if E is an S-set and B_E is finite. The existence of S-sets for which B_E is infinite was first proved by Varopoulos [13],

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and later refined by Korner [7], Katznelson and Korner [4], and Katznelson and McGehee [5], [6]. Conversely, there are well-known examples of non-S-sets E where B_E is finite (see [2]). Primarily, we are interested in studying for which sets E the condition $B_E < \infty$ holds. Without loss of generality, we will assume that all of our sets are compact.

In [8], where certain perfect symmetric sets are shown to be bounded-S-sets with $B_E = 1$, the following property is utilized.

$$\text{(P)} \quad \begin{cases} \text{There exists an increasing sequence } \{E_k\} \text{ of finite} \\ \text{subsets of } E \text{ such that } f \in C(E) \text{ and } \sup_k \|f\|_{A(E_k)} < \infty \\ \text{imply that } f \in A(E) \text{ and } \|f\|_{A(E)} = \sup_k \|f\|_{A(E_k)}. \end{cases}$$

Property (P) is well-adapted to the sets considered in [8], but is not as applicable to the question of bounded synthesis in general. For example, let E be an independent set. Then, by Kronecker's theorem, for every countable collection $\{E_k\}$ of finite subsets of E whose union is dense in E we have

$$\sup_k \|f\|_{A(E_k)} = \sup_k \|f\|_{C(E_k)} = \|f\|_{C(E)} \quad \text{for all } f \text{ in } C(E).$$

Property (P) holding for the set E would then imply that $A(E) = C(E)$, that is, E must be a Helson set. Thus, at least in the case of independent sets, property (P) is too restrictive. Furthermore, there exist independent, non-Helson sets E for which B_E is either finite or infinite, so that an attempt to modify this property by requiring that $\sup_k \|f\|_{A(E_k)} < \infty$ only for those f in $A(E)$ is also insufficient for our investigations of bounded synthesis. We note that since there exist independent Helson sets for which spectral synthesis fails [7], property (P) does not itself imply spectral synthesis for E. What property (P) does give us is that every pseudomeasure in $N(E)$ is synthesizable by a PM-norm-bounded sequence of measures $\{\mu_k\}$ satisfying $\operatorname{supp} \mu_k \subseteqq E_k$, $k = 1, 2, 3, \ldots$ (cf. Lemma 1). This condition, as evidenced in the result of Meyer, can be considered a weakened form of the property defining Herz sets. A closed set E is a Herz set if it satisfies the Herz criterion ([2, p. 76], or [10, pp. 166–169]): there exist discrete subgroups $G_k \subseteqq \mathrm{R}$ generated by positive elements x_k, $k \geqq 1$, such that $\bigcup_k G_k$ is dense in R and for each k the set $H_k = \{x \in G_k\colon \operatorname{dist}(x, E) < x_k\}$ is contained in E. Herz sets are sets of bounded synthesis: every pseudomeasure with support in E is the w^*-limit of a PM-norm-bounded sequence of measure supported by the sets H_k. For our study, we introduce the following definition.

DEFINITION 1. A sequence of subsets $\{E_k\}_{k=1}^{\infty}$ of a set E is said to satisfy the Bochner-Herz property, or property (BH), if there exists a constant b such that $\|\varphi\|_{A(E)} \leqq b \cdot \sup_k \|\varphi\|_{A(E_k)}$ for every φ in $A(E)$.

The infimum of all b satisfying the condition above will be called the Bochner-Herz constant for the sequence $\{E_k\}$. It is clear that if (BH) holds then the union $\bigcup_{k=1}^{\infty} E_k$ must be dense in E. Since $F \subseteqq E$ implies that $\|\varphi\|_{A(F)} \leqq \|\varphi\|_{A(E)}$ for all φ in A, if property holds for some sequence $\{F_j\}$ then it also holds for the sequence $\{E_k\}$, where $E_k = \bigcup_{j=1}^{k} F_j$. We will always assume that $E_k \subseteqq E_{k+1}$ for all k.

EXAMPLES. 1. Let E be any (closed) set, and fix $x_0 \in E$. For $k \geqq 1$, let

$$E_k = \{x_0\} \cup \{x \in E : \operatorname{dist}(x, E) \geqq 2^{-k}\}.$$

We can find functions $\{\varphi_k\} \subseteqq A$ so that $\|\varphi_k\| \leqq 4$, $x_0 \notin \operatorname{supp} \varphi_k$, and $\varphi_k = 1$ on a neighborhood of $E_k \setminus \{x_0\}$. It is now easy to see that $\{E_k\}$ satisfies property (BH). In Theorem 2 we show that this example works if we replace $\{x_0\}$ by any closed countable set.

2. Let E be a Helson set with Helson constant

$$c = \sup\{\|f\|_{A(E)} : \|f\|_{C(E)} \leqq 1\}.$$

If $\{F_k\}$ is a collection of finite subsets of E with $F_k \subseteqq F_{k+1}$ and $\overline{\bigcup F_k} = E$, then

$$\|f\|_{A(E)} \leqq c\|f\|_{C(E)} = c \cdot \sup_k \|f\|_{C(E_k)} \leqq c \cdot \sup_k \|f\|_{A(E_k)}.$$

Thus, property (BH) holds for any increasing sequence of subsets whose union is dense in a Helson set. Note that the existence of a Helson non-S-set again shows that property (BH) does not imply spectral synthesis for the set E.

3. Let E_0 be an independent Helson set disobeying spectral synthesis. Katznelson and McGehee [6] have shown that there exists a sequence of finite sets $\{H_j\}_{j=1}^{\infty}$, with $\operatorname{gp}(\bigcup_j H_j) \cap E_0 = \emptyset$, so that $E = E_0 \cup \bigcup_j H_j$ is a set of synthesis but not of bounded synthesis. Since $B_F = 1$ for any Helson set F, and since $B_{F \cup H} \leqq 3B_F$ for any finite set H, we can set $E_k = E_0 \cup \bigcup_{j=1}^{k} H_j$ and obtain $\sup_k B_{E_k} \leqq 3$. Therefore, by Proposition 1(i), we see that property (BH) must fail for the sequence $\{E_k\}$.

Our interest in the Bochner-Herz property can be found in the following lemma, which is an easy application of a classical theorem of Banach.

LEMMA 1. *Property* (BH) *holds for E and an increasing sequence of subsets $\{E_k\}_{k=1}^{\infty}$ with constant b if and only if, for every $S \in N(E)$, there exist $S_k \in N(E_k)$, $k \geqq 1$, satisfying $\|S_k\|_{PM} \leqq b\|S\|_{PM}$ and $S_k \xrightarrow{w^*} S$.*

PROOF. Consider the linear space N_0 of $N(E)$ defined by $N_0 = \bigcup_{k \geqq 1} N(E_k)$. For any $\varphi \in A$, the condition $\|\varphi\|_{A(E)} \leqq b \cdot \sup_k \|\varphi\|_{A(E_k)}$ holds if and only if, for each $\varepsilon > 0$, there exists an $S \in N_0$ such that $\|S\|_{PM} \leqq b + \varepsilon$ and $|\langle S, \varphi\rangle| = \|\varphi\|_{A(E)}$. Applying the result of Banach ([1, p. 213]) now gives the result.

Note that since $N(E)$ is the w^*-closure of $M(E)$, $N(E)$ is always realized as the w^*-closure of the set $\bigcup_k \bar{M}(E_k)$ and hence of $\bigcup_k N(E_k)$. However, the synthesizing measures are not, in general, PM-norm bounded.

PROPOSITION 1. (i) *Suppose* $E = \overline{\bigcup_k E_k}$, *with* $E_k \subseteq E_{k+1}$ *and suppose that property* (BH) *holds for the sequence* $\{E_k\}$. *If* $\sup_k B_{E_k}$ *is finite, then* B_E *is finite.*

(ii) *Suppose that* $E = \bigcup_k E_k$ *and that* B_E *is finite. Then the sequence* $\{E_k\}$ *satisfies* (BH) *with Bochner-Herz constant* $b \leqq B_E$.

PROOF. (i) Let $S \in N(E)$. Since (BH) holds, by Lemma 1 we can find $S_k \in N(E_k)$ such that $\|S_k\|_{PM} \leqq b\|S\|_{PM}$ and $S_k \xrightarrow{w^*} S$. Set $\sup_k B_{E_k} = B$. Then for each S_k there exists a sequence $\{\mu_{k,n}\}_{n \geqq 1} \subseteq M(E_k)$ satisfying

$$\sup_n \|\mu_{k,n}\|_{PM} \leqq B\|S_k\|_{PM} \leqq Bb\|S\|_{PM}, \quad \text{and} \quad \mu_{k,n} \underset{n}{\xrightarrow{w^*}} S_k.$$

Since $A(E)$ is separable, it is clear that we can find a subsequence $\{\nu_k\}$ from $\{\mu_{k,n}\}$ satisfying $\operatorname{supp} \nu_k \subseteq E_k$ and $\|\nu_k\|_{PM} \leqq Bb\|S\|_{PM}$ for $k \geqq 1$, and $\nu_k \xrightarrow{w^*} S$. Thus, B_E is finite with $B_E \leqq Bb$.

(ii) Let $\varphi \in A(E)$ and $\varepsilon > 0$ be given. There is an S in $N(E)$ with $\|S\|_{PM} = 1$ such that $\|\varphi\|_{A(E)} < (1+\varepsilon)|\langle S, \varphi\rangle|$. Since B_E is finite, we can find a measure μ in $M(E)$ with $\|\mu\|_{PM} \leqq B_E\|S\|_{PM} = B_E$ and $|\langle S, \varphi\rangle| < (1+\varepsilon)|\langle \mu, \varphi\rangle|$. Because μ is supported by the countable union $\bigcup_k E_k$, there is an integer K so that the measure $\nu = \mu|_{E_k}$ satisfies

$$\|\nu\|_{PM} < (1+\varepsilon)\|\mu\|_{PM} \leqq (1+\varepsilon)B_E, \quad \text{and}$$
$$|\langle \mu, \varphi\rangle| < (1+\varepsilon)|\langle \nu, \varphi\rangle|.$$

We now have

$$\begin{aligned} \|\varphi\|_{A(E)} &\leqq (1+\varepsilon)|\langle S, \varphi\rangle| < (1+\varepsilon)^2|\langle \mu, \varphi\rangle| \\ &\leqq (1+\varepsilon)^3|\langle \nu, \varphi\rangle| \leqq (1+\varepsilon)^3\|\nu\|_{PM}\|\varphi\|_{A(E_k)} \\ &\leqq (1+\varepsilon)^4 B_E \cdot \sup_k \|\varphi\|_{A(E_k)}. \end{aligned}$$

Since φ in $A(E)$ and $\varepsilon > 0$ are arbitrary, this proves (ii).

We remark that the hypothesis in (ii) cannot be weakened to the case where $E = \overline{\bigcup_k E_k}$. To see this, we only need consider an independent set E which is the support of a measure $\mu \neq 0$ whose Fourier-Stieltjes transform $\hat{\mu}$ vanishes at infinity (see [9]). Then E is a non-Helson set (see [2, Theorem 4.5.2]). Furthermore, it is well-known that $B_E = 1$ whenever every portion of E supports a non-zero measure whose transform vanishes at infinity (cf. [6]). If each E_k is a finite subset of E, then, as we noted earlier, $\sup_k \|\varphi\|_{A(E_k)} = \|\varphi\|_{C(E)}$ for each φ in $A(E)$. Since E is non-Helson, the $A(E)$ and $C(E)$ norms are not equivalent on $A(E)$, and so property (BH) must fail.

It is not unexpected, based on Lemma 1, that we should be able to draw similarities between bounded synthesis properties and the Bochner-Herz property. We in fact employ or modify many of the methods used to prove results in bounded synthesis. However, these methods take us only so far, mostly because many constructions demonstrating the failure of bounded synthesis begin with the existence of a non-synthesizable pseudomeasure, that is, an element supported by a set E but not in the w^*-closure of $M(E)$, whereas we know that $N(E)$ is always equal to the w^*-closure of $\bigcup_k N(E_k)$. Let us denote by PF the space of pseudofunctions, so that

$$PF = \left\{ S \in PM : \limsup_{x \to \infty} |\hat{S}(x)| = 0 \right\},$$

and set $PF_0(E) = PF \cap M(E)$ and $PF_1(E) = PF \cap N(E)$. If $PF_0(E) \neq \{0\}$, then E is called an M_0-set, and if $PF_1(E) \neq \{0\}$ then E is called an M_1-set. Every M_0-set is necessarily an M_1-set, but Pyateckii-Sapiro showed that there exists an M_1-set which is not an M_0-set (see [2, Theorem 4.4.2]). We will say that E is a UM_0-set (UM_1-set) if every portion of E is an M_0-set (respectively an M_1-set). For classical results concerning M_0- and M_1-sets, see [15, Chapter IX]. Finally, note that if E is a UM_1-set, then $PF_1(E)$ is a closed subspace of PF whose dual space is $A(E)$.

LEMMA 2. *Let E be a UM_1-set, and suppose that $\{E_k\}_{k \geqq 1}$ is an increasing sequence of subsets with $E = \bigcup_k E_k$. If $S \in PF_1(E)$, then there exist pseudofunctions $S_k \in N(E_k)$, $k \geqq 1$, such that $\|S_k - S\|_{PM} \to 0$.*

PROOF. Let N_1 denote the PM-normed closure of the space $\bigcup_k PF_1(E_k)$; N_1 is then also a closed subspace of $PF_1(E)$. If $S \notin N_1$, by the Hahn-Banach theorem there is a φ in $A(E)$ = dual space of $PF_1(E)$ such that $\langle T, \varphi \rangle = 0$ for all T in N_1 and $\langle S, \varphi \rangle \neq 0$. For a fixed $x \in E$, let $F = E \cap I$, where I is any closed interval containing x in its interior, and set $W_k = E_k \cap F$. Note that the set F, being a portion of a UM_1-set, is also a UM_1-set. Since $F = \bigcup_k W_k$, the Baire category theorem now implies that there exists an index n and a portion F' of F so that $F' \subseteqq W_n$. But then F' satisfies $PF_1(F') \neq \{0\}$ and so we have $PF_1(W_n) \neq \{0\}$ as well. Hence there exists a pseudofunction $S_F \in PF_1(W_n) \subseteqq N_1$ with $\hat{S}_F(0) = \|S_F\|_{PM} = 1$. By choosing a nested sequence of portions $\{F_j\}$ with $\bigcap_j F_j = \{x\}$, we can find $\{S_j\} \subseteqq N_1$ so that $S_j \xrightarrow{w^*} \delta_x$ in the topology $\sigma(PM, A)$. It follows that the w^*-closure of N_1 in $N(E)$ contains all discrete measures supported by E, and therefore must equal $N(E)$. In particular, $S \in N(E)$, and since $\langle T, \varphi \rangle = 0$ for all T in N_1, we obtain $\langle S, \varphi \rangle = 0$. This contradiction proves the lemma.

REMARK. The condition $E = \bigcup_k E_k$ can be weakened to $E = \overline{\bigcup_k E_k}$ if we know that each E_k is a UM_1-set. Furthermore, if each E_k is a UM_0-set, then we can choose the pseudofunctions S_k as measures.

LEMMA 3. *Let E be a UM_1-set. Then for every S in $N(E)$ there exists a sequence of pseudofunctions $\{S_k\} \subseteqq N(E)$ such that $\|S_k\|_{PM} \leqq \|S\|_{PM}$ for all $k \geqq 1$ and $S_k \overset{w^*}{\to} S$.*

PROOF. Since E is a UM_1-set, the second dual of $PF_1(E)$ is $N(E)$. Thus, for every S in $N(E)$, $\|S\|_{PM} \leqq 1$, and for every finite collection $\{\varphi_1, \ldots, \varphi_n\} \subseteqq A(E)$, there exists a sequence $\{S_k\} \subseteqq PF(E)$ such that $\|S_k\|_{PM} \leqq 1$ and

$$\langle S_k, \varphi_j \rangle \underset{k}{\to} \langle S, \varphi_j \rangle \quad \text{for} \quad j = 1, \ldots, n$$

(see [14, Theorem IV.8.3]). Since $A(E)$ is separable, the result now follows.

THEOREM 1. *Suppose E is a UM_1-set and that $E = \bigcup_k E_k$ for an increasing sequence $\{E_k\}$ of subsets of E. Then $\{E_k\}$ satisfies property* (BH) *with constant* 1. *In particular, for every S in $N(E)$ there is a sequence $\{S_k\}$ of pseudofunctions with $\operatorname{supp} S_k \subseteqq E_k$, $\|S_k\|_{PM} \leqq \|S\|_{PM}$ for all k, and such that $S_k \overset{w^*}{\to} S$.*

PROOF. If $S \in N(E)$, then by Lemma 3 there exist pseudofunctions P_n, $n \geqq 1$, such that $\operatorname{supp} P_n \subseteqq E$, $\|P_n\|_{PM} \leqq \|S\|_{PM}$, and $P_n \overset{w^*}{\to} S$. By Lemma 2, each P_n is realized as the PM-normed limit of a seuqnce $\{P_{n,k}\}_{k \geqq 1}$ of pseudofunctions with $\operatorname{supp} P_{n,k} \subseteqq (\operatorname{supp} P_n) \cap E_k \subseteqq E_k$. A simple diagonalization proces applied to $\{P_{n,k}\}$ (and multiplying by appropriate scaling factors $r_k \to 1$ if necessary) produces the required sequence $\{S_k\}$.

An easy consequence is the following result.

COROLLARY 1. *If E is a UM_1-set, then B_E is finite if and only if there exists an increasing sequence of subsets satisfying $E = \bigcup_k E_k$ and such that $\sup_k B_{E_k}$ is finite.*

Thus we find that the UM_1-set condition guarantees (bounded) synthesis by pseudofunctions in much the same way that the UM_0-set condition guarantees bounded synthesis by measures. One might consider to what extent either of these conditions can be weakened. For example, let E satisfy the condition: given any portion F of E and any $\varepsilon > 0$, there exists a non-zero measure μ with support in F such that

$$\limsup_{x \to \infty} |\hat{\mu}(x)| < \varepsilon \|\mu\|_{PM}.$$

It is not difficult to show that such sets have $B_E = 1$. We can then ask whether every set E satisfying the following similar condition always has the

Bochner-Herz property: given any portion F of E and any $\varepsilon > 0$, there exists a non-zero pseudomeasure S in $N(F)$ such that

$$\limsup_{x \to \infty} |\hat{S}(x)| < \varepsilon \|S\|_{PM}.$$

The answer is yes, for the reason that every set satisfying this condition is necessarily a UM_1-set. To see this, recall a result of Pyateckii-Sapiro (see [2, Theorem 4.3.4]) which says that every set $\bar{E}$ with $PF_1(E) = \{0\}$ is the countable union of closed sets F_k for which there exist numbers $\eta_k = \eta(F_k) > 0$ satisfying

$$\limsup |\hat{S}(x)| \geqq \eta_k \|S\|_{PM}$$

for all S in $N(F_k)$. Since E, as a closed subset of R, is a complete metric space, the Baire category theorem would imply that there is a neighborhood V and an index n such that the nonempty portion $E \cap V$ of E is contained in F_n. For all S in $E \cap V$ we would then have

$$\limsup |\hat{S}(x)| \geqq \eta_n \|S\|_{PM}.$$

But this contradicts our original assumption on E. Thus, every such set is a UM_1-set. For sets which are not UM_1-sets, we know that other conditions are needed for a sequence of subsets of E to satisfy property (BH). Define the limit set of a sequence $\{E_k\}$ as the intersection $E_0 = \bigcap_{k=1}^{\infty} \overline{E \setminus E_k}$. It is not surprising that the set E_0 plays a major role in whether or not (BH) holds. For example, if E_0 is a non-S-set, then (BH) may fail even if $E \setminus E_0$ is a countable set (Example 3). In the case when $E = \bigcup_k E_k$, it is clear that for every x in E_0 there is an index k_x so that $x \in E_k$ for all $k \geqq k_x$. We will be concerned primarily with increasing sequences $\{E_k\}$ for which $E = \bigcup_k E_k$ and $E_0 \subseteqq E_1$, and then consider what conditions on E_0 imply that property (BH) holds.

Before stating our next result, we recall some basic facts about pseudomeasures. Let f be a function in A with $\operatorname{supp} f \subseteqq [-1, 1]$, $f = 1$ on a neighborhood of 0, and $\|f\| \leqq 3$. For $n \geqq 1$, define $f_n \in A$ by $f_n(x) = f(nx)$. For every $t \in \mathrm{R}$, define $f_{t,n} \in A$ by $f_{t,n} = f_n * \delta_t$. We have $\|f_{t,n}\| \leqq 3$ for all $n \geqq 1$ and all t. Let S be a pseudomeasure. Then for each t the sequence $\{Sf_{t,n}\}_{n=1}^{\infty} \subseteqq PM$ is bounded in norm by $3\|S\|_{PM}$, and converges w^* to 0 for every $t \in \mathrm{R} \setminus J$, where J is a countable set contained in the support of S. By passing to a subsequence of $\{f_n\}$, we may assume there exist complex numbers a_t so that $Sf_{t,n} \xrightarrow{w^*} a_t \delta_t$ for each t in R. Then, whenever F is a finite set and $\mu = \sum_{t \in F} b_t \delta_t$ is in $M(F)$, we have

$$S(f_n * \mu) \underset{n}{\xrightarrow{w^*}} \mu_S = \sum a_t b_t \delta_t$$

and

$$\|\mu_S\|_{PM} \leqq \left(\limsup_n \|f_n * \mu\|\right)\|S\|_{PM}.$$

THEOREM 2. *Suppose* $E = \bigcup_k E_k$, $E_k \subseteqq E_{k+1}$, *and that* $E_0 = \bigcap_k \overline{E \setminus E_k}$ *is a countable set. Then* $\{E_k\}$ *satisfies the Bocher-Herz property.*

PROOF. Let $\varphi \in A(E)$ and set $\alpha = \sup_k \|\varphi\|_{A(E_k)}$. Let $\varepsilon > 0$ be given, and let $S \in N(E)$, $\|S\|_{PM} \leqq 1$, have $|\langle S, \varphi\rangle| > (1-\varepsilon)\|\varphi\|_{A(E)}$. By our remarks above, we can find a sequence $\{f_n\}_{n=1}^{\infty} \subseteqq A$ such that $Sf_{t,n}$ converges to $a_t\delta_t$ for each t in R. For every $k = 1, 2, 3, \ldots$, let $\varphi_k \in A$ satisfy $\varphi_k = \varphi$ on E_k and

$$\|\varphi_k\| < \|\varphi\|_{A(E_k)}(1+k^{-1}).$$

Since the pseudomeasures $\{S\varphi_k\}_{k \geqq 1}$ are bounded in norm, we can find a subsequence which converges w^* to a pseudomeasure S_1 with $\|S_1\|_{PM} \leqq \alpha\|S\|_{PM} \leqq \alpha$. Furthermore, since $S \in N(E)$, and since for any open set V disjoint from E_0 we have, for k sufficiently large, $S\varphi_k = S\varphi$ on V (that is, $f \in A$ and $\operatorname{supp} f \subseteqq V$ imply $\langle S\varphi_k, f\rangle = \langle S\varphi, f\rangle$), we find that $\operatorname{supp}(S\varphi - S_1) \subseteqq E_0$. Having support in a countable set, $S_0 = S\varphi - S_1$ is almost periodic, and so there exists a finite set $F_0 \subseteqq E_0$ and a measure μ with $\operatorname{supp}\mu = F_0$ and $\|S_0 - \mu\|_{PM} < \varepsilon$. By using a standard construction (see [5, Lemma 1]), given any finite set H_0, we can find a measure ν_H with the following properties:

(1) $\limsup_n \|\nu_H * f_n\| \leqq 3$;
(2) $\nu_H(\{t\}) = 1$ for all t in H_0;
(3) $\operatorname{supp}\nu_H = H \subseteqq \operatorname{gp}(H_0)$ and H is finite;

here $\operatorname{gp}(H_0)$ denotes the algebraic group generated by H_0. We first apply this construction to the set F_0 to obtain a measure ν_F with $\operatorname{supp}\nu_F = F \subseteqq \operatorname{gp}(F_0)$, and then set $H_0 = F \cap E$. We can now also find a measure ν_H for the set H_0 with $\operatorname{supp}\nu_H$ in a finite set H.

Consider the sequence $\{S(f_n * \nu_H)\}_{n=1}^{\infty}$. By our choice of $f_n \in A$, we see that

$$S(f_n * \nu_H) \xrightarrow[n]{w^*} \nu = \sum_{t \in H \cap E'H} \nu_H(\{t\})a_t\delta_t.$$

The measure ν satisfies $\|\nu\|_{PM} \leqq 3$, and $\operatorname{supp}\nu \subseteqq H \cap (\operatorname{supp} S)$ is finite. We write

$$\mu = S_0 + (\mu - S_0) = S\varphi - S_1 + (\mu - S_0)$$
$$= (S-\nu)\varphi + \nu\varphi - S_1 + (\mu - S_0),$$

and, since $\operatorname{supp}\mu = F_0$, obtain

$$[(S-\nu)\varphi + \nu\varphi - S_1 + (\mu - S_0)](f_n * \nu_F) \xrightarrow[n]{w^*} \mu.$$

But

$$(\operatorname{supp} \nu_F) \cap E \subseteqq F \cap E = H_0 \subseteqq \{t : \nu_H(\{t\}) = 1\},$$

so that $(S-\nu)(f_n * \nu_F) \xrightarrow[n]{w^*} 0$. Since $\operatorname{supp} \nu$ is a finite subset of E, we have $\|\nu\varphi\|_{PM} \leqq 3\alpha$, hence

$$\begin{aligned} \|\mu\|_{PM} &\leqq 3(\|\nu\varphi\|_{PM} + \|S_1\|_{PM} + \|\mu - S_0\|_{PM}) \\ &\leqq 3(3\alpha + \alpha + \varepsilon) \\ & 12\alpha + 3\varepsilon. \end{aligned}$$

But then $S\varphi = S_1 + (S_0 - \mu) + \mu$ satisfies $\|S\varphi\|_{PM} \leqq 13\alpha + 4\varepsilon$. In particular,

$$\begin{aligned} (1-\varepsilon)\|\varphi\|_{A(E)} &\leqq |\langle S, \varphi\rangle| = |\langle S\varphi, 1\rangle| \\ &\leqq |(S\varphi)\hat{}(0)| \\ &\leqq \|S\varphi\|_{PM} \leqq 13\alpha + 4\varepsilon. \end{aligned}$$

Since ε is arbitrary, we have

$$\|\varphi\|_{A(E)} \leqq 13 \sup_k \|\varphi\|_{A(E_k)}.$$

Since we are interested in determining when B_E is finite for a given set E, we try to combine the conditions required for (BH) to hold and the bounded synthesis properties of the subsets E_k. The basic idea is that we attempt to build the set E from subsets E_k in such a way which guarantees that B_E is finite. Most constructions of sets E for which B_E is infinite are attained by producing disjoint portions E_k of E where $B_{E_k} \to \infty$. The example of Korner [7] further demonstrates that even if F_1 and F_2 are S-sets with $B_{F_1} = B_{F_2} = 1$ and $F_1 \cap F_2 = \{0\}$, it is possible that $B_{F_1 \cup F_2}$ is infinite. Here, we can consider E_0 as the set $F_1 \cap F_2 = \{0\}$, and let

$$E_k = E_0 \cup \{x \in F_1 \cup F_2 : \operatorname{dist}(x, 0) \geqq \varepsilon_k\},$$

where $\varepsilon_k \to 0$. Then (BH) holds but $\sup_k B_{E_k}$ is infinite. Our concern is to find conditions on E and $\{E_k\}$ which imply that B_E is finite. Our initial results lead us to some interesting consequences concerning the spectral synthesis properties of the union of two S-sets. First, we give a lemma which is well-known from the study of the union problem for Helson sets.

LEMMA 4. *Let E be a Helson set with Helson constant c. Then for every closed set F such that* $\operatorname{gp}(E) \cap F = \phi$ *and any measure μ supported by $E \cup F$*

we have

$$\|\mu|_E\|_{PM} \leqq c^2\|\mu\|_{PM}.$$

PROOF. See [2, Theorem 2.1.3].

THEOREM 3. *Let* $E = \bigcup_k E_k$, $E_k \subseteqq E_{k+1}$, *and suppose the limit set* $E_0 = \bigcap_k \overline{E \setminus E_k}$ *is a relatively open subset of* E_k *for each* k. *Suppose further that*

(i) E_0 *is a Helson S-set;*
(ii) $B = \sup_k B_{E_k}$ *is finite;*
(iii) $\mathrm{gp}(E_0) \cap E = E_0$.

Then B_E *is finite.*

PROOF. By (ii) and Proposition 1(i), it is sufficient to show that the sequence $\{E_k\}$ satisfies the Bochner-Herz property. Let $\varphi \in A$. Since E_0 is an S-set, there is a $\varphi_0 \in A$ such that $\|\varphi_0\| < 2\|\varphi\|_{A(E_0)}$ and $\varphi_0 = \varphi$ on a neighborhood of E_0. If there exists a constant b for which property (BH) holds whenever $\psi \in A$ has $\psi = 0$ on a neighborhood of E_0, we obtain

$$\begin{aligned} \|\varphi\|_{A(E)} &\leqq \|\varphi_0\| + \|\varphi_0 - \varphi\|_{A(E)} \\ &\leqq 2\|\varphi\|_{A(E_0)} + b\cdot \sup_k\|\varphi_0 - \varphi\|_{A(E_k)} \\ &\leqq 2\|\varphi\|_{A(E_0)} + b\|\varphi_0\| + b\cdot \sup_k\|\varphi\|_{A(E_k)} \\ &\leqq (2+3b)\sup_k\|\varphi\|_{A(E_k)}. \end{aligned}$$

Thus, we can assume that $\varphi = 0$ in a neighborhood of E_0. Now choose $\varepsilon > 0$ and let $S \in N(E)$, $\|S\|_{PM} \leqq 1$, such that $(1-\varepsilon)\|\varphi\|_{A(E)} < |\langle S, \varphi\rangle|$. Set

$$\alpha = \sup_k \|\varphi\|_{A(E_k)}.$$

We now proceed as in the proof of Theorem 2 to obtain pseudomeasures S_0 and S_1 satisfying $S_0 = S\varphi - S_1$, $\|S_1\|_{PM} \leqq \alpha$, and $\operatorname{supp} S_0 \subseteqq E_0$. Note that $S_0 \in M(E_0) \subseteqq N(E_k)$ for all k. Since φ vanishes on a neighborhood of E_0, $\operatorname{supp} S\varphi$ is disjoint from E_0 and therefore is contained in some E_{k_1}. But $S \in N(E)$, so there is an index $k \geqq k_1$ so that $S\varphi \in N(E_k)$, and therefore $S_1 = S\varphi - S_0 \in N(E_k)$. By (ii), there exists a sequence of measures $\{\mu_n\}$ in $M(E_k)$ satisfying

$$\sup_n \|\mu_n\|_{PM} \leqq B\|S_1\|_{PM} \leqq B\alpha \quad \text{and} \quad \mu_n \xrightarrow{w^*} S_1.$$

Set $\nu_n = \mu_n|_{E_0}$; by Lemma 4, we have $\|\nu_n\|_{PM} \leqq c^2\|\mu_n\|_{PM} \leqq c^2 B\alpha$ for each n, where c is the Helson constant of the set E_0. Since E_0 is a relatively open subset of E_k, it is clear that

$$\nu_n \xrightarrow{w^*} S_0 = S_1|_{E_0}.$$

Hence $\|S_0\|_{PM} \leqq c^2 B\alpha$. Writing $S\varphi = S_0 + S_1$, we find that $\|S\varphi\|_{PM} \leqq (1+c^2B)\alpha$. We conclude as in Theorem 2 that

$$\|\varphi\|_{A(E)} \leqq (1+c^2B)\sup_k \|\varphi\|_{A(E_k)}.$$

This proves the theorem.

REMARKS. 1. The above proof estimates the Bochner-Herz constant b as $b \leqq 5+3c^2B \leqq 8c^2B$.

2. Theorem 3 remains valid if we replace (iii) by the slightly weaker condition

(iii′) there is a countable set $H \subseteqq E$ such that $\mathrm{gp}(H) \cap E_0 = \emptyset$ and $\mathrm{gp}(E_0) \cap E \subseteqq E_0 \cup H$.

This follows because, given any measure μ with support in some E_k, we can use the result of ([5, Lemma 1]) to extract a measure ω with $\mathrm{supp}\,\omega \subseteqq H$, $\|\omega\|_{PM} \leqq 3\|\mu\|_{PM}$, and $\mathrm{supp}(\mu-\omega) \cap H = \emptyset$. If then $\{\mu_n\}$ synthesizes $S_1 \in N(E_k)$ as in the above proof, the measures $\nu_n = (\mu_n - \omega_n)|_{E_0}$ satisfy

$$\|\nu_n\|_{PM} \leqq c^2\|\mu_n - \omega_n\|_{PM} \leqq 4c^2\|\mu_n\|_{PM} \leqq 4c^2B\|S_1\|_{PM} \quad \text{and} \quad \nu_n \xrightarrow{w^*} S_0.$$

Thus, the conclusion of the theorem still holds. The Bochner-Herz constant will now satisfy $b \leqq 20c^2B$.

3. Example 3 shows that Theorem 3 is false if we do not require that E_0 be an S-set.

We now obtain the following corollary concerning the union problem for sets of spectral synthesis.

COROLLARY 2. *Let K_1 and K_2 be S-sets. Suppose there exist closed sets $F_k \subseteqq K_1 \cup K_2$, $k \geqq 1$, such that $F_k \subseteqq F_{k+1}$, $K_1 \cup K_2 = \overline{\bigcup_k F_k}$, and $K_1 \cap K_2 = \bigcap_k (K_1 \cup K_2)\setminus F_k$. Suppose further that*

(i) *$K_1 \cap K_2$ is a relatively open subset of F_k for each k;*
(ii) *$K_1 \cap K_2$ is a Helson S-set;*
(iii) *$\mathrm{gp}(K_1 \cap K_2) \cap (K_1 \cup K_2) = K_1 \cap K_2$;*
(iv) *$B = \sup_k B_{F_k}$ is finite;*
(v) *$K_1 \cup K_2$ has measure 0.*

Then $K_1 \cup K_2$ is a set of spectral synthesis.

PROOF. Set $E_0 = K_1 \cap K_2$, $F = K_1 \cup K_2$, and assume that F disobeys spectral synthesis. Then there exists a pseudomeasure S, $\|S\|_{PM} \leqq 1$,

$\operatorname{supp} S \subseteqq F$, and a function φ in A with $\varphi = 0$ on F such that $\langle S, \varphi \rangle \geqq M$, where M is a positive constant to be determined later. Our proof consists of applying the method of Katznelson and McGehee ([6, Theorem VI]; see Example 3) in order to obtain a contradiction. We may assume, as done in the proof of ([6, Theorem VI]), that F contains no rational multiples of π. Since F has measure 0, their construction yields an increasing sequence of finite sets $H_k \subseteqq \{r\pi : r \text{ rational}\}$, $k \geqq 1$, for which all of the limit points of $\bigcup_k H_k$ are in F, and so that the set $E = F \cup \bigcup_k H_k$ is an S-set and yet $B_E \geqq M$. In particular, we obtain

(4) $\qquad \|\varphi\|_{A(E)} \geqq M,$ and

(5) $\qquad |\langle \mu, \varphi \rangle| \leqq \|\mu\|_{PM}$ for each measure μ supported by E.

Set $H = \bigcup_k H_k$, and note that H is a countable set satisfying condition (iii′) in Remark 2 above for the sets E and E_0. We now proceed to define sets E_k for which the conditions of Theorem 3 hold. First, note that since $F = \bigcup_k F_k$ is a totally disconnected set, we can find closed sets $D_k \subseteqq F_k$ and compact neighborhoods V_k of D_k, with $V_k \subseteqq V_{k+1}$, satisfying

(6) $\qquad E_0 \cap V_k = \phi;$

(7) $\qquad F \cap V_k = D_k;$

(8) $\qquad F = E_0 \cup \bigcup_k D_k;$

(9) $\qquad E_0 = \bigcap_k \overline{F \setminus D_k}.$

For each k, let $P_k = H_k \cup (V_k \cap H)$, and set $E_k = F_k \cup P_k$. Then $E = \bigcup_k E_k$ and $E_k \subseteqq E_{k+1}$; also, by (i), (6), and the fact that H_k is a finite set, E_0 is a relatively open subset of E_k.

Now suppose that $T \in N(E_k)$ for some k. Let P_0 be any finite subset of P_k, and let $\nu = \nu_P$ be a measure determined by ([5, Lemma 1]) satisfying conditions (1), (2), and (3) for some finite set $P \subseteqq \operatorname{gp}(P_0) \subseteqq \operatorname{gp}(H)$. The sequence $\{T(\nu * f_n)\}_n \subseteqq PM$, being uniformly bounded in norm, has a subsequence that converges w^* to a measure μ whose PM norm is bounded by $3\|T\|_{PM}$, which has finite support in the set $P \cap \operatorname{supp} T \subseteqq P_k$, and for which $\mu = T$ in a neighborhood of P_0. By taking an increasing family of finite subsets which exhaust P_k, we can find a sequence $\{\mu_n\} \subseteqq M(P_k)$ with $\sup_n \|\mu_n\|_{PM} \leqq 3\|T\|_{PM}$ and which converges w^* to a pseudomeasure T_P for which $\operatorname{supp}(T - T_p) \subseteqq F_k$. Set $T_1 = T - T_P$. Note that D_k, being both an open and closed subset of F disjoint from E_0, is an S-set. Hence T_1, having support in the disjoint sets D_k and $F_k \setminus D_k$, equals T on a neighborhood of $F_k \setminus D_k$. Since $T \in N(E_k)$, this implies that $T_1 \in N(F_k)$. We therefore find that, for each k and every T in $N(E_k)$, T is the w^*-limit of measures from $M(E_k)$ whose norms are bounded by $4B_{F_k}\|T\|_{PM}$. Thus, $\sup_k B_{E_k} \leqq 4B$. Theorem 3 and Remark 2 following now imply that the sequence $\{E_k\}$ satisfies the Bochner-Herz property with constant $b \leqq 80c^2B$, where c is the Helson constant of the set E_0.

However, whenever $T = T_1 + T_P \in N(E_k)$ as above, since $\varphi = 0$ on $K_1 \cup K_2 \supseteqq F_k \supseteqq \operatorname{supp} T_1$, we have $\langle T_1, \varphi\rangle = 0$. Thus, (5) gives

$$\begin{aligned} |\langle T, \varphi\rangle| &= |\langle T_1, \varphi\rangle + \langle T_P, \varphi\rangle| = |\langle T_P, \varphi\rangle| \\ &= \lim_n |\langle \mu_n, \varphi\rangle| \\ &\leqq 3\|T\|_{PM} \end{aligned}$$

for all T in $N(E_k)$. We conclude that $\|\varphi\|_{A(E_k)} \leqq 3$ for all $k \geqq 1$, that is

$$\sup_k \|\varphi\|_{A(E_k)} \leqq 3.$$

This, together with (4), forces $b \geqq M/3$. By choosing $M > 240c^2B$, we obtain our contradiction. Thus, $K_1 \cup K_2$ is a set of spectral synthesis.

We note that Corollary 2 remains valid when $K_1 \cap K_2 \subseteqq E_0 \subseteqq K_1$. The condition that $K_1 \cup K_2$ has measure zero is required for the construction of Katznelson and McGehee. Even though one would not expect the measure of a set to be directly related to its spectral synthesis properties in general, it is likely that condition (v) is consistent with the requirements imposed by the conditions (ii)–(iv).

EXAMPLE. 4. Let E_1 and E_2 be disjoint perfect sets whose union is a Kronecker set ([8, §5.1.2]). Then $A(E_1 + E_2)$ is isometric to the tensor algebra $V = C(E_1 \hat{\otimes} C(E_2)$ via the canonical identification of $E_1 + E_2$ with $E_1 \times E_2$ (see [12]). Let $g: E_1 \to E_2$ be a nondecreasing continuous function; it is then easy to check that the set

$$F = \{x + y : x \in E_1, y = g(x)\}$$

is a Kronecker set, and so a Helson S-set. Let K_1 and K_2 be S-set subsets of $E_1 + E_2$ whose intersection $E_0 = K_1 \cap K_2$ is contained in F. Then E_0 is a Helson S-set satisfying $\operatorname{gp}(E_0) \cap (K_1 \cup K_2) = E_0$. Set

$$E_0^* = \{(x, y) : x \in E_1, y = g(x)\}.$$

For any neighborhood W^* of E_0^* we can find a finite collection $R_1, \ldots, R_p$ of disjoint rectangular sets which are both open and closed in $E_1 \times E_2$, whose projections onto E_1 and E_2 are mutually disjoint, and for which $E_0^* \subseteqq \bigcup_j R_j \subseteqq W^*$. Therefore, there exist functions $\{f_n^*\}$ in V for which $\sup_n \|f_n^*\|_V < \infty$, and open and closed neighborhoods W_n^* of E_0^*, such that

$\bigcap_n W_n^* = E_0^*$, $f_n^* = 1$ on W_n^*, and $f_n^* = 0$ off W_n^* (see [11, Theorem 1.2]). Hence, there exist functions $\{f_n\}$ in $A(E_1+E_2)$, $\sup_n \|f_n\|_{A(E_1+E_2)} < \infty$, and neighborhoods W_n of E_0 such that $\bigcap_n W_n = E_0$, $f_n = 1$ on $(E_1+E_2) \cap W_n$, and $f_n = 0$ on $(E_1+E_2) \cap W_n^c$. Let K denote the union $K_1 \cup K_2$, and set $G_k = K \cap W_k^c$, $k \geqq 1$. Since K_1 and K_2 are S-sets, G_k is an S-set. If $\sup_k B_{G_k} < \infty$, then Corollary 2 shows that K is an S-set. But even when $\sup_k B_{G_k}$ is not finite, we can still use Corollary 2 to show that K is an S-set. For suppose that $S \in PM(K)$ is not synthesizable on K. Then for each index k we can apply the Herz criterion to find a countable set H_k, all of whose limit points are in G_k, so that $G_k' = G_k \cup H_k$ has $B_{G_k'} = 1$, and yet $S \notin N(K \cup \bigcup_k H_k)$. Furthermore, since E_0 is an independent set, the H_k can be chosen so that $\mathrm{gp}(\bigcup_k H_k) \cap \mathrm{gp}(E_0) = \{0\}$. Now let $E = K_1' \cup K_2'$, where, for $j = 1, 2$, we write $H_k = H(k,1) \cup H(k,2)$, with all of the limit points of $H(k,j)$ lying in K_j, and $K_j' = K_j \cup \bigcup_k H(k,j)$, and then set $F_k = E \cap W_k^c$. It is easy to show that all of the hypotheses of the corollary are satisfied for $K_1' \cup K_2'$ and the sequence $\{F_k\}$. Hence $K_1' \cup K_2'$ is an S-set, and this contradicts the fact that $S \notin N(K \cup \bigcup_k H_k)$. Thus, $K_1 \cup K_2$ is a set of spectral synthesis.

The sets E_1 and E_2 used in Example 4 can be generalized slightly by using the results of Kaijser [3], where a topological isomorphism is established between $A(E_1 + \cdots + E_n)$ and the tensor algebra $C(E_1)\hat{\otimes}\cdots\hat{\otimes}C(E_n)$ for certain Helson sets $E_1 \cup \cdots \cup E_n$. The set F in the example can also be generalized (e.g., the finite uniton of such sets also works). The property used in the proof of Corollary 2, and what is implied by the conditions (ii)–(iv), is that, for each k and any S in $N(F_k)$, the norm $\|S|_{E_0}\|_{PM}$ is bounded by $\|S\|_{PM}$ times a fixed constant independent of S and k. The functions $\{f_n\}$ in Example 4 provide us with this property for the set E_0 relative to E_1+E_2. We therefore make the following definition (cf. Ditkin sets, [2, pp. 71–73]). For f in A, we write $\mathrm{supp}_E f$ for the closure of the set $\{x \in E : f(x) \neq 0\}$.

DEFINITION 2. A compact subset F of a closed set E is called an E-Ditkin set, if there exist functions $\{f_n\} \subseteqq A(E)$, with $\sup_n \|f_n\|_{A(E)} < \infty$, and neighborhoods W_n of F so that $f_n = 1$ on $E \cap W_n$, $\mathrm{supp}_E f_{n+1} \subseteqq \mathrm{supp}_E f_n$, and $\bigcap_n \mathrm{supp}_E f_n = F$.

Note that unlike Ditkind sets, which are sets of spectral synthesis, an E-Ditkin set might not be an S-set even if E is an S-set. In fact, Example 3 shows that while E_0 is an E-Ditkin set, we still have $PM(E_0) \cap N(E) \neq N(E_0)$. We now give versions of the last two results obtained with E-Ditkin sets.

THEOREM 4. *Let* $E = \bigcup_k E_k$, $E_k \subseteqq E_{k+1}$, *and suppose that* $E_0 = \bigcap_k \overline{E \setminus E_k}$ *is an E-Ditkin set. If* E_0 *is an S-set, then the sequence* $\{E_k\}$ *satisfies property* (BH).

COROLLARY 3. *Let K_1 and K_2 be S-sets of measure zero. Suppose there exists a closed set E_0 with $K_1 \cap K_2 \subseteqq E_0 \subseteqq K_1$ satisfying*

(i) *E_0 is an S-set, and*
(ii) *E_0 is a $(K_1 \cup K_2)$-Ditkin set.*

Then $K_1 \cup K_2$ is a set of spectral synthesis.

Finally, we remark that the proof of Corollary 2 can be adapted to obtain the following result concerning pseudofunctions. However, this fact has a simple proof that we present below.

PROPOSITION 2. *Let K_1 and K_2 be S-sets whose intersection $K_1 \cap K_2$ is a U-set* (i.e., $PF(K_1 \cap K_2) = \{0\}$). *Then every pseudofunction supported by $K_1 \cup K_2$ is synthesizable on $K_1 \cup K_2$.*

PROOF. Let K_1 and K_2 be as in the hypotheses, and let $S \in PF(K_1 \cup K_2)$. If $\varphi \in A$ for which $\varphi = 0$ on $K_1 \cup K_2$, it is enough to show that $S\varphi = 0$. Since K_1 and K_2 are S-sets, we have $\operatorname{supp} S\varphi \subseteqq K_1 \cap K_2$. But S being a pseudofunction implies that $S\varphi$ is also a pseudofunction. Hence, $K_1 \cap K_2$ being a U-set forces $S\varphi = 0$.

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