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PHRAGMÉN-LINDELÖF THEOREMS FOR SUBHARMONIC FUNCTIONS ON THE UNIT DISK

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Dedicated to Professor Maurice Heins on the occasion of his retirement.

1. Introduction.

Let Δ denote the open unit disk $\{|z| < 1\}$ and C its circumference $\{|z| = 1\}$. Let u be a subharmonic function defined in Δ, u^+ the subharmonic function $\max\{u, 0\}$,

$$u_*(\zeta) = \liminf_{r \to 1} u(r\zeta), \quad \zeta \in C,$$

and

$$M(r; u) = \max\{u(r\zeta) : \zeta \in C\}.$$

If Ω is a subset of Δ and $\zeta \in C$, then

$$\zeta\Omega = \{\zeta : z \in \Omega\}$$

is the rotate of Ω by ζ. Assuming 1 is a limit point of Ω, define

$$u_\Omega^*(\zeta) = \limsup_{z \to \zeta} u(z), \quad z \in \zeta\Omega.$$

For the special case when Ω is the radius $[0, 1)$ of Δ, we write u^* instead of u_Ω^*.

The classical maximum principle for subharmonic functions defined in Δ may be stated as follows.

$$\textit{If } u_\Delta^*(\zeta) \leqq 0 \textit{ for all } \zeta \in C, \textit{ then } u(z) \leqq 0 \textit{ for all } z \in \Delta.$$

The idea of Phragmén and Lindelöf [12] was to allow a small exceptional set and add a growth condition so that the same conclusion can be drawn.

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For example:

If E is a finite subset of C such that $u_{\Delta}^{*}(\zeta) \leqq 0$, $\zeta \in C-E$, and

$$u(z) = o(|\zeta - z|^{-1}) \quad \textit{as} \quad z \to \zeta \textit{ for each } \zeta \in E,$$

then $u(z) \leqq 0$ for all $z \in \Delta$.

(Basic exposition of this subject can be found in [17; pp. 176–186] and [9; Chapter 4, section 8, and Chapter 5, section 9].)

The central questions that we consider here are the following.

(1). What refinements can be made in the classical maximum principle when a growth condition is placed on $M(r;u)$?

(2). What kinds of Phragmén-Lindelöf exceptional sets are allowable with such a growth condition?

Consider first the extreme case where u is assumed to be bounded above. Littlewood [11] showed that in this case, u has finite radial limits ($-\infty < u_{*} = u^{*} < +\infty$) a.e. with respect to linear Lebesgue measure on C with the essential supremum of u^{*} equal to the supremum of u. This leads to a *radial* Phragmén-Lindelöf theorem:

If E is a subset of C having linear measure $|E| = 0$ and u is bounded above with $u^{*}(\zeta) \leqq 0$ for each $\zeta \in C-E$, then $u(z) \leqq 0$ for all $z \in \Delta$.

Thus under the assumption that there exists a constant $c < +\infty$ such that $M(r;u) \leqq c$, we need only consider the *radial* limit superior $u^{*}(\zeta)$ for $\zeta \in C$, and we can allow a fairly large exceptional set, any set of linear measure 0.

In [7], Dahlberg considered the question of what growth conditions on $M(r;u)$ still give rise to *radial* Phragmén-Lindelöf theorems. The following is a version of [7; Theorem 2]:

If E is a countable subset of C,

$$M(r;u) = o[(1-r)^{-2}] \quad \textit{as } r \to 1,$$

$$u^{+}(r\zeta) = o[(1-r)^{-1}] \quad \textit{as } r \to 1 \quad \textit{for each } \zeta \in E,$$

and

$$u^{*}(\zeta) \leqq 0 \quad \textit{for } \zeta \in C-E,$$

then $u(z) \leqq 0$ for all $z \in \Delta$.

(In this section we shall not state results in their greatest generality. More complete versions of some of them appear in section 2.) Dahlberg also showed that $o[(1-r)^{-2}]$ is the critical "global" growth rate for such radial maximum principles by giving an example of an unbounded harmonic function h with

$$M(r;|h|) = O[(1-r)^{-2}] \quad \text{as} \quad r \to 1$$

and

$$h^*(\zeta) = h_*(\zeta) = 0 \quad \text{for each } \zeta \in C.$$

Standard examples of positive harmonic functions show that the "local" growth rate $o[(1-r)^{-1}]$ placed along radii ending in the exceptional set E cannot be weakened to $O[(1-r)^{-1}]$, and that the result is false if E is any *uncountable* Borel set.

A second result of Dahlberg dealt with functions of particular slow growth [7; Theorem 4]. Two sharp generalizations were given by the authors in [5]. For the statement of the one given below and in the sequel, let ω denote an arbitrary continuous increasing concave-downward function on $[0, 2\pi]$ vanishing at 0 and satisfying $\omega'(0) = \infty$ and let H_ω denote the Hausdorff measure on C with generating function ω. Recall that for each Borel subset E of C, we have

$$H_\omega(E) = \lim_{t\to 0}\left[\inf\left\{\sum_{A\in \mathcal{O}} \omega(|A|)\right\}\right],$$

where the infimum is taken over all countable covers $\mathcal{O}$ of E by open arcs A having linear measure $|A| \leqq t$. (See [13] for general information concerning Hausdorff measures.) The next result is a version of [5; Theorem 2] (cf. [7; Theorem 4] where $\omega(t) = t^\alpha$, $0 < \alpha < 1$, and E is a countable union of closed sets).

If E is a Borel subset of C with $H_\omega(E) = 0$,

$$M(r;u^+) = O[\omega(1-r)/(1-r)] \quad \textit{as } r \to 1,$$

and $u^*(\zeta) \leqq 0$ for $\zeta \in C-E$, then $u(z) \leqq 0$ for all $z \in \Delta$.

The condition $H_\omega(E) = 0$ was shown to be sharp. Note that here, the global growth rate is slower than both the global and local growth rate in the previous result.

To summarize the radial case, the maximal global growth rate for a radial maximum principle is $o[(1-r)^{-2}]$. As long as a local growth rate of $o[(1-r)^{-1}]$ is placed along radii ending in the exceptional set, we can allow countable Phragmén-Lindelöf exceptional sets. (Here and in the sequel, the *local* 'o' or 'O' growth rate is *not* required to occur uniformly over the points of the exceptional set E.) For slower global growth rates that are $o[(1-r)^{-1}]$, a precise accounting of allowable Phragmén-Lindelöf exceptional sets is given in terms of Hausdorff measure.

In view of these results, it is natural to ask whether the global growth rate $o[(1-r)^{-2}]$ coupled with a local growth condition slower than $o[(1-r)^{-1}]$ allows larger exceptional sets. We show that

if for each radius ending in the exceptional set E, the radial growth condition is $O[\omega(1-r)/(1-r)]$, *then we can allow E to be a countable union of* ω*-sets.*

These sets are studied in [6] and are defined as follows. If K is a closed subset of C with $|K| = 0$ and (I_k) is an enumeration of the component arcs of $C-K$, then K is said to be an ω-set provided $\sum\omega(|I_k|) < \infty$.

Suppose now that we pass to faster global growth rates so that a radial maximum principle is not possible. Can we still improve the classical maximum principle? We show that this is always possible regardless of the growth condition.

Consider first the global growth condition

$$M(r;u) = o[(1-r)^{-\pi/\alpha}] \quad \text{as } r \to 1,$$

where $\alpha \in (0, \pi/2)$. Let Ω_α denote a sector in Δ with vertex 1 that is symmetric about the x-axis and has angular opening $\pi - 2\alpha$. We prove the following sharp sectorial maximum principle:

If u satisfies the global growth condition just given and $u^*_{\Omega_\alpha}(\zeta) \leqq 0$ *for each* $\zeta \in C$, *then* $u(z) \leqq 0$ *for all* $z \in \Delta$.

With regard to Phragmén-Lindelöf exceptional sets E, we can again allow countable unions of ω-sets provided the local growth condition

$$O[\omega(|\zeta - z|)/|\zeta - z|] \quad \text{as } z \to \zeta \text{ is satisfied in } \zeta\Omega_\alpha \text{ for each } \zeta \in E.$$

More generally, for a given global growth condition on $M(r;u)$, we can always find a tangential set Ω that is "thinner" than Δ at 1 such that $u^*_\Omega(\zeta) \leqq 0$ for all $\zeta \in C$ implies $u(z) \leqq 0$ for every $z \in \Delta$. Fairly accurate estimates of the relationship between the growth rate and the tangentiality of Ω are made possible by a theorem of Warschawski [18; Theorem XI (A)]. In addition we show that countable unions of ω-sets can be taken as exceptional sets E if ω is continuously differentiable and the local growth condition $O[\omega'(|\zeta - z|)]$ is imposed in $\zeta\Omega$ for each $\zeta \in E$.

Returning finally to the classical case where the global growth condition is completely dropped and $\Omega = \Delta$, we arrive back at the Phragmén-Lindelöf theorem originally given but we find that the exceptional set E can be taken to be any set of linear measure 0.

The paper is organized as follows. In section 2 the main results are stated in full generality. In section 3 we modify proofs given by Dahlberg to prove Proposition 1. This result is a refinement of the classical maximum principle in the presence of a global growth condition (with exceptional set $E = \phi$). In section 4 we define auxiliary functions depending on ω having certain growth properties towards points of an allowed exceptional set E. In section 5 we use these auxiliary functions in conjunction with Proposition 1 to prove our main results. Explicit computations are also given in which growth conditions are estimated using Warschawski's theorem. Section 6 is devoted to applications concerning level sets and the zero sets of the radial-limit functions of analytic functions of prescribed growth. In section 7 we generalize a Phragmén-Lindelöf theorem of Dahlberg for the unit ball in R^n, and we conclude with a discussion of the question of whether the size of the exceptional sets in our main results is best possible.

2. Main results.

Along with the notation established in section 1, we will need the following. Let P denote the Poisson kernel

$$P(z,t) = \frac{1-|z|^2}{|e^{it}-z|^2}, \quad z \in \Delta, t \in \mathsf{R},$$

where R is the set of real numbers, and corresponding to each Borel measure μ on C, let

$$P[d\mu](z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(z,t)d\mu(t), \quad z \in \Delta.$$

If the integration is taken over an arc A of C, we shall interpret this to be $P[d\nu]$ where ν is the Borel measure on C for which $\nu|_A = \mu|_A$ and $\nu|_{C-A} \equiv 0$. When $d\mu$ is of the form $f(e^{it})dt$ where $f \in L^1$ (that is, f is a real-valued function defined a.e. on C and integrable with respect to linear Lebesgue measure). The notation $[Pf](z)$ will sometimes be used instead of $P[d\mu](z)$.

The following is a more general statement of Dahlberg's radial Phragmén-Lindelöf theorem [7; Theorem 2] stated in a simplified form in section 1.

THEOREM A. *Let E be a countable subset of C. If*

(1) $\quad M(r;u) = o[(1-r)^{-2}]$ *as* $r \to 1$,

(2) $\quad$ *there exists* $g \in L^1$ *such that* $u_* \leqq g$ a.e.,

(3) $\quad u^*(\zeta) < \infty$ *for each* $\zeta \in C - E$, *and*

$$(4) \qquad u^+(r\zeta) = o[(1-r)^{-1}] \text{ as } r \to 1 \text{ for every } \zeta \in E,$$

then $u^* \in L^1$, $u_* = u^*$ a.e., *and* $u \leqq Pu_*$.

For the statement of our main result concerning the sectorial and radial cases, we use the notation

$$(5) \quad \Omega_\alpha = \{z \in \Delta : |z-1/2| \leqq 1/2, \alpha + \frac{\pi}{2} \leqq \operatorname{Arg}(z-1) \leqq -\alpha + 3\pi/2\}, \quad \alpha \in (0, \pi/2].$$

In particular, $\Omega_{\pi/2}$ denotes the radius $[0, 1)$.

THEOREM 1. *Let E be a countable union of ω-sets, $\alpha \in (0, \pi/2]$, and $\Omega = \Omega_\alpha$. If u satisfies* (2),

$$(3') \qquad u^*_\Omega(\zeta) < \infty \quad \text{for each } \zeta \in C - E,$$

$$(6) \qquad M(r; u^+) = o[(1-r)^{-\pi/\alpha}] \quad \text{as } r \to 1,$$

and for each $\zeta \in E$ there exists a positive constant $c = c(\zeta)$ such that

$$(7) \qquad u(z) \leqq c\frac{\omega(|\zeta - z|)}{|\zeta - z|}, \quad z \in \zeta\Omega,$$

then $u^* \in L^1$, $u_* = u^*$ a.e., *and* $u \leqq Pu_*$.

Notice that Theorem A can be regarded as the limiting case of Theorem 1 where $\alpha = \pi/2$ and $\omega \equiv 1$.

Combining (2) and (3') by requiring $u^*_\Omega(\zeta) \leqq 0$ for each $\zeta \in C - E$ leads to the radial and sectorial Phragmén-Lindelöf theorems stated in section 1. The global growth condition is sharp in the sense that 'o' cannot be changed to 'O'. We shall demonstrate this using examples in section 5.

The next theorem is a generalized form of our result for faster global growth conditions. In it, the global growth rate is given in terms of the growth of a certain conformal map. In order to state this result, we shall need to establish some additional notation. Let

$$\Gamma : [0, 1] \to \{z \in \Delta : 0 \leqq \operatorname{Arg} z \leqq \pi/2\} \cup \{0, 1\}$$

be a Jordan arc such that $|\Gamma(t)| = t$ for each $t \in [0, 1]$. Let $\Omega^* = \Omega^*(\Gamma)$ be the Jordan region contained in Δ that is bounded by Γ, $\{\zeta \in C : 0 \leqq \operatorname{Arg} \zeta \leqq \pi/2\}$, and $\{iy : 0 \leqq y \leqq 1\}$. Thus Ω^* consists of those points in Δ that are are in the first quadrant but outside of Γ. Let

$$\Omega = \Omega(\Gamma) = \{z \in \Delta : -\pi/2 \leqq \operatorname{Arg} z \leqq \pi/2 \quad \text{and} \quad z, \bar{z} \notin \Omega^*\},$$

where $\bar{z}$ denotes the complex conjugate of z. We call Γ (along with Ω *and* Ω^*)

tangential if

$$\lim_{t\to 1}\,[\operatorname{Re}\Gamma(t)-1]/\operatorname{Im}\Gamma(t) = 0.$$

For an arbitrary allowed Γ, define φ to be a homeomorphism that maps the closure $\overline{\Omega^*}$ of Ω^* onto $\{\operatorname{Re} w \geqq 0\} \cup \{\infty\}$ (in the extended w-plane) taking Ω^* conformally onto $\{\operatorname{Re} w > 0\}$ with $\varphi(0) = 0$ and $\varphi(1) = \infty$. (The existence of φ follows, with the aid of an appropriate Möbius transformation, from a theorem of Carathéodory concerning Riemann mappings onto simply-connected Jordan regions. In fact, φ would be uniquely determined by specifying $\varphi(i)$ to be a value on the positive imaginary axis. See standard texts such as [9] and [14] for details.) For each $\lambda \in [-\pi/2, \pi/2]$, let

$$R_\lambda = \{\operatorname{Arg} w = \lambda\} \cup \{0, \infty\}$$

and let Γ_λ be the Jordan arc with image $\varphi^{-1}(R_\lambda)$ and parametrization given as follows: $\Gamma_\lambda(s)$ is the unique point $z_s \in \varphi^{-1}(R_\lambda)$ such that $|\varphi(z_s)| = s$ for each $s \in [0, \infty]$. Observe that $\Gamma_{-\pi/2}$ is a reparametrization of Γ and the image of $\Gamma_{\pi/2}$ is

$$\{\zeta \in C : 0 \leqq \operatorname{Arg}\zeta \leqq \pi/2\} \cup \{iy : 0 \leqq y \leqq 1\}.$$

For every $\lambda \in (-\pi/2, \pi/2]$, define $\Omega^*(\lambda)$ to be the Jordan region in Ω^* bounded by $\Gamma_{-\pi/2}$ and Γ_λ, and let

$$\mu_\lambda(r) = \inf\{|\varphi(z)| : |z| = r, z \in \Omega^*(\lambda)\}, \quad r \in [0, 1). \tag{8}$$

THEOREM 2. *Suppose that ω has a continuous first derivative on* $(0, 2\pi]$ *and that E is a countable union of ω-sets. Assume that an arbitrary allowed Γ along with Ω, Ω^*, and φ are given. If u satisfies* (2), (3′),

$$M(r; u) = o[\mu_\lambda(r)] \quad \text{as } r \to 1 \tag{9}$$

for some $\lambda \in (-\pi/2, \pi/2]$, and for each $\zeta \in E$ there exists a positive constant $c = c(\zeta)$ such that

$$u(z) \leqq c\omega'(|\zeta - z|), \quad z \in \zeta\Omega, \tag{10}$$

then $u^* \in L^1$, $u_* = u^*$ a.e., *and $u \leqq Pu_*$.*

Once again this can be put in a simpler form by requiring $u^*_\Omega(\zeta) \leqq 0$ for each $\zeta \in C - E$ instead of (2) and (3′).

With the aid of a theorem of Warschawski (see Theorem C of section 5), we can obtain estimates of the global growth rate (8) under very modest hypotheses on Γ. To describe these hypotheses, we introduce a polar coordinate system. Let ϱ and ϕ denote the radial and angular coordinates in the z-plane relative to a polar axis parallel to the positive imaginary axis and pointing in the same direction with pole at 1. We assume that in some neighborhood of 1, the arc Γ admits a polar representation

$$\phi = \Phi^{+}(\varrho), \quad 0 < \varrho < a,$$

where $a \in (0, 1)$. Also let

$$\phi = \Phi^{-}(\varrho) = \sin^{-1}(\varrho/2), \quad 0 < \varrho < a,$$

(a polar representation of the arc of C contained in $\{\operatorname{Im} z > 0\}$ with one endpoint 1 and the other a distance of a from the point 1). Set

$$\Theta(\varrho) = \Phi^{+}(\varrho) - \Phi^{-}(\varrho), \quad 0 < \varrho < a,$$

and suppose that $\Theta(\varrho)$ is a monotone nondecreasing continuously differentiable function on $(0, a)$ satisfying

$$\lim_{\varrho \to 0} \varrho[d\Theta(\varrho)/d\varrho] = 0.$$

Define

$$\Lambda(\varrho) = \varrho\Theta(\varrho), \quad 0 < \varrho < a.$$

Note that $\Lambda(\varrho)$ is a strictly-increasing continuously-differentiable function. The next result is proved using Theorem 2 and Warschawski's theorem.

THEOREM 3. *Let $\varepsilon \in (0, \pi)$. Theorem 2 remains valid when* (9) *is replaced by*

$$\text{(11)} \qquad M(r; u) = o\left\{\exp\left[(\pi - \varepsilon) \int_{(1-r)(1+\varepsilon)}^{\Lambda(a)} \frac{(\Lambda^{-1})'(t)}{t} dt\right]\right\} \quad \text{as } r \to 1.$$

In the opposite direction, there exists an unbounded nonnegative subharmonic function u defined in Δ such that

$$\text{(12)} \qquad M(r; u) = o\left\{\exp\left[(\pi + \varepsilon) \int_{(1-r)(1-\varepsilon)}^{\Lambda(a)} \frac{(\Lambda^{-1})'(t)}{t} dt\right]\right\} \quad \text{as } r \to 1,$$

with $\lim_{z \to \zeta} u(z) = 0$ *for every* $\zeta \in C - \{1\}$ *and* $u(z) = 0$ *for each* $z \in \Omega$.

If $\Theta(\varrho) = \varrho^{\beta}$ where $\beta \in (0, \infty)$, then Γ has the same order of tangency relative to C at 1 as the the curve given in cartesian coordinates by $y = x^{\beta+1}$, $x \geqq 0$, relative to the x-axis at the origin. A direct computation shows that in this case (11) and (12) can be replaced by

$$M(r; u) = 0\left\{\exp\left[\frac{\pi \mp \varepsilon}{\beta} \frac{1}{(1-r)^{\beta/(\beta+1)}}\right]\right\} \quad \text{as } r \to 1.$$

Given any decreasing positive function v on $(0, 1]$ such that $\lim_{t\to 0} v(t) = \infty$, we can use (11) to find $\Theta(\varrho)$ with associated growth rate to ∞ faster than $v(1-r)$ as $r \to 1$. This leads to the following Phragmén-Lindelöf theorem for functions of a specified growth.

THEOREM 4. *Suppose that v is as stated above, ω is continuously differentiable, and E is a countable union of ω-sets. Then there exists a Jordan arc Γ along with the associated set Ω having the following property. If u satisfies* (2), (3′),

$$M(r;u) = o[v(1-r)] \quad \textit{as } r \to 1, \tag{13}$$

and for each $\zeta \in E$ there exists a positive constant $c = c(\zeta)$ such that (10) *holds, then $u^* \in L^1$, $u^* = u_*$* a.e., *and $u \leqq Pu_*$.*

COROLLARY 4. *Suppose that E, v, and Ω are as stated in Theorem 4. If $u^*_\Omega(\zeta) \leqq 0$ for each $\zeta \in C - E$, the global growth condition* (13) *is satisfied, and for each $\zeta \in E$, there exists $c = c(\zeta)$ such that* (10) *holds, then $u(z) \leqq 0$ for all $z \in \Delta$.*

Finally, we give a generalized version of the classical Phragmén-Lindelöf theorem where no global growth condition is imposed.

THEOREM 5. *Let E be a subset of C having linear measure* 0. *If u satisfies* (2), (3′) *with $\Omega = \Delta$, and*

$$u(z) = o(|\zeta - z|^{-1}) \quad \textit{as } z \to \zeta,\ \zeta \in E,$$

then $u^* \in L^1$, $u^* = u_*$ a.e., *and $u \leqq Pu_*$.*

3. Modification of Dahlberg's results.

The two results proved in this section are related to Dahlberg's [7; Lemma 1] and [7; Theorem 2]. We shall use the notation established in sections 1 and 2. Proposition 1 of this section is just Theorem 2 for the case when $E = \emptyset$.

We start by recalling a standard Phragmén-Lindelöf theorem (see for example [17; section 5.62, pp. 178–179] for the version for analytic functions) and then give an immediate corollary that is applicable to the present context.

In the following theorem, let $H = \{\operatorname{Re} w > 0\}$ and let ∂H denote its boundary, the imaginary axis.

THEOREM B. *Let v be a subharmonic function defined in the right half-plane H. If*

$$\limsup_{z\to\zeta} v(z) \leqq 0, \quad \zeta \in \partial H,$$

and

$$\sup\{v^+(w): w \in H, |w| = r\} = o(r) \quad \text{as } r \to \infty,$$

then $v(w) \leqq 0$ *for all* $w \in H$.

COROLLARY B. *Let* $\varphi: \Omega^* \to H$ *be the conformal mapping given in section* 2. *If* N *is a neighborhood of* 1 *such that*

$$\limsup_{z \to \zeta} u(z) \leqq 0, \quad \zeta \in \partial(\Omega^* \cap N) - \{1\}, \tag{14}$$

and

$$u^+(z) = o[|\varphi(z)|] \quad \text{as } z \to 1, \tag{15}$$

then $u(z) \leqq 0$ *for all* $z \in \Omega^* \cap N$.

The corollary follows upon applying Theorem B to the subharmonic function

$$v(w) = \begin{cases} (u \circ \varphi^{-1})^+(w), & w \in \varphi(N \cap \Omega^*), \\ 0, & w \in \varphi(N - \Omega^*). \end{cases}$$

The following lemma is an analogue of [7; Lemma 1] with a somewhat different proof. It gives conditions involving $M(r;u)$ which lead to the conclusion that (15) holds.

LEMMA 1. *Let* A *be an open arc in* $C \cap \{\operatorname{Im} z > 0\}$ *with one endpoint* 1. *Suppose that* (9) *is satisfied for some* $\lambda \in (-\pi/2, \pi/2)$. *If*

$$\limsup_{z \to \zeta} u(z) \leqq 0, \quad \zeta \in A, \tag{16}$$

then (15) *holds.*

PROOF. Let λ be as in the statement of the lemma. Then Γ_λ divides Ω^* into two Jordan regions: $\Omega^*(\lambda)$ which is bounded by $\Gamma_{-\pi/2}$ and Γ_λ, and $\Omega'(\lambda) = \Omega^* - \overline{\Omega^*(\lambda)}$. It suffices to verify that (15) holds in each of these regions separately. The result for $\Omega^*(\lambda)$ is an immediate consequence of the definition of $\mu_\lambda(r)$ and the hypothesis on $M(r;u)$, so we proceed directly to $\Omega'(\lambda)$.

For each $\theta \in [0, \pi/2)$ such that $e^{i\theta} \in A$, let

$$\Gamma_{\lambda,\theta}(s) = e^{i\theta}\Gamma_\lambda(s), \quad s \in [0, \infty],$$

and

$$\varphi_\theta(z) = \varphi(ze^{i\theta}), \quad z \in e^{i\theta}\Omega^*.$$

Then $\Gamma_{\lambda,0} = \Gamma$, $\varphi_0 = \varphi$, and φ_θ maps $e^{i\theta}\Omega'(\lambda)$ onto $\{\lambda < \operatorname{Arg} w < \pi/2\}$ with

$$\operatorname{Re}[\varphi_\theta \circ \Gamma_{\lambda,\theta}] = \cos\lambda |\varphi \circ \Gamma_\lambda| \tag{17}$$

and

(18) $$|\varphi \circ \Gamma_\lambda(s)| \geqq \mu_\lambda(|\Gamma_\lambda(s)|), \quad s \in [0, \infty).$$

Let $\varepsilon > 0$. Since $M(r; u) = o[\mu_\lambda(r)]$ as $r \to 1$, there exists $r_0 \in (0, 1)$ such that

(19) $$u(z) \leqq \varepsilon\mu_\lambda(|z|), \quad r_0 < |z| < 1.$$

Furthermore, there exists $s_o \in (0, \infty)$ such that for each $s \in (s_o, \infty)$ we have $|\Gamma_\lambda(s)| > r_o$, $\theta_o \equiv \operatorname{Arg} \Gamma_\lambda(s_o) > \operatorname{Arg} \Gamma_\lambda(s)$, and $\exp(i\theta_o) \in A$. Choose $\theta_1 \in (\theta_o, \pi/2)$ such that $\exp(i\theta_1) \in A$. By (16) there is a positive constant M_ε such that

(20) $$u(z) \leqq M_\varepsilon, \quad z \in \Delta, \ \theta_o \leqq \operatorname{Arg} z \leqq \theta_1.$$

For each $\theta \in (0, \theta_0)$, let $s_\theta \in (s_o, \infty)$ be the largest number for which $\operatorname{Arg} \Gamma_{\lambda,\theta}(s_\theta) = \theta_o$. Define $\Omega'_\theta(\lambda)$ to be the subregion of $\Omega'(\lambda)$ bounded by $\{\Gamma_{\lambda,\theta}(s) : s_\theta \leqq s \leqq \infty\}$,

$$\{z \in \Omega'(\lambda) : \operatorname{Arg} z = \theta_o, |z| \geqq \Gamma_{\lambda,\theta}(s_\theta)|\},$$

and the subarc of A with endpoints $\exp(i\theta)$ and $\exp(i\theta_o)$.

Suppose now that $Z \in \Omega'(\lambda)$ with $|Z - 1| < |\exp(i\theta_o) - 1|/2$. Then for $\theta \in (0, \theta_o)$ sufficiently small, $Z \in \Omega'_\theta(\lambda)$. For each such θ, consider the subharmonic function

$$v_\theta(z) = u(z) - \frac{\varepsilon}{\cos\lambda} \operatorname{Re}[\varphi_\theta(z)] - M_\varepsilon, \quad z \in \Omega'_\theta(\lambda).$$

By (16)–(20) it follows that $v_\theta \leqq 0$ on $\partial\Omega'_\theta(\lambda)$ and we can apply the maximum principle to conclude that $v_\theta(z) \leqq 0$, $z \in \Omega'_\theta(\lambda)$. In particular,

$$u(Z) \leqq \frac{\varepsilon}{\cos\lambda} |\varphi(Z)| + M_\varepsilon.$$

Taking the limit as $\theta \to 0$, we get

$$u(Z) \leqq \frac{\varepsilon}{\cos\lambda} |\varphi(Z)| + M_\varepsilon.$$

Since M_ε is a fixed constant (depending on ε) and $\varphi(z) \to \infty$ as $z \to 1$ in $\Omega'(\lambda)$, we conclude that (15) holds for $\Omega'(\lambda)$. This completes the proof.

We note that versions of Corollary B and Lemma 1 hold with obvious modifications when Ω^* is replaced by $\tilde{\Omega}^* = \{z : \bar{z} \in \Omega^*\}$, $\zeta\Omega^*$, or $\zeta\tilde{\Omega}^*$ for any $\zeta \in C$. In applying any one of these versions we shall simply refer to Corollary B or Lemma 1.

The following proposition and its proof are closely related to [7; Theorem 2] and Dahlberg's proof. We shall therefore only sketch the proof here.

PROPOSITION 1. *If u satisfies* (2), (3′) *with* $E = \phi$, *and* (9) (*for some* $\lambda \in (-\pi/2, \pi/2)$), *then* $u^* \in L^1$, $u^* = u_*$ a.e. *and* $u \leqq Pu_*$.

The next lemma will be used in the proof.

LEMMA 2. *Let A be an open arc in C and*

$$S(A) = \{r\zeta : 0 < r < 1, \zeta \in A\}.$$

If V is a subharmonic function defined in Δ that is bounded above in $S(A)$ such that $V_* \leqq 0$ a.e. *in A, then*

$$\limsup_{z \to \zeta} V(z) = 0, \quad \zeta \in A.$$

See [7; p. 307 first paragraph] for a proof of Lemma 2.

PROOF OF PROPOSITION 1. Let $f = (u^+)_*$. By (2) we have $f \in L^1$. Let $v = u^+ - Pf$ and define

$$O = \{\zeta \in C : \limsup_{z \to \zeta} v(z) \leqq 0\}.$$

We show first that $O = C$. Suppose, to get a contradiction, that $R \equiv C - O \neq \phi$. Let

$$F_j = \{\zeta \in C : v(z) \leqq j \textit{ for } z \in \zeta\Omega\}, \quad j = 1, 2, \ldots .$$

Then each F_j is closed. In general, this follows from the continuity properties of subharmonic functions (see [7; p. 307 second paragraph]); an elementary proof can be given if $(0, 1)$ is contained in the interior of Ω. From (3′) we see that u is bounded in $\zeta\Omega$ for each $\zeta \in C$ and hence $\bigcup F_j = C$.

By the Baire category theorem, there is an open arc A and a positive integer j such that $\phi \neq A \cap R \subseteqq F_j$. Lemma 2 implies that $A \cap R$ has empty interior (with respect to C) so that $\bar{A} \cap R$ is a closed nowhere-dense set. Let (A_n) be an enumeration of the component arcs of $O \cap A$ and let η_n and ξ_n be the endpoints of A_n. Let $\mathscr{E}$ be the set of all these endpoints. Since $R \cap A \subseteqq F_j$, there exists a positive integer $k \geqq j$ such that $\mathscr{E} \subseteqq F_k$. Thus

$$v(z) \leqq k, \quad z \in \zeta\Omega, \zeta \in \mathscr{E}. \tag{21}$$

Let p_n be the midpoint of the arc A_n. Choose $q_n \in \Delta$ along the radius $[0, p_n]$ so the radial segment $[q_n, p_n]$ divides the region $S(A_n) - (\eta_n\Omega \cup \xi_n\Omega)$ into two symmetric subregions. (The sector $S(A_n)$ is defined in Lemma 2.) By (21) and the fact that $v(z) \leqq M$ on $[q_n, p_n]$ for some positive constant M, we can apply Lemma 1 and Corollary B, and then the maximum principle to conclude that

$$v(z) \leqq \max(k, M), \quad z \in S(A_n) - (\eta_n\Omega \cup \xi_n\Omega).$$

From Lemma 2 and the maximum principle, we now deduce that

$$v(z) \leqq k, \quad z \in S(A_n). \tag{22}$$

Since n was arbitrary, we conclude that $v(z) \leqq k$ for all $z \in \bigcup S(A_n)$. Since we also have $R \cap A \subseteqq F_k$, it follows that $v(z) \leqq k$ for all $z \in R(A)$. Lemma 2 implies that $A \subseteqq O$ which leads to the contradiction that $R \cap A = \emptyset$. This completes the proof that $O = C$.

Since $O = C$, we have $u^+ \leqq Pf$. Therefore, there exists a real Borel measure μ on C such that $P[d\mu]$ is the least harmonic majorant of u in Δ. From a theorem of Littlewood it follows that $u_* = u^*$ a.e. in C and $d\mu = u_* dt + d\mu_s$. We also conclude that $u \leqq Pu_*$. The proof of Proposition 1 is thereby completed.

4. Exceptional sets and auxiliary functions.

To deal with the cases where $E \neq \emptyset$ in Theorems 1, 2, and 4, we shall employ certain auxiliary functions that are related to ones used previously in [5]. As in [5], we start by assuming without loss of generality that ω has a continuous monotone nonincreasing derivative on $(0, 2\pi]$. This reduction (which is needed only for Theorem 1) is possible since there always exists such a function $\bar{\omega}$ for which $\bar{\omega}/4 \leqq \omega \leqq \bar{\omega}$ (see [6; Theorem 2.1]). The following is the main result of this section. It concerns the existence of a positive harmonic function $h(z)$ that approaches ∞ at specified rates as z approaches points of the exceptional set E with unrestricted approach or sectorially.

PROPOSITION 2. *Let E be a countable union of ω-sets. Then there exists a positive harmonic function $h = PF$ with $F \in L^1$ such that*

$$\frac{\omega(|\zeta - z|)}{|\zeta - z|} = o[h(z)] \quad \text{as } z \to \zeta,\ z \in \zeta\Omega_\alpha,\ \zeta \in E, \tag{23}$$

for every $\alpha \in (0, \pi/2]$, and

$$\omega'(|\zeta - z|) = o[h(z)] \quad \text{as } z \to \zeta,\ z \in \Delta,\ \zeta \in E. \tag{24}$$

PROOF. Initially, let us suppose that E is an infinite ω-set, that is, E is a closed subset of C with $|E| = 0$ and $\sum \omega(|I_k|) < \infty$, where $(I_k)_1^\infty$, is an enumeration of the component arcs of $C - E$. Let $\mathscr{E}_k$ be the set of endpoints of I_k for each k and let (x_k) be sequence of positive numbers such that $\lim_{k \to \infty} x_k = \infty$ and $\sum x_k \omega(|I_k|) < \infty$. For each k we select a function $g_k: (0, 2\pi] \to R$ that is continuous and monotone nonincreasing such that $g_k \omega'$ is integrable,

$g_k(2\pi) = x_k$, and $\lim_{t\to 0} g_k(t) = \infty$. Thus

$$x_k\omega(t) \leqq \int_0^t \omega'(s)g_k(s)ds < \infty, \quad t\in[0,2\pi]. \tag{25}$$

Furthermore, we choose g_k decreasing quickly enough so that

$$\int_0^{|I_k|} \omega'(t)g_k(t)dt \leqq 2x_k\omega(|I_k|). \tag{26}$$

Define $F: C \to R \cup \{+\infty\}$ by

$$F(\zeta) = \begin{cases} \omega'[\operatorname{dist}(\zeta, \mathscr{E}_k)]g_k[\operatorname{dist}(\zeta, \mathscr{E}_k)], & \zeta\in C-E, \\ +\infty, & \zeta\in E. \end{cases}$$

Then F is a positive (or infinite) function that is continuous in the extended sense. From (26) and the fact that E is an ω-set, it follows that $F\in L^1$. It is also verified without difficulty that if $\zeta\in E$, then for open arcs A with one endpoint ζ we have

$$\lim_{|A|\to 0} \frac{\int_A F(e^{it})dt}{\omega(|A|)} = +\infty. \tag{27}$$

Let $h = PF$. Then h is a positive harmonic function on Δ with a continuous extension to $\bar{\Delta}$ (also denoted h) such that $h|_C = F$. We consider now the growth of $h(z)$ as z approaches points of E from within Δ.

We verify (23) first. Let $\zeta\in E$ and $z\in\zeta\Omega_\alpha$. Observe that

$$P(z,t) \geqq \frac{1-|z|}{|e^{it}-z|}\,\frac{1}{|e^{it}-z|.} \quad t\in\mathbf{R}.$$

Now let A be the open arc with midpoint ζ and $|A| = 2(1-|z|)$. Geometric considerations show that the quotient $(1-|z|)/|e^{it}-z|$ is bounded below by a positive constant independent of $e^{it}\in A$ or the choice of $z\in\zeta\Omega_\alpha$ (though it will depend on $\alpha\in(0,\pi/2]$). It follows that there exists a positive constant $c = c(\alpha)$ such that

$$h(z) \geqq \frac{1}{2\pi}\int_A P(z,t)F(e^{it})dt \geqq c\,\frac{\int_A F(e^{it})dt}{1-|z|}.$$

Putting this inequality together with (27), we arrive at (23).

Next we prove that (24) holds. From the definition of F we see that if $\zeta\in E$, then

$$\omega'(|\zeta-\eta|) = o[F(\eta)] \quad as\ \eta \to \zeta,\ \eta \in C. \tag{28}$$

Let $z \in \Delta$ with $|z-\zeta| < 1/2$ and let B be an open arc with endpoints ζ_1 and ζ_2 such that

$$|\zeta_1-\zeta| < |\zeta_2-\zeta| = |z-\zeta| \quad \text{and} \quad |\zeta_1-\zeta_2| = 1-|z|.$$

Then both $(1-|z|)/|e^{it}-z|$, $e^{it} \in B$, and $\int_B |e^{it}-z|^{-1} dt$ are bounded below by a positive constant (independent of allowed z). Thus there exists a positive constant c (independent of allowed z) such that

$$h(z) \geqq \frac{1}{2\pi} \int_B P(z,t)F(e^{it})dt \geqq c \min\{F(e^{it}) : e^{it} \in B\}.$$

From the definition of B and (28), we can now conclude that (24) holds.

The requirement made at the outset that E is infinite is clearly not essential to the proofs given above, so we may assume that the result is proved for E an arbitrary (nonempty) ω-set. To complete the proof, assume that $E = \bigcup_{k=1}^{\infty} E_k$ where each E_k is an ω-set. Then for each k, there exists h_k and F_k relative to E_k as in the statement of Proposition 2. Let (y_k) be a sequence of positive numbers such that $F = \sum y_k F_k \in L^1$. Then $h = \sum y_k h_k = PF$ has the required properties and Proposition 2 is established.

5. Proofs of main results.

PROOF OF THEOREM 2. If $E = \emptyset$, then Proposition 1 applies directly. Suppose that $E \neq \emptyset$ and let $h = PF$ be as in Proposition 2 relative to E. Then $h^* = h_* = F$ a.e., $h^* \in L^1$, and $h = Ph_*$. Applying Proposition 1 to $u-h$ ($\leqq u$), we have $(u-h)^* \in L^1$, $(u-h)^* = (u-h)_*$ a.e., and $u-h \leqq P(u-h)_*$. Combining these with the observations just made concerning h, we obtain the required conclusions for u.

PROOF OF THEOREM 1. Assume first that (6) is replaced by (9). In this case essentially the same argument as we gave for the proof of Theorem 2 gives the required conclusion. To return from (9) to (6), it suffices to carry out the elementary estimates (using the fact that $|\varphi(\varrho e^{i\phi})|$ behaves essentially like $\varrho^{-\pi/\alpha}$) to show that for any $\lambda \in (-\pi/2, \pi/2]$ there exist positive constants c_1 and c_2 such that

$$c_1 \mu_\lambda(r) \leqq (1-r)^{-\pi/\alpha} \leqq c_2 \mu_\lambda(r), \quad r \in (1/2, 1).$$

This leads to the desired result.

To see that the 'o' global growth condition cannot be weakened to a 'O'

condition in Theorem 1, consider the function

$$u(z) = \begin{cases} \operatorname{Re}\varphi(z), & z \in \Omega^*, \\ 0, & z \in \Delta - \Omega^*. \end{cases} \tag{29}$$

Then u is a nonnegative subharmonic function with

$$M(r;u) = O[(1-r)^{-\pi/\alpha}] \quad \text{as } r \to 1$$

such that

$$\lim_{z\to\zeta} u(z) = 0, \quad \zeta \in C - \{1\},$$

and

$$u(z) = 0, \quad z \in \Omega_\alpha.$$

In order to prove Theorem 3, we shall first need to state the appropriate parts of the theorem of Warschawski cited section 1. The version given below contains very minor modifications of the original statement so it can be applied directly in the present context.

THEOREM C. *Under the hypotheses stated preceding Theorem* 3, *the following results hold.*

(I) *There exists a neighborhood of* 1 *such that* Γ_λ *is representable in the form*

$$\phi = \psi(\varrho) - \Theta(\varrho)\lambda/\pi + o[\Theta(\varrho)] \quad \textit{as } \varrho \to 0,$$

where $\psi(\varrho) = \frac{1}{2}[\Phi^+(\varrho) + \Phi^-(\varrho)]$ *and the 'o' condition is uniform over all* $\lambda \in [-\pi/2, \pi/2]$.

(II) $$\lim_{\varrho\to 0} = \frac{\log|\varphi(z)|}{\displaystyle\int_\varrho^a \frac{ds}{\Lambda(s)}} = \pi \textit{ uniformly for all } z \in \Omega^* \textit{ with } |z| = \varrho.$$

PROOF OF THEOREM 3. We start with some computations which derive a relationship between $\varrho = |z-1|$ and $r = |z|$ when $z \in \Gamma_\gamma$ for any $\gamma \in [-\pi/2, \pi/2]$. Theorem C(I) asserts that for each such γ, the Jordan arc Γ_γ is representable in the form

$$\phi_\gamma = \tilde{\Theta}_\gamma(\varrho) + \sin^{-1}(\varrho/2) \quad \text{as } \varrho \to 0,$$

where

$$\tilde{\Theta}_\gamma(\varrho) = c(\gamma)\Theta(\varrho) + o[\Theta(\varrho)]$$

and $c(\gamma) = 1/2-\gamma/\pi$. Elementary geometric considerations show that

$$1-r^2 = \varrho(2\sin\phi-\varrho).$$

Therefore,

$$\sin\phi_\gamma = (\varrho/2)\cos[\tilde{\Theta}_\gamma(\varrho)]+\sqrt{1-\varrho^2/4}\sin[\tilde{\Theta}_\gamma(\varrho)]$$

so that

$$\begin{aligned} 1-r^2 &= \varrho\{\varrho(\cos[\tilde{\Theta}_\gamma(\varrho)]-1)+\sqrt{4-\varrho^2}\sin[\tilde{\Theta}_\gamma(\varrho)]\} \\ &= \varrho\{-(\varrho/2)[\tilde{\Theta}_\gamma(\varrho)]^2+\sqrt{4-\varrho^2}[\tilde{\Theta}_\gamma(\varrho)]\} \\ &= c(\gamma)\sqrt{4-\varrho^2}\Lambda(\varrho)[1+o(1)] \quad \text{as } \varrho\to 0, \end{aligned}$$

where $\Lambda(\varrho) = \varrho\Theta(\varrho)$. It follows that

(30) $$1-r = c(\gamma)\Lambda(\varrho)[1+o(1)] \quad \text{as } \varrho\to 0.$$

Also, by Theorem C(II) and the change of variables $t = \Lambda(s)$, we have

(31) $$|\varphi(z)| = \exp\left\{\pi\int_{\Lambda(\varrho)}^{\Lambda(a)}\frac{(\Lambda^{-1})'(t)}{t}dt[1+o(1)]\right\} \quad \text{as } \varrho\to 0.$$

Observe that the 'o' conditions in (30) and (31) are both independent of γ.

To prove the first assertion of the theorem we shall estimate $\mu_\lambda(r)$. Let $\lambda\in[-\pi/2,\pi/2)$ and consider $z\in\Omega^*(\lambda)$ (the region defined preceding the definition (8) of $\mu_\lambda(r)$) with $|z| = r$. Note that for z within $\Omega^*(\lambda)$, the condition $\varrho\to 0$ is equivalent to $r\to 1$. Hence for $\gamma\in[-\pi/2,\lambda]$ and $z\in\Gamma_\gamma$ with $|z| = r$, it follows from (30) that

(32) $$\varrho(z) = \Lambda^{-1}\left[\frac{1-r}{c(\gamma)}(1+o(1))\right] \quad \text{as } r\to 1,$$

where the 'o' does not depend on the allowed γ. Equation (31) also holds with '$\varrho\to 0$' replaced by '$r\to 1$' and a similar remark concerning the 'o' condition.

Given $\varepsilon>0$, select $\lambda\in[-\pi/2,\pi/2)$ close enough to $-\pi/2$ and $\varepsilon'\in(0,\varepsilon)$ sufficiently small so that $(1+\varepsilon')/c(\lambda) < 1+\varepsilon$. Then for $z\in\Omega^*(\lambda)$, $|z| = r$, and r sufficiently close to 1, we have

$$|\varphi(z)| \geqq \exp\left\{(\pi-\varepsilon)\int_{(1-r)(1+\varepsilon)}^{\Lambda(a)}\frac{(\Lambda^{-1})(t)}{t}dt\right\}.$$

The required conclusion is now obtained using this inequality, the definition of $\mu_\gamma(r)$, and Theorem 2.

For the proof of the second assertion of the theorem, consider the positive harmonic function $h = \operatorname{Re}\varphi$ defined in Ω^*. By (30), the fact that $c(\lambda) \leqq 1$, and (29), we can find $\delta > 0$ such that $\varrho(z) < \delta$ implies

$$1-r \leqq \Lambda(\varrho)/(1-\varepsilon)$$

and

$$h(z) \leqq \exp\left\{(\pi+\varepsilon)\int\limits_{\Lambda(\varrho)}^{\Lambda(a)} \frac{(\Lambda^{-1})'(t)}{t}\,dt\right\}.$$

This leads to the inequality

$$h(z) \leqq \exp\left\{(\pi+\varepsilon)\int\limits_{(1-r)(1-\varepsilon)}^{\Lambda(a)} \frac{(\Lambda^{-1})'(t)}{t}\,dt\right\} \tag{33}$$

for all $z \in \Omega^*$ with $\varrho(z) < \delta$. Because h is continuously 0 at each point of $\partial\Omega^* - \{1\}$, we have $h(z) < \varepsilon$ whenever $\varrho(z) \geqq \delta$ and r is sufficiently close to 1. Hence (33) holds for all $z \in \Omega^*$ with $|z| = r$ when r is sufficiently close to 1. Taking u to be the subharmonic function defined by (29), we see that u has the required properties. In this connection, note that the growth rate appearing on the right-hand side of (33) satisfies a 'o' condition relative to the same growth condition with any slightly larger ε. This completes the proof of Theorem 3.

We omit the proof of Theorem 4 since this result is a straightforward consequence of Theorem 3. Theorem 5 for the case $E = \emptyset$ is proved using the compactness of C and the classical maximum principle to conclude that u is bounded above, and then the desired conclusion is derived as before. The case when $E \neq \emptyset$ can be reduced to showing that $v = u - Pu_*$ satisfies $\limsup_{z\to\zeta} v(z) \leqq 0$ for each $\zeta \in E$. Suppose that $\zeta_0 \in E$ and this property does not hold for ζ_0. Using Lemma 2 and the local growth hypothesis of ζ_0, one shows that $\limsup_{z\to\zeta} v(z) \leqq 0$ for every ζ in an open arc containig ζ_0 except possibly ζ_0 (that is, ζ_0 is isolated). Applying the classical Phragmén-Lindelöf Theorem (stated in section 1 for a finite exceptional set), we contradict our assumption concerning ζ_0. This completes the proof.

6. Applications.

In this section we briefly discuss applications of the results given in sections 1 and 2 to level sets and radial-limit zero sets.

A. *Level Sets.* We shall give below some extensions of results appearing in [2]. The proofs are essentially the same except for the use of improved Phragmén-Lindelöf theorems.

Let $\mathscr{L}(r;u)$ denote the level set $\{z \in \Delta : u(z) = r\}$ where u is a *continuous* subharmonic function. Consider first a component Φ of $\{u(z) > r\}$ and define

$$U(z) = \begin{cases} u(z), & z \in \Phi, \\ r, & z \in \Delta - \Phi. \end{cases}$$

Then U is again a subharmonic function which satisfies the same local and global growth conditions (to $+\infty$) as u. This fact leads to the following extension of results appearing in [2; section 3].

THEOREM 6. *Suppose that* $r \in (\inf u, \sup u)$ $\alpha \in (0, \pi/2)$, *and* $\Omega = \Omega_\alpha$ *(as defined in* (5)). *Let* Φ *b a component of* $\{u(z) > r\}$.

(I) *If u is bounded above, then $(\partial\Phi) \cap C$ is a perfect subset of C that is locally of positive linear measure at each of its points.*

(II) *If $M(r;u) = O[\omega(1-r)/(1-r)]$ as $r \to 1$, then $(\partial\Phi) \cap C$ is a perfect subset of C that is locally of positive H_ω-measure at each of its points.*

(III) *If $M(r;u) = o[(1-r)^{-\pi/\alpha}]$ as $r \to 1$ and $u(z) = O[\omega(|\zeta - z|)/|\zeta - z|]$ as $z \to \zeta$ in $\zeta\Omega$ for each $\zeta \in C$, then $(\partial\Phi) \cap C$ is a perfect subset of C such that for every open arc A, the set $\partial\Phi \cap A$ is either empty or is not a countable union of ω-sets.*

(IV) *If $M(r;u) = o[(1-r)^{-\pi/\alpha}]$ as $r \to 1$ and $u(z) = o(|\zeta - z|^{-1})$ as $z \to \zeta$ in $\zeta\Omega$ for each $\zeta \in C$, then $(\partial\Phi) \cap C$ is a perfect subset of C.*

An analogue of (III) also can be given for more general global growth rates when the local growth condition is given in terms of ω'.

Next, suppose that Φ is a component of $\{u(z) < r\}$ (sometimes called a level tract). By the maximum principle, Φ must be simply-connected. If $u = \log|f|$ where f is a nonconstant analytic function, then $(\partial\Phi) \cap \Delta$ consists locally of analytic arcs. In general, this part of the boundary of Φ may behave quite badly as it approaches the circumference C. For example, it may spiral outward and cluster to every point of C. However, this behavior is not possible of f is in the MacLane class $\mathscr{L}$. By definition, $\mathscr{L}$ is the class of nonconstant analytic functions f defined in Δ such that for every $r \in (\inf|f|, \sup|f|)$, we have $\lim_{t \to 1} S_r(t) = 0$, where

$$S_r(t) = \sup[\operatorname{diam} l : l \textit{ is a component of } \{t < |z| < 1\} \bigcap \mathscr{L}(r; |f|)], \quad t \in (0,1).$$

Here, $\operatorname{diam} l = \sup\{|z-w| : z,w \in l\}$. In [10], Hornblower proved that

$$\int_0^1 \log^+\log^+ M(r;|f|)dr < \infty$$

is a sufficient condition for $f \in \mathscr{L}$. This leads to the following generalization of [2; Theorem 4.1].

THEOREM 7. *Let* $\alpha \in (0, \pi/2]$ *and* $\Omega = \Omega_\alpha$. *Suppose that* $s \in (\inf|f|, \sup|f|)$ *and* Φ *is a component of* $\{|f(z)| < s\}$. *If* $u = \log^+|f|$ *satisfies*

$$M(r;u) = o[(1-r)^{-\pi/\alpha}] \quad \textit{as } r \to 1$$

and

$$u(z) = o(|\zeta - z|^{-1}) \quad \textit{as } z \to \zeta \textit{ in } \zeta\Omega \textit{ for each } \zeta \in C,$$

then Φ *is a simply-connected Jordan region.*

Recall that a nonconstant analytic function f is said to be in the class $A^{-\infty}$ if the subharmonic function $u = \log^+|f(z)|$ satisfies $M(r;u) = O[-\log(1-r)]$ as $r \to 1$ and that every analytic function f of bounded characteristic has $M(r;u) = O[(1-r)^{-1}]$ as $r \to 1$. We therefore obtain the next corollary.

COROLLARY 7. *Let* f *be a nonconstant analytic function*, $s \in (\inf|f|, \sup|f|)$, *and let* Φ *be a component of* $\{|f(z)| < s\}$. *If*

(I) $f \in A^{-\infty}$

or

(II) f *is of bounded characteristic and satisfies*

$$\log^+|f(r\zeta)| = o[(1-r)^{-1}] \quad \textit{as } r \to 1,\ \zeta \in C,$$

then Φ *is a simply-connected Jordan region.*

Thus, for example, if $f, 1/f \in A^{-\infty}$, then every component Φ of $\Delta - \mathscr{L}(r;|f|)$ is a simply-connected Jordan region with $(\partial\Phi) \cap C$ a perfect subset of C that is locally of positive H_v-measure at each of its points, where $v(t) = t\log(2\pi e/t)$. Notice that the 'o' condition in (II) cannot be relaxed to 'O' as the function $f(z) = \exp[(1+z)/(1-z)]$ shows.

We turn now to an application of Theorem 6 to analytic functions which share a level set of their moduli. For a general study of analytic functions sharing level curves and tracts along with an account of the history of this problem, see the recently published work of Stephenson [16]. Specifically, we consider Blaschke products that share a level set of their moduli with (bounded) nonvanishing analytic functions. We relate a certain boundary exceptional set of such a Blaschke product B with the growth of $|\log|g||$ when g is a nonvanishing analytic function such that $|B|$ and $|g|$ share a level set.

We start by briefly recalling some background concerning the functions involved. More details are given in [2; section 2]. An inner function is, by definition, a bounded analytic function having radial limits of modulus 1 a.e.. Every inner function I has a canonical factorization $I = \eta BS$, where η is a constant of modulus 1. B is a Blaschke product, and S is a singular inner function. The singular inner function S is a nonvanishing function such that $-\log|S| = P[d\mu]$, where μ is a finite positive Borel measure that is singular with respect to linear Lebesgue measure. Also, any bounded analytic function g has a canonical factorization $g = IO$, where I is an inner function and O is a bounded outer function.

To each nonconstant inner function I, associate the boundary exceptional set

$$E(I) = \left\{\zeta \in C : \limsup_{r \to 1} |I(r\zeta)| < 1\right\}.$$

THEOREM 8. *If I is a nonconstant inner function such that $E(I)$ has zero H_ω-measure in a neighborhood of one of its points, then I cannot share a level set of its modulus with a nonvanishing analytic function f satisfying*

$$|\log|f(z)|| = O\left[\frac{\omega(1-|z|)}{1-|z|}\right] \quad as\ |z| \to 1. \tag{34}$$

In the opposite direction, for each Borel set E with $H_w(E) > 0$ and for every $r \in (0,1)$, there exists a Blaschke product B with $E(B) \subseteqq E$ and a nonvanishing bounded analytic function g satisfying (34) (*with $f = g$*) *such that $\mathscr{L}(r;|B|) = \mathscr{L}(r;|g|)$.*

PROOF. A result of Hall (see [2; Lemma 3.1]) shows that any component of $\{|I(z)| < r\}$ must have $(\partial\Phi) \cap C \subseteqq E(I)$ when I is a nonconstant inner function and $r \in (0,1)$. On the other hand, Theorem 6 (II) implies that if f is a nonvanishing analytic function satisfying (34), then every component Φ of $\Delta - \mathscr{L}(r;|g|)$ has the property that $(\partial\Phi) \cap C$ is a perfect set that is locally of positive H_ω-measure at each of its points. Putting these two facts together, we arrive at he first assertion.

Suppose now that E is a Borel set with $H_\omega(E) > 0$. Then we can find a finite positive Borel measure μ supported in E and singular with respect to linear Lebesgue measure such that

$$P[d\mu](z) \leqq c\,\frac{\omega(1-|z|)}{1-|z|}, \quad z \in \Delta.$$

for some positive constant c. Let S be the singular inner function for which $-\log|S| = P[d\mu]$ and let

$$L_a(z) = (a-z)/(1-\bar{a}z), \quad a,z \in \Delta.$$

By a result of Frostman (see [2; Theorem 6.2]), there exists $a \in \Delta$ with $r^2 > |a|$ and a constant η of modulus 1 such that $B = \eta L_\alpha \circ S$ is a Blaschke product. Then $g = rL_{(a/r)} \circ (B/r)$ has the required properties. This completes the proof.

Concerning the functions B and g constructed in the preceding paragraph, the following facts follow from results appearing in [2; section 6]. The function g must be a *nontrivial* product of a singular inner function and a bounded outer function, and B shares *exactly* one level set of its modulus with g. We also note that for the case when $\liminf_{t\to 0}\omega(t)/[t\omega'(t)] > 1$ (e.g., $\omega(t) = t^\alpha$, $0 < \alpha < 1$), the condition $\sum\omega(1-|a_k|) < \infty$ on the zeros a_k (enumerated according to multiplicity) of B in a neighborhood of a point of $E(B)$ implies that $E(B)$ has zero H_ω*-measure* in an open arc containing that point (see [4]).

B. *Radial-limit zero sets of analytic functions of prescribed growth.* Let ν be a positive decreasing function defined on $(0.1]$ such that $\lim_{t\to 0}\nu(t) = \infty$. Barth and Schneider [1] showed that there exists an analytic function $f \not\equiv 0$ such that $M(r;\log|f|) \leqq \nu(1-r)$, $r \in [0, 1)$, having radial limits equal to 0 a.e. in C. Using the radial Phragmén-Lindelöf theorem for functions of slow growth stated in section 1 we obtain the folowing result relevant to the case when $\nu(t) = c\omega(t)/t$, where c is a positive constant.

THEOREM 9. *Let $f \not\equiv 0$ be an analytic function such that*

$$M(r;\log|f|) \leqq c\omega(1-r)/(1-r), \quad r \in [0, 1),$$

for some constant $c > 0$, and let E be the radial-limit zero set of f. Then in any open arc A such that $|E \cap A| > 0$, we have $H_\omega(A-E) > 0$.

Since there exists a first-category subset E of C such that $|E| = 2\pi$ and $H_\omega(C-E) = 0$, Theorem 9 shows that not every first-category subset of C is the radial-limit zero set of an analytic function $f \not\equiv 0$ satisfying the growth condition given in the theorem. It follows that a converse to the Lusin-Privalov radial uniqueness theorem is not possible using functions of a prescribed (slow) growth. (See [3; section 5(3)] for a discussion of this question.)

7. Conclusion.

In this section we give a generalization of a Phragmén-Lindelöf theorem for functions of slow growth to the unit ball B^n of R^n and discuss the question of whether the exceptional sets given in the theorems of section 2 are best possible.

In [8], Dahlberg proved a Phragmén-Lindelöf theorem for Lipschitz domains in R^n, $n \geqq 3$. In particular, his theorem for the unit ball B^n is the following [8; Theorem, p. 306].

THEOREM D. *Let E be a closed subset of the sphere $\partial B^n = S^{n-1}$ having vanishing α-dimensional Hausdorff measure, where $0 < \alpha < n-1$. If U is a subharmonic function defined in B^n such that $M(r;U) = O[(1-r)^{\alpha+1-n}]$ as $r \to 1$ and*

$$\limsup_{z \to \zeta} U(z) \leqq 0, \quad \zeta \in S^{n-1} - E, \tag{35}$$

then $U(z) \leqq 0$ for all $z \in B^n$.

Here, $M(r;U)$ is defined analogously to the two-dimensional case. Using auxiliary functions in essentially the same way as [5], Theorem D can be sharpened; the set E need only be required to be a Borel set (not necessarily closed), and the allowable global growth rates can be expanded in conjunction with a wider class of Hausdorff measures. In the statement of this result, let

$$\omega(t) = \int_0^t f(s)s^{n-2}ds, \quad t \in [0,2], \tag{36}$$

where f is a monotone nonincreasing continuous function on $(0,2]$ with $f(2) > 0$ such that $f(s)s^{n-2}$ is integrable. For each open ball

$$D = D(\eta, r) = \{z \in S^{n-1} : \|z - \eta\| < r\}, \quad \eta \in S^{n-1}, r > 0,$$

associate the auxiliary function

$$h_D(z) = \int_0 P(z,\zeta) f[\operatorname{dist}(\zeta, \partial D)] d\sigma(\zeta), \quad z \in B^n,$$

where P is the Poisson kernel

$$P(z,\zeta) = \frac{1 - \|z\|^2}{\|\zeta - z\|^n}, \quad z \in B^n, \zeta \in S^{n-1},$$

$\|\cdot\|$ and 'dist' are the Euclidean norm and distance in R^n, and σ is the usual surface measure in S^{n-1}. Employing cones in B^n with base D instead of the triangular sets in Δ used in [5], we can make all of the analogous estimates of the auxiliary functions. Because a radial maximum principle is not available in higher dimensions, we need the stronger condition (35) in place of the corresponding assumption along radii. We obtain the following result using the same method but without appeal to a radial maximum principle.

THEOREM 10. *Let E be a Borel subset of S^{n-1} such that $H_\omega(E) = 0$ (with ω defined in* (36)). *If U is subharmonic in B^n such that* (35) *holds and*

$$M(r;U) = O\left[\frac{\omega(1-r)}{(1-r)^{n-1}}\right] \quad as\ r \to 1,$$

then $U(z) \leqq 0$ for all $\in B^n$.

Theorem 10 is shown to be sharp is essentially the same way as [5; Theorem 2].

Finally we turn to the question of whether the size of the Phragmén-Lindelöf exceptional sets E in Theorems 1, 2 and 4 is best possible. For concreteness, let us consider the radial version of Theorem 1 in the following form. Suppose that

$$M(r;u) = o[(1-r)^{-2}] \quad \text{as } r \to 1, \quad u^*(\zeta) \leqq 0, \zeta \in C - E,$$

and

$$u^+(r\zeta) = O[\omega(1-r)/(1-r)] \quad \text{as } r \to 1 \quad \text{for each } \zeta \in E.$$

We would like to characterize the Borel sets E for which we can conclude $u(z) \leqq 0$ for all $z \in \Delta$. By the sharpness of [5; Theorem 2], a necessary condition is that $H_\omega(E) = 0$. We have proven that a sufficient condition is that E be a countable union of ω-sets. (Notice that this gives a second proof that an ω-set has H_ω-measure 0; see also [6; Corollary 4.2]). On the other hand, it can be shown using [6; Theorem 4.1] that there are sets of H_ω-measure 0 which are not countable unions of ω-sets. Thus the problem of giving the desired characterization remains unresolved.

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