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UNIFORM HOMEOMORPHISMS BETWEEN UNIT BALLS IN L_p-SPACES

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Introduction.

It was proved by Mazur [4] that for $p,q \geqq 1$ the spaces L_p and L_q, l_q are homeomorphic. From this work it also follows that the unit balls $B(L_p)$ and $B(L_q), B(l_q)$ are uniformly homeomorphic. However, in Lindenstrauss [3] and Enflo [1] the nonexistence of a uniform homeomorphism between L_p and L_q was established. Enflo also proved that L_1 and l_1 are not uniformly homeomorphic [2]. From the argument it also follows that the unit balls are not Lipschitz equivalent.

In this paper we study uniform homeomorphisms between $B(L_p)$ and $B(l_q)$.

For a uniform homeomorphism $T: B(X) \xrightarrow{\text{onto}} B(Y)$ we define the modulus of continuity δ_T by

$$\delta_T(\varepsilon) = \sup\{\|T(x_1) - T(x_2)\| : \|x_1 - x_2\| \leqq \varepsilon\}.$$

In Section 1 we prove that if $X = L_1$ and $Y = l_1$, then $\delta_{T^{-1}}(\delta_T(\varepsilon)) \geqq K\varepsilon|\log \varepsilon|$ for a sequence of $\varepsilon \to 0$.

In Section 2 we prove that for $X = L_p, l_p$, $Y = L_q$, l_q and $1 \leqq p < q \leqq 2$ we have

$$\delta_{T^{-1}}(\delta_T(\varepsilon)) \geqq K\varepsilon^{p/q} \quad \text{for all } \varepsilon.$$

We also prove that if $T: B(L_p) \to B(L_q)$ or $T: B(l_p) \to B(l_q)$ then this result is sharp. In fact by using the Mazur map we obtain a $T: B(L_p) \to B(L_q)$ (or $B(l_p) + B(l_q)$) such that for $1 \leqq p < q$ we have $\delta_{T^{-1}}(\delta_T(\varepsilon)) \leqq K\varepsilon^{p/q}$ for all ε.

By using the Mazur map we also construct a uniform homeomorphism

$$T: B(L_p) \to B(L_2) \to B(l_2) \to B(l_q)$$

and give estimates of $\delta_{T^{-1}}(\delta_T(\varepsilon))$.

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1.

We first prove the following lemma which will be used several times in this paper. We put $L_p(0,1) = L_p$.

LEMMA 1.1. *Let* $T: B(X) \xrightarrow{\text{onto}} B(Y)$ *be a uniform homeomorphism. Then there exists a* $K > 0$ *such that*

$$\delta_T(\varepsilon) \geqq K\varepsilon \quad \text{for all } \varepsilon \leqq 1.$$

PROOF. For every ε, $0 < \varepsilon \leqq 1$, let $\delta_T(\varepsilon) = K(\varepsilon)\varepsilon$ and assume that $\inf\{K(\varepsilon)\} = 0$. Then we can find a subsequence $\{K(\varepsilon_n)\}$ such that $\varepsilon_n \to 0$, $K(\varepsilon_n) \to 0$, when $n \to \infty$.

Obviously $K(1) = \delta_T(1) \leqq N\delta_T(1/N)$ for every integer N. Now, for every n let N_n be the integer such that $1/(N_n+1) < \varepsilon_n \leqq 1/N_n$. Then we have

$$K(1)/(N_n+1) \leqq \delta_T(1/(N_n+1)) \leqq \delta_T(\varepsilon_n) = K(\varepsilon_n)\varepsilon_n \leqq K(\varepsilon_n)/N_n.$$

Thus $K(\varepsilon_n) \geqq K(1)N_n/(N_n+1)$ which for large n contradicts the assumption $K(\varepsilon_n) \to 0$.

Hence let $K = \inf\{K(\varepsilon)\} > 0$ and Lemma 1.1 is proved.

THEOREM 1.1. *Let* $T: B(L_1) \xrightarrow{\text{onto}} B(l_1)$ *be a uniform homeomorphism. Then there exists a* $K > 0$ *and a sequence* $\{\varepsilon_n\}$ *with* $\varepsilon_n \to 0$ *such that*

$$\delta_{T^{-1}}(\delta_T(\varepsilon_n)) \geqq K\varepsilon_n|\log \varepsilon_n| \quad \textit{for all } \varepsilon_n.$$

For the proof of Theorem 1.1 we need the following lemmas.

LEMMA 1.2. *If* $\delta_T(\varepsilon)(\varepsilon|\log\varepsilon|)^{-1} \to 0$ *when* $\varepsilon \to 0$, *then for every* ε, $0 < \varepsilon \leqq 1$, *the sequence*

$$\{\varepsilon/2^n : n = 0, 1, 2, 3, \ldots\}$$

contains an infinite subsequence $\{\varepsilon_n\}$ *such that*

$$\delta_T(\varepsilon_n/2) \leqq \delta_T(\varepsilon_n)/2 + \delta_T(\varepsilon_n/|\log \varepsilon_n|).$$

LEMMA 1.3. *Let* $a, x, y \in B(l_1)$ *and assume that*

$$|\|a-x\| - \|x-y\|/2| \leqq \varepsilon \quad \textit{and} \quad |\|a-y\| - \|x-y\|/2| \leqq \varepsilon.$$

Then there is a metric midpoint m *between* x *and* y *such that* $\|m-a\| \leqq 3\varepsilon$.

The main idea of the proof of Theorem 1.1 is as follows: If T satisfies the assumption in Lemma 1.2, then T maps metric midpoints on "almost" metric midpoints. The set of metric midpoints is a compact set in l_1 but not in L_1. By Lemma 1.3, this gives that we can find well separated points in L_1 such that the distance between their images is much smaller. We give the details below.

PROOF OF LEMMA 1.2. Given $\varepsilon > 0$ set

$$\delta_T(\varepsilon/2^n) = K(\varepsilon/2^n)(\varepsilon/2^n)|\log(\varepsilon/2^n)|.$$

Then for infinitely many n we have $K(\varepsilon/2^n) \geqq K(\varepsilon/2^{n+1})$. Otherwise we for some N would have

$$0 < K(\varepsilon/2^N) < K(\varepsilon/2^{N+1}) < K(\varepsilon/2^{N+2}) < \cdots,$$

which gives a contradiction, since by assumption $K(\varepsilon/2^n) \to 0$ whe $n \to \infty$. Hence let $\{\varepsilon_n\}$ be an infinite subsequence of $\{\varepsilon/2^n : n = 0, 1, 2, \ldots\}$ such that $K(\varepsilon_n) \geqq K(\varepsilon_n/2)$. Then we have

$$\begin{aligned}\delta_T(\varepsilon_n/2) &= K(\varepsilon_n/2)(\varepsilon_n/2)|\log(\varepsilon_n/2)| \\ &\leqq K(\varepsilon_n)\varepsilon_n(|\log \varepsilon_n| + \log 2)/2 \\ &= (\delta_T(\varepsilon_n) + K(\varepsilon_n)\varepsilon_n \log 2)/2.\end{aligned}$$

Let $m(n)$ be such that

$$2^{m(n)} < |\log \varepsilon_n| \leqq 2^{m(n)+1}.$$

Then we get

$$\begin{aligned}\delta_T(\varepsilon_n/|\log \varepsilon_n|) &\geqq \delta_T(\varepsilon_n/2^{m(n)+1}) \geqq \delta_T(\varepsilon_n)/2^{m(n)+1} \\ &= K(\varepsilon_n)\varepsilon_n|\log \varepsilon_n|/2^{m(n)+1} \geqq K(\varepsilon_n)\varepsilon_n/2.\end{aligned}$$

Thus we have

$$\delta_T(\varepsilon_n/2) \leqq \delta_T(\varepsilon_n)/2 + \delta_T(\varepsilon_n/|\log \varepsilon_n|)\log 2$$

and Lemma 1.2 is poved.

PROOF OF LEMMA 1.3. Without loss of generality we can assume $y = 0$. Let

$$A = \{w \in l_1 : \|x - w\| + \|w\| = \|x\|\}.$$

One can easily check that $w \in A$ if and only if

$$\operatorname{sign}(w_n) = \operatorname{sign}(x_n) \quad \text{and} \quad |w_n| \leqq |x_n| \quad \text{for all } n.$$

Without loss of generality we assume $x_n \geqq 0$ for all n. Now, let

$$b = \begin{cases} a_n & \text{if } 0 \leqq a_n \leqq x_n \\ 0 & \text{if } a_n < 0 \\ x_n & \text{if } x_n < a_n. \end{cases}$$

Then $b \in A$ and $\|a - x\| + \|a\| = \|x\| + 2\|a - b\|$.

By assumption this gives;

$$\|a-b\| \leqq \varepsilon \quad \text{and} \quad \|x\|/2-2\varepsilon \leqq \|b\| \leqq \|x\|/2+2\varepsilon. \tag{1}$$

We let

$$m = b\|x\|/2\|b\| \quad \text{if } \|b\| \geqq \|x\|/2$$

and

$$m = b+(x-b)(\|x\|-2\|b\|)/2\|x-b\| \quad \text{if } \|b\| < \|x\|/2.$$

One can easily see that $m \in A$ and since $\|m\| = \|x\|/2$ we have that m is a metric midpoint between 0 and x. Furthermore, from (1) we get

$$\|m-a\| \leqq \varepsilon + \|m-b\| \leqq 3\varepsilon,$$

and the proof is complete.

Proof of Theorem 1.1. We first assume there for some $K_1 > 0$ exists a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ such that

$$\delta_T(\varepsilon_n) \geqq K_1\varepsilon_n|\log \varepsilon_n| \quad \text{for all } n.$$

By Lemma 1.1 there is a $K_2 > 0$ such that $\delta_{T^{-1}}(\varepsilon) \geqq K_2\varepsilon$ for all $\varepsilon \leqq 1$. Thus, for all n we have

$$\delta_{T^{-1}}(\delta_T(\varepsilon_n)) \geqq K_2\delta_T(\varepsilon_n) \geqq K_1K_2\varepsilon_n|\log \varepsilon_n|$$

and Theorem 1.1 is proved for this case.

Now, if we can not find such a sequence for any $K_1 > 0$, then δ_T satisfies the assumption in Lemma 1.2.

Let $K_1 > 0$ and $K(\varepsilon)$ be such that

$$K_1\varepsilon \leqq \delta_T(\varepsilon) = K(\varepsilon)\varepsilon|\log \varepsilon| \quad \text{for all } \varepsilon \leqq 1.$$

Given any fixed $\varepsilon > 0$, let $\{\varepsilon_n\}$ be a sequence as in Lemma 1.2. Since

$$K(\varepsilon_n/|\log \varepsilon_n|) \to 0 \quad \text{when } n \to \infty$$

we can assume that

$$K(\varepsilon_n/|\log \varepsilon_n|) < K_1/8 \quad \text{and} \quad |\log \varepsilon_n| > 1 \quad \text{for all } n.$$

Now, given ε_n in the sequence and r, $0 < r < \delta_T(\varepsilon_n/|\log \varepsilon_n|)/\delta_T(\varepsilon_n)$, we can find $f,g \in B(L_1)$ with $\|f-g\| \leqq \varepsilon_n$ such that

$$\|T(f)-T(g)\| > (1-r)\delta_T(\varepsilon_n).$$

Then we have $\|f-g| \geqq \varepsilon_n/2$. To see this we assume $\|f-g\| < \varepsilon_n/2$. Then by

Lemma 1.2 we get

$$(1-r)\delta_T(\varepsilon_n) < \|T(f)-T(g)\| \leqq \delta_T(\varepsilon_n/2) \leqq \delta_T(\varepsilon_n)/2+\delta_T(\varepsilon_n/|\log\varepsilon_n|).$$

From this we get

$$\begin{aligned} K_1\varepsilon_n/2 &\leqq \delta_T(\varepsilon_n)/2 \leqq r\delta_T(\varepsilon_n)+\delta_T(\varepsilon_n/|\log\varepsilon_n|) < 2\delta_T(\varepsilon_n/|\log\varepsilon_n|) \\ &= 2K(\varepsilon_n/|\log\varepsilon_n|)\varepsilon_n(|\log\varepsilon_n|+\log|\log\varepsilon_n|)/|\log\varepsilon_n| \\ &\leqq 4\varepsilon_n K(\varepsilon_n/|\log\varepsilon_n|) \quad \text{if} \quad |\log\varepsilon_n| > 1. \end{aligned}$$

This contradicts the assumption $K(\varepsilon_n/|\log\varepsilon_n|) < K_1/8$.

Now, we can find a sequence $\{M_k\}$ of metric midpoints between f and g such that

$$\|M_k-M_i\| = \|f-g\|/2 \quad \text{for } k \neq i.$$

Since we here also need $\|M_k\| \leqq 1$ we give for the sake of completeness an example of such a sequence.

We first assume $f > g \geqq 0$. For every $k = 1, 2, 3, \ldots$, let $a_{k,1} = 0$ and for $n = 2, 3, \ldots, 2^k+1$, let $a_{k,n} \in [0, 1]$ such that

$$a_{k,n} < a_{k,n+1} \quad \text{and} \quad \int_{a_{k,n}}^{a_{k,n+1}} |f(x)-g(x)| = \|f-g\|/2^k.$$

We define

$$M_k = \begin{cases} f(x) & x \in (a_{k,n}, a_{k,n+1}) \text{ and } n \text{ odd} \\ g(x) & x \in (a_{k,n}, a_{k,n+1}) \text{ and } n \text{ even.} \end{cases}$$

One can easily check that

$$\|M_k-f\| = \|M_k-g\| = \|f-g\|/2$$

and

$$\|M_i-M_k\| = \|f-g\|/2 \quad \text{if } k \neq i.$$

Furthermore we have

$$\begin{aligned} \|M_k\| &= \sum_{n\,\text{odd}} \int_{a_{k,n}}^{a_{k,n+1}} f(x) + \sum_{n\,\text{even}} \int_{a_{k,n}}^{a_{k,n+1}} g(x) \\ &= \int_0^1 g(x) + \sum_{n\,\text{odd}} \int_{a_{k,n}}^{a_{k,n+1}} f(x)-g(x) = \int_0^1 g(x)+\|f-g\|/2 \\ &= (\|f\|+\|g\|)/2 \leqq 1. \end{aligned}$$

The general case follows by using the same definitions as above for the restrictions of f, g to the sets $\{x : g(x) > f(x) \geqq 0\}$, $\{x : f(x) > 0 > g(x)\}, \ldots$.

We now prove that $T(M_k)$ is an "almost" metric midpoint between $T(f)$ and $T(g)$: Since $\|f - M_k\| \leqq \varepsilon_n/2$, we by Lemma 1.2 get

$$\begin{aligned}\|T(f)-T(M_k)\| &\leqq \delta_T(\varepsilon_n/2) \leqq \delta_T(\varepsilon_n)/2+\delta_T(\varepsilon_n/|\log \varepsilon_n|)\\ &\leqq \|T(f)-T(g)\|/2+r\delta_T(\varepsilon_n)+\delta_T(\varepsilon_n/|\log \varepsilon_n|)\\ &\leqq \|T(f)-T(g)\|/2+2\delta_T(\varepsilon_n/|\log \varepsilon_n|).\end{aligned}$$

Similarly we have

$$\|T(g)-T(M_k)\| \leqq \|T(f)-T(g)\|/2+2\delta_T(\varepsilon_n/|\log \varepsilon_n|).$$

This gives

$$|\|T(f)-T(M_k)\|-\|T(f)-T(g)\|/2| \leqq 2\delta_T(\varepsilon_n/|\log \varepsilon_n|)$$

and

$$|\|T(g)-T(M_k)\|-\|T(f)-T(g)\|/2| \leqq 2\delta_T(\varepsilon_n/|\log \varepsilon_n|).$$

Since the metric midpoints between $T(f)$ and $T(g)$ is a compact set and $\{M_k\}$ is not, we by Lemma 1.3 can find k,i with $k \neq i$ and a metric midpoint m between $T(f)$ and $T(g)$ such that

$$\|T(M_k)-T(M_i)\| \leqq \|T(M_k)-m\|+\|T(M_i)-m\| \leqq 12\delta_T(\varepsilon_n/|\log \varepsilon_n|).$$

Hence we get

$$\delta_{T^{-1}}(12\delta_T(\varepsilon_n/|\log \varepsilon_n|)) \geqq \|M_k - M_i\| = \|f-g\|/2 \geqq \varepsilon_n/4.$$

Let $\varepsilon_n/|\log \varepsilon_n| = \varepsilon_{n'}$. Since $|\log \varepsilon_n| > 1$, we get

$$\varepsilon_{n'}|\log \varepsilon_{n'}| = \varepsilon_n(|\log \varepsilon_n|+\log|\log \varepsilon_n|)/|\log \varepsilon_n| \leqq 2\varepsilon_n.$$

Thus we have

$$\delta_{T^{-1}}(\delta_T(\varepsilon_{n'})) \geqq \delta_{T^{-1}}(12\delta_T(\varepsilon_n/|\log \varepsilon_n|))/12 \geqq \varepsilon_n/48 \geqq \varepsilon_{n'}|\log \varepsilon_{n'}|/96$$

and the proof of Theorem 1.1 is complete.

2.

For $1 \geqq p,q$ we let $M_{p,q}$ denote the Mazur map of L_p onto L_q that is

$$M_{p,q}(f)(x) = \text{sign}(f(x))|f(x)|^{p/q},$$

and we let $m_{p,q}$ denote the Mazur map of l_p onto l_q. We put $L_p = L_p(0,1)$.

THEOREM 2.1. *Let $1 \leqq p < q \leqq 2$ and let $T : B(L_p) \xrightarrow{\text{onto}} B(L_q)$ be a uniform* *homeomorphism. Then there exists a* $K > 0$ *such that*

$$\delta_{T^{-1}}(\delta_T(\varepsilon)) \geqq K\varepsilon^{p/q} \text{ for all } \varepsilon \leqq 1.$$

THEOREM 2.2. *Let* $1 \leqq p < q$ *and let* $T = M_{p,q}: B(L_p) \to B(L_q)$. *Then there exists a* $K > 0$ *such that*

$$\delta_{T^{-1}}(\delta_T(\varepsilon)) \leqq K\varepsilon^{p/q} \quad \text{for all } \varepsilon \leqq 1.$$

REMARK. In Theorem 2.1 we have the same estimate for $T: B(L_p) \to B(l_q)$, $B(l_p) \to B(l_q)$, $B(l_p) \to B(L_q)$, and in Theorem 2.2 we have the same estimate for $T = m_{p,q}: B(l_p) \to B(l_q)$.

Let F be an isometry between L_2 and l_2. Then $T = m_{2,q} \circ F \circ M_{p,2}$ is a uniform homeomorphism of $B(L_p)$ onto $B(l_q)$ satisfying the following theorem.

THEOREM 2.3.

(a) *If* $1 \leqq p \leqq 2 < q$, *then* $\delta_{T^{-1}}(\delta_T(\varepsilon)) \leqq K\varepsilon^{p/q}$.
(b) *If* $1 \leqq p < q \leqq 2$, *then* $\delta_{T^{-1}}(\delta_T(\varepsilon)) \leqq K\varepsilon^{pq/4}$.
(c) *If* $2 \leqq p < q$, *then* $\delta_{T^{-1}}(\delta_T(\varepsilon)) \leqq K\varepsilon^{4/pq}$.

In the proof to Theorem 2.1 we shall use a construction similar to one used by Enflo in [1]. We first recall a result from this paper.

A set of 2^n points in L_p is called an n-dimensional cube if each of the points is indexed by an n-vector whose components are 0 or 1. In an n-dimensional cube a pair of points is called an edge if the indexes of the points differ in only one component and a pair of points is called an n-diagonal if the indexes of the points differ in all components.

THEOREM. *Let C be an n-dimensional cube in* L_p, $1 \leqq p \leqq 2$, *and let*

$$s_{\max} = \max\{\|f-g\| : (f,g) \text{ edge in } C\}$$

$$d_{\min} = \min\{\|f-g\| : (f,g) \text{ n-diagonal in } C\}.$$

Then we have $n^{1/p}s_{\max} \geqq d_{\min}$.

PROOF OF THEOREM 2.1. We construct an n-dimensional cube C_n as follows. Divide $[0, 1]$ into n intervals of length $1/n$ and let the cube be the set of functions which takes the value 1 or 0 on each interval. We let a function be indexed by the n-vector whose mth component is equal to the value of the function on the mth interval. In this cube every edge has length $1/n^{1/p}$ and every n-diagonal has length 1.

Let

$$s_n = \max\{\|T(f)-T(g)\| : (f,g) \text{ edge in } C_n\}$$

and

$$d_n = \min\{\|T(f)-T(g)\| : (f,g)\ n\text{-diagonal in } C_n\}.$$

Then, for the image of C_n under T we have by Theorem above

$$n^{1/q}s_n \geqq d_n.$$

Obviously we have $s_n \leqq \delta_T(1/n^{1/p})$, and since T^{-1} is uniformly continuous we have $\inf_n\{d_n\} = d > 0$.

Thus

$$\delta_T(1/n^{1/p}) \geqq d/n^{1/q} \quad \text{for } n = 2,3,4,\ldots.$$

Now, for every $\varepsilon > 0$, we set $\delta_T(\varepsilon) = K(\varepsilon)\varepsilon^{p/q}$. Then for $1/(n+1) < \varepsilon^p \leqq 1/n$ we have

$$d/(n+1)^{1/q} \leqq \delta_T(1/(n+1)^{1/p}) \leqq \delta_T(\varepsilon) = K(\varepsilon)\varepsilon^{p/q} \leqq K(\varepsilon)/n^{1/q}.$$

From this we get

$$\inf\{K(\varepsilon) : \varepsilon > 0\} = K_1 > 0.$$

Let $K_2 > 0$ such that $\delta_{T^{-1}}(\varepsilon) \geqq K_2\varepsilon$ for all $\varepsilon \leqq 1$. Then we have

$$\delta_{T^{-1}}(\delta_T(\varepsilon)) \geqq K_2\delta_T(\varepsilon) \geqq K_1K_2\varepsilon^{p/q} \quad \text{for all } \varepsilon \leqq 1$$

and the proof of Theorem 2.1 is complete.

PROOF OF THEOREM 2.2. From Mazur [4] we obtain the following inequalities. If $1 \leqq p \leqq q$ then

(1) $\|M_{p,q}(f)-M_{p,q}(g)\| \leqq 2\|f-g\|^{p/q}$ for all $f,g \in B(L_p)$

(2) $\|M_{q,p}(f)-M_{q,p}(g)\| \leqq (q/p)2^{q/p-1}\|f-g\|$ for all $f,g \in B(L_q)$

Thus, we have $\delta_T(\varepsilon) \leqq 2\varepsilon^{p/q}$ and since $T^{-1} = M_{q,p}$, we have

$$\delta_{T^{-1}}(\varepsilon) \leqq (q/p)2^{q/p-1}\varepsilon$$

and this proves Theorem 2.2.

PROOF OF THEOREM 2.3. The estimates (1) and (2) in the proof of Theorem 2.2 also hold for the map $m_{p,q}$.

In case (a) we have $p \leqq 2$, $2 < q$ and hence by (1) we have

$$\|m_{2,q}\circ F\circ M_{p,2}(f)-m_{2,q}\circ F\circ M_{p,2}(g)\| \leqq K_1\|f-g\|^{p/q} \text{ for all } f,g \in B(L_p),$$

Thus $\delta_T(\varepsilon) \leqq K_1\varepsilon^{p/q}$ and $\delta_{T^{-1}}(\varepsilon) \leqq K_2\varepsilon$ and case (a) is proved. Case (b) and (c) are proved similarly.

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