MATH. SCAND. 62 (1988), 303-326

COMMUTATORS AND GENERATORS

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Abstract.

Let $\mathscr{B}$ be a Banach space ,σ a C_0-group of isometries of $\mathscr{B}$ with generator H, and $\mathscr{D} \subseteqq D(H)$ a σ-invariant core of H. Suppose $K: \mathscr{D} \to \mathscr{B}$ is a dissipative operator satisfying

1. $$\|Ka\| \leqq k_0(\|Ha\| \vee \|a\|), \qquad a \in \mathscr{D},$$

2. $$\|[\sigma_t, K]a\| \leqq k_1|t|(\|Ha\| \vee \|a\|), \quad a \in \mathscr{D}, t \in \mathbf{R}$$

for some $k_0, k_1 \geqq 0$. Then it follows that the closure $\bar{K}$ of K generates a C_0-semigroup of contractions τ. Furthermore if $K\mathscr{D} \subseteqq D(H)$, a property which follows from Conditions 1 and 2 if $\mathscr{B}$ is reflexive, then $\tau D(H) \subseteqq D(H)$. Generalizations of these results are discussed and applications to Hilbert space theory and commutative, and noncommutative, C*-algebras are given.

1. Introduction.

Let σ be a C_0-group of isometries of a Banach space $\mathscr{B}$ with infinitesimal generator H and introduce the subspaces $\mathscr{B}_n = D(H^n)$, and $\mathscr{B}_\infty = \bigcap_{n \geq 1} \mathscr{B}_n$. Then $\mathscr{B}_n$ is a Banach space with respect to the norm

$$\|a\|_n = \sup_{0 \leq m \leq n} \|H^m a\|,$$

and $\mathscr{B}_\infty$ is a Frechet space when equipped with the family of norms $\{\|\cdot\|_n\}_{n \geq 1}$. Our aim is to examine properties of operators K from $\mathscr{B}_\infty$ into $\mathscr{B}$ which are continuous in the sense

$$\|Ka\| \leqq c\|a\|_p \tag{1.1}$$

for some $c \geqq 0, p \geqq 0$, and all $a \in \mathscr{B}_\infty$. Typically one is interested in

Received April 14, 1987.

smoothness properties, dissipativity, or generation criteria. For motivation let us first consider the "smoothness" condition $K\mathscr{B}_\infty \subseteqq \mathscr{B}_1$.

Let $a \in \mathscr{B}_\infty$. Then $Ka \in \mathscr{B}_1$ if, and only if, $\lim(1-\sigma_t)Ka/t$ exists as $t \to 0$. But it follows from the continuity hypothesis (1.1) that

$$\lim_{t\to 0} K(1-\sigma_t)a/t = KHa$$

exists and hence $Ka \in \mathscr{B}_1$ if, and only if, $\lim_{t\to 0}[\sigma_t, K]a/t$ exists. Thus the property $K\mathscr{B}_\infty \subseteqq \mathscr{B}_1$ is directly characterized by the behaviour of the commutators $[\sigma_t, K]a$ for small t. In fact by the uniform boundedness theorem, it follows that if $K\mathscr{B}_\infty \subseteqq \mathscr{B}_1$ then

$$\|[\sigma_t, K]a\| \leqq k|t|\,\|a\|_q, \quad a \in \mathscr{B}_\infty,\ t \in \mathrm{R},$$

for some $k \geqq 0$, $q \geqq 0$. If $\mathscr{B}$ is reflexive there is also a converse. In this case σ is a C_0^*-group and hence $Ka \in \mathscr{B}_1$ if, and only if,

$$\sup_{t\neq 0} \|(1-\sigma_t)Ka/t\| < +\infty$$

(see, for example, [5, Proposition 3.1.23]). Therefore by the above reasoning, $K\mathscr{B}_\infty \subseteqq \mathscr{B}_1$ if, and only if,

$$\|[\sigma_t, K]a\| = O(t) \tag{1.2}$$

as $t \to 0$, for each $a \in \mathscr{B}_\infty$. This illustrates the fundamental nature of commutator bounds, but it is less evident that such bounds are also of relevance to the discussion of generation properties of K.

Glimm and Jaffe [7] appear to have been the first to prove a generator result from hypotheses similar to (1.1) and (1.2). They proved that if $\mathscr{B}$ is a Hilbert space, H is positive, and K is a symmetric operator satisfying $K\mathscr{B}_\infty \subseteqq \mathscr{B}_1$ and

$$\|Ka\| \leqq k_0\|a\|_1 \quad , \quad a \in \mathscr{B}_\infty \tag{1.3}$$

$$\|[\sigma_t, K]a\| \leqq k_1\|a\|_1\,|t|, \quad a \in \mathscr{B}_\infty,\ t \in \mathrm{R}, \tag{1.4}$$

then K is essentially self-adjoint, i.e. the closure of K generates a C_0-group of isometries of $\mathscr{B}$. (In fact this statement is an equivalent rephrasing of the Glimm-Jaffe result, Theorem 1.2.) Subsequently many authors [6], [12], [13] proved variants of this theorem, but all these generalizations were for symmetric operators on Hilbert space and positivity of H was fundamental. Our main result is a version of the Glimm-Jaffe theorem, in which these assumptions are relaxed. Our theorem is for a dissipative operator K on a Banach space $\mathscr{B}$, and there is no spectral restraint on the generator H. We assume (1.4) and a slightly weaker form of (1.3) and then conclude that the closure of K generates a C_0-semigroup of contractions τ. The property $K\mathscr{B}_\infty \subseteqq \mathscr{B}_1$ then implies the invariance $\tau\mathscr{B}_1 \subseteqq \mathscr{B}_1$ and is automatically satisfied if $\mathscr{B}$ is reflexive, e.g. if $\mathscr{B}$ is a Hilbert space.

In the Hilbert space context, Faris and Lavine [6] proved an invariance property of τ. They established that $\tau D(h) \subseteqq D(h)$, where $D(h)$ denotes the quadratic form domain of H, that is $D(h) = D((1+H)^{1/2})$, but even in this special context the property $\tau D(H) \subseteqq D(H)$ appears to be new.

2. Generator theorems.

In this section we prove the result mentioned in the introduction. We use the above notation but it is also convenient to consider the subspace $\mathscr{B}_{1/2}$ of $\mathscr{B}$ introduced in [15], namely

$$\mathscr{B}_{1/2} = \{a\,;\sup_{t\neq 0}\|(1-\sigma_t)a\|/|t| < +\infty\},$$

equipped with the norm

$$\|a\|_{1/2} = \|a\| \vee \sup_{t\neq 0}\|(1-\sigma_t)a\|/|t|.$$

Note that $\mathscr{B}_1 \subseteqq \mathscr{B}_{1/2}$, and if $a \in \mathscr{B}_1$, then one automatically has $\|a\|_{1/2} = \|a\|_1$. In fact if $\mathscr{B}$ is reflexive, then $\mathscr{B}_1 = \mathscr{B}_{1/2}$ and $\|\cdot\|_1 = \|\cdot\|_{1/2}$, but in general the spaces are distinct.

Remark. Although $\sigma\mathscr{B}_{1/2} \subseteqq \mathscr{B}_{1/2}$, the restriction of σ to $\mathscr{B}_{1/2}$ is not necessarily strongly continuous with respect to the norm $\|\cdot\|_{1/2}$. If $\mathscr{B} = C_0(\mathrm{R}^2)$ and σ is the group whose action is given in radial co-ordinates by

$$(\sigma_t f)(r,\theta) = f(r,\theta+t/r),$$

then $\sigma|_{\mathscr{B}_{1/2}}$ is not strongly continuous. Nevertheless $\sigma|_{\mathscr{B}_1}$ is strongly continuous with respect to the $\|\cdot\|_1$ norm.

Theorem 2.1. *Let $\mathscr{D} \subseteqq \mathscr{B}$ denote a σ-invariant core of H. If $K : \mathscr{D} \to \mathscr{B}$ is a dissipative operator satisfying the following two conditions:*

1. *for each $\varepsilon > 0$ there is a $\kappa_\varepsilon > 0$ such that*

$$\|Ka\| \leqq \varepsilon\|a\|_2 + \kappa_\varepsilon\|a\|_1, \quad a \in \mathscr{D},$$

2. *there is a $k_1 > 0$ such that*

$$\|[\sigma_t, K]a\| \leqq k_1\|a\|_1|t|, \quad a \in \mathscr{D},\ t \in \mathrm{R},$$

then the closure $\bar{K}$ of K generates a C_0-contraction semigroup τ on $\mathscr{B}$. Moreover $\tau\mathscr{B}_{1/2} \subseteqq \mathscr{B}_{1/2}$ and

$$\|\tau_t a\|_{1/2} \leqq e^{k_1 t}\|a\|_{1/2}, \quad t \geqq 0, a \in \mathscr{B}_{1/2}.$$

Finally if $\bar{K}\mathscr{D}_1 \subseteqq \mathscr{B}_1$, *where* $\mathscr{D}_1 \subseteqq \mathscr{B}_2$ *is a* σ*-invariant core of* H, *then* $\tau\mathscr{B}_1 \subseteqq \mathscr{B}_1$ *and* $\tau|_{\mathscr{B}_1}$ *is a* C_0*-semigroup satisfying*

$$\|\tau_t a\|_1 \leqq e^{k_1 t}\|a\|_1, \quad t \geqq 0, a \in \mathscr{B}_1.$$

PROOF. First note that if $a \in \mathscr{B}_2$ then

$$\sigma_t a = a + tHa + \int_0^t ds (t-s)\sigma_s H^2 a$$

and hence

$$\|Ha\| \leqq (t/2)\|H^2 a\| + (2/t)\|a\|.$$

This demonstrates that Condition 1 is equivalent to

1′. for each $\varepsilon > 0$ there is a $\kappa_\varepsilon > 0$ such that

$$\|Ka\| \leqq \varepsilon\|H^2 a\| + \kappa_\varepsilon\|a\|.$$

Second, for $\alpha > 0$ introduce the regularization K_α by $D(K_\alpha) = \mathscr{D}$ and

$$K_\alpha = \frac{1}{\alpha}\int_0^\alpha ds \sigma_s K \sigma_{-s}.$$

It follows automatically that K_α satisfies Condition 1′.

Next note that $-H^2$ generates a C_0-contraction semigroup ϱ, the Gaussian semigroup associated with σ by the definition

$$\varrho_t a = (4\pi t)^{-1/2}\int_{-\infty}^{\infty} ds e^{-s^2/4t}\sigma_s a, \quad t > 0.$$

Since K is dissipative, K_α is dissipative. Therefore, by Condition 1′ and perturbation theory (see, for example, [5, Theorem 3.1.32]), the operators

$$H_{\alpha,\beta} = K_\alpha - \beta H^2, \quad \alpha, \beta > 0,$$

generate C_0-contraction semigroups $\tau^{\alpha,\beta}$. If $r_{\alpha,\beta}(\varepsilon) = (I + \varepsilon H_{\alpha,\beta})^{-1}$, then $\|r_{\alpha,\beta}(\varepsilon)\| \leqq 1$. Our immediate aim is to obtain estimates on $\|r_{\alpha,\beta}(\varepsilon)\|_n$ for $n = 1, 2$.

As a preliminary, note that K, and K_α, can be extended to $\mathscr{B}_2$ by continuity using Condition 1. It immediately follows that the extensions, which we also denote by K, and K_α, must satisfy Conditions 1 and 2.

LEMMA 2.2. *If* $\varepsilon k_1 < 1$ *then*

$$\|r_{\alpha,\beta}(\varepsilon)a\|_1 \leqq \|a\|_1(1-\varepsilon k_1)^{-1}, \quad a \in \mathscr{B}_1.$$

PROOF. One has $\|r_{\alpha,\beta}(\varepsilon)a\| \leqq \|a\|$. But from the identity

$$\frac{(\sigma_t-1)}{t} r_{\alpha,\beta}(\varepsilon)a = r_{\alpha,\beta}(\varepsilon)\frac{(\sigma_t-1)}{t}a + \varepsilon r_{\alpha,\beta}(\varepsilon)[K_\alpha, \sigma_t/t]r_{\alpha,\beta}(\varepsilon)a,$$

one obtains the estimate

$$\left\|\frac{(\sigma_t-1)}{t} r_{\alpha,\beta}(\varepsilon)a\right\| \leqq \left\|\frac{(\sigma_t-1)}{t}a\right\| + \varepsilon k_1\|r_{\alpha,\beta}(\varepsilon)a\|_1 \tag{2.1}$$

by use of Condition 2 for K_α. Therefore taking the limit $t \to 0$ with $a \in \mathscr{B}_1$, one deduces that

$$\|r_{\alpha,\beta}(\varepsilon)a\|_1 \leqq \|a\|_1 + \varepsilon k_1\|r_{\alpha,\beta}(\varepsilon)a\|_1,$$

which immediately yields the desired result.

LEMMA 2.3. *If* $2\varepsilon k_1 < 1$ *and* $a \in \mathscr{B}_2$, *then*

$$\|r_{\alpha,\beta}(\varepsilon)a\|_2 \leqq \{\|a\|_2 + 2\varepsilon k_1\alpha^{-1}\|a\|_1(1-\varepsilon k_1)^{-1}\}(1-2\varepsilon k_1)^{-1}.$$

PROOF. First note the identity

$$\frac{(\sigma_t-1)^2}{t^2} r_{\alpha,\beta}(\varepsilon)a = r_{\alpha,\beta}(\varepsilon)\frac{(\sigma_t-1)^2}{t^2}a - \varepsilon r_{\alpha,\beta}(\varepsilon)\frac{[\sigma_t,[\sigma_t,K_\alpha]]}{t^2} r_{\alpha,\beta}(\varepsilon)a -$$

$$-2\varepsilon r_{\alpha,\beta}(\varepsilon)\frac{[\sigma_t,K_\alpha]}{|t|}\frac{(\sigma_t-1)}{|t|} r_{\alpha,\beta}(\varepsilon)a,$$

which yields the estimate

$$\left\|\frac{(\sigma_t-1)^2}{t^2} r_{\alpha,\beta}(\varepsilon)a\right\| \leqq \left\|\frac{(\sigma_t-1)^2}{t^2}a\right\| + 2\varepsilon k_1\left\|\frac{(\sigma_t-1)}{t} r_{\alpha,\beta}(\varepsilon)a\right\|_1 + \tag{2.2}$$

$$+\varepsilon\left\|\frac{[\sigma_t,[\sigma_t,K_\alpha]]}{t^2} r_{\alpha,\beta}(\varepsilon)a\right\|.$$

But if $b \in \mathscr{B}_2$ one also has the identity

$$\frac{1}{t^2}[\sigma_t,[\sigma_t,K_\alpha]]b = \sigma_t\left\{\frac{1}{\alpha t^2}\int_{\alpha-t}^{\alpha} ds\sigma_s[\sigma_t,K]\sigma_{-s} - \frac{1}{\alpha t^2}\int_{-t}^{0} ds\sigma_s[\sigma_t,K]\sigma_{-s}\right\}b.$$

Hence applying Condition 2, one finds

$$\left\|\frac{[\sigma_t, [\sigma_t, K_\alpha]]}{t^2} b\right\| \leqq \frac{2k_1}{\alpha} \|b\|_1.$$

Therefore combining these estimates gives

$$\left\|\frac{(\sigma_t-1)^2}{t^2} r_{\alpha,\beta}(\varepsilon)a\right\| \leqq \left\|\frac{(\sigma_t-1)^2}{t^2} a\right\| + 2\varepsilon k_1 \left\|\frac{(\sigma_t-1)}{t} r_{\alpha,\beta}(\varepsilon)a\right\|_1 +$$
$$+ 2\varepsilon k_1 \alpha^{-1} \|r_{\alpha,\beta}(\varepsilon)a\|_1.$$

But $r_{\alpha,\beta}(\varepsilon)a \in \mathscr{B}_2$ for all $a \in \mathscr{B}$. Thus taking the limit $t \to 0$ with $a \in \mathscr{B}_2$, one finds

$$\|H^2 r_{\alpha,\beta}(\varepsilon)a\| \leqq \|H^2 a\| + 2\varepsilon k_1 \|H r_{\alpha,\beta}(\varepsilon)a\|_1 + 2\varepsilon k_1 \alpha^{-1} \|r_{\alpha,\beta}(\varepsilon)a\|_1,$$

and combining this with Lemma 2.2 gives

$$\|r_{\alpha,\beta}(\varepsilon)a\|_2 \leqq \|a\|_2 + 2\varepsilon k_1 \|r_{\alpha,\beta}(\varepsilon)a\|_2 + 2\varepsilon k_1 \alpha^{-1} \|a\|_1 (1-\varepsilon k_1)^{-1}.$$

Next we use the estimates to prove strong convergence of $r_{\alpha,\beta}(\varepsilon)$ as $\beta \to 0$, then $\alpha \to 0$. Since the resolvents are contractions it suffices to prove convergence on the dense set $\mathscr{B}_2$. But

$$\begin{aligned}\|(r_{\alpha,\beta_1}(\varepsilon) - r_{\alpha,\beta_2}(\varepsilon))a\| &\leqq \varepsilon|\beta_1 - \beta_2| \|H^2 r_{\alpha,\beta_2}(\varepsilon)a\| \\ &\leqq \varepsilon|\beta_1 - \beta_2| \|r_{\alpha,\beta_2}(\varepsilon)a\|_2\end{aligned}$$

for $a \in \mathscr{B}_2$. Moreover $\|r_{\alpha,\beta}(\varepsilon)a\|_2$ is uniformly bounded in β, whenever $2\varepsilon k_1 < 1$, by Lemma 2.3. Therefore $r_{\alpha,\beta}(\varepsilon)$ converges strongly as $\beta \to 0$, for all $\alpha > 0$, whenever $2\varepsilon k_1 < 1$. Now fixing β one has the estimate

$$\begin{aligned}\|(r_{\alpha_1,\beta}(\varepsilon) - r_{\alpha_2,\beta}(\varepsilon))a\| &\leqq \varepsilon\|(K_{\alpha_1} - K_{\alpha_2}) r_{\alpha_2,\beta}(\varepsilon)a\| \\ &\leqq \varepsilon\|(K_{\alpha_1} - K) r_{\alpha_2,\beta}(\varepsilon)a\| + \varepsilon\|(K_{\alpha_2} - K) r_{\alpha_2,\beta}(\varepsilon)a\|.\end{aligned}$$

But for $b \in \mathscr{B}_2$

$$(2.3) \qquad \begin{aligned}\|(K_\alpha - K)b\| &\leqq \frac{1}{\alpha} \int_0^\alpha ds \|\sigma_s[K, \sigma_{-s}]b\| \\ &\leqq \frac{1}{\alpha} \int_0^\alpha ds k_1 s \|b\|_1 = \frac{k_1 \alpha \|b\|_1}{2}.\end{aligned}$$

Therefore if $\varepsilon k_1 < 1$

$$\|(r_{\alpha_1,\beta}(\varepsilon) - r_{\alpha_2,\beta}(\varepsilon))a\| \leqq \varepsilon k_1 \frac{(\alpha_1+\alpha_2)}{2} \|r_{\alpha_2,\beta}(\varepsilon)a\|_1$$

$$\leqq \varepsilon k_1 \frac{(\alpha_1+\alpha_2)}{2} \|a\|_1 (1-\varepsilon k_1)^{-1}$$

by Lemma 2.2. This establishes that the strong limit

$$r(\varepsilon) = \lim_{\alpha\to 0} \lim_{\beta\to 0} r_{\alpha,\beta}(\varepsilon)$$

exists for $2\varepsilon k_1 < 1$.

Next remark that if $a \in \mathscr{B}_2$ then

$$\|(r_{\alpha,\beta}(\varepsilon)-1)a\| \leqq \varepsilon(\|K_\alpha a\| + \beta\|H^2 a\|).$$

Hence if $\beta < 1$ there is by Condition 1 a $k > 1$ such that

$$\|(r_{\alpha,\beta}(\varepsilon)-1)a\| \leqq \varepsilon k \|a\|_2$$

uniformly in α and β. Thus

$$\lim_{\varepsilon\to 0} r_{\alpha,\beta}(\varepsilon) = 1$$

uniformly in α and β.

It now follows from this uniformity, by the Trotter-Kato theorem, (see [9, Chapter IX, Theorem 2.17]), that

$$r(\varepsilon) = (I+\varepsilon\hat{K})^{-1}$$

where $\hat{K}$ is the generator of a contraction semigroup τ. Moreover

$$\lim_{\alpha\to 0} \lim_{\beta\to 0} \tau_t^{\alpha,\beta} a = \tau_t a, \quad a \in \mathscr{B},$$

uniformly for t in finite intervals. Next we identify $\hat{K}$ with the closure of K.

Now if $a \in D(\hat{K})$, set $b = (I+\varepsilon\hat{K})a$, and then $a = r(\varepsilon)b$. Next choose $b_n \in \mathscr{B}_2$ such that $\|b_n - b\| \to 0$ as $n \to \infty$ and set

$$a_{\alpha,\beta,n} = r_{\alpha,\beta}(\varepsilon)b_n.$$

It follows that $a_{\alpha,\beta,n} \in \mathscr{B}_2$ and

$$\lim_{n\to\infty} \lim_{\alpha\to 0} \lim_{\beta\to 0} \|a_{\alpha,\beta,n} - a\| = 0.$$

But one also has

$$(I+\varepsilon K)a_{\alpha,\beta,n} - b_n = \varepsilon(K-K_\alpha)r_{\alpha,\beta}(\varepsilon)b_n + \varepsilon\beta H^2 r_{\alpha,\beta}(\varepsilon)b_n.$$

Hence using the estimate (2.3), one finds

$$\|(I+\varepsilon K)a_{\alpha,\beta,n}-b_n\| \leqq \varepsilon k_1\alpha\|r_{\alpha,\beta}(\varepsilon)b_n\|_1+\varepsilon\beta\|r_{\alpha,\beta}(\varepsilon)b_n\|_2.$$

Thus by Lemmas 2.2 and 2.3

$$\lim_{n\to\infty}\lim_{\alpha\to 0}\limsup_{\beta\to 0}\|(I+\varepsilon K)a_{\alpha,\beta,n}-b\| = 0.$$

This proves that $a\in D(\bar{K})$ and

$$(I+\varepsilon\bar{K})a = b = (I+\varepsilon\hat{K})a.$$

Therefore $\bar{K}$ is an extension of $\hat{K}$. But since $\hat{K}$ is a generator, it has no proper dissipative extension, and hence $\bar{K}=\hat{K}$, i.e. the closure $\bar{K}$ of K generates the C_0-contraction semigroup τ.

To prove $\tau\mathscr{B}_{1/2}\subseteqq\mathscr{B}_{1/2}$, we first recall that if $a\in\mathscr{B}_1$, then $a\in\mathscr{B}_{1/2}$ and $\|a\|_1=\|a\|_{1/2}$. But since $r_{\alpha,\beta}(\varepsilon)a\in\mathscr{B}_2$ for all $a\in\mathscr{B}$, one has

$$\|r_{\alpha,\beta}(\varepsilon)a\|_{1/2} = \|r_{\alpha,\beta}(\varepsilon)a\|_1.$$

Hence if $a\in\mathscr{B}_{1/2}$ and $\varepsilon k_1 < 1$ one has

$$\left\|\frac{(\sigma_t-1)}{t}r_{\alpha,\beta}(\varepsilon)a\right\| \leqq \|r_{\alpha,\beta}(\varepsilon)a\|_{1/2} \leqq \|a\|_{1/2}(1-\varepsilon k_1)^{-1}$$

by the estimate (2.1). Consequently

$$\left\|\frac{(\sigma_t-1)}{t}r(\varepsilon)a\right\| \leqq \|a\|_{1/2}(1-\varepsilon k_1)^{-1}.$$

This establishes that $r(\varepsilon)\mathscr{B}_{1/2}\subseteqq\mathscr{B}_{1/2}$ and

$$\|r(\varepsilon)a\|_{1/2} \leqq \|a\|_{1/2}(1-\varepsilon k_1)^{-1}.$$

Therefore by iteration

$$\left\|\frac{(\sigma_t-1)}{t}r(s/n)^n a\right\| \leqq \|a\|_{1/2}(1-(s/n)k_1)^{-n}, \quad s\geqq 0.$$

Thus in the limit $n\to\infty$,

$$\left\|\frac{(\sigma_t-1)}{t}\tau_s a\right\| \leqq \|a\|_{1/2}e^{k_1 s},$$

which implies $\tau\mathscr{B}_{1/2}\subseteqq\mathscr{B}_{1/2}$ and

$$\|\tau_s a\|_{1/2} \leqq \|a\|_{1/2}e^{k_1 s}.$$

Now consider the final statement of the theorem. Since $\mathscr{B}_2$ is a core for $\bar{K}$, it follows from Condition 1 that $\mathscr{D}_1$ is a core for $\bar{K}$, and dissipativity is equivalent to the requirement that

$$\|(1+\varepsilon(\bar{K}+k_1)a\| \geqq (1+\varepsilon k_1)\|a\|, \quad a \in \mathscr{D}_1.$$

Hence

$$\|(1+\varepsilon(\bar{K}+k_1))a\| \geqq (1+\varepsilon k_1)\|a\| - \varepsilon k_1\|a\|_1, \quad a \in \mathscr{D}_1. \tag{2.4}$$

Next for $a \in \mathscr{D}_1$ one has

$$\left\|\frac{(\sigma_t-1)}{t}(1+\varepsilon(\bar{K}+k_1))a\right\| \geqq \left\|(1+\varepsilon(\bar{K}+k_1))\frac{(\sigma_t-1)}{t}a\right\| - \varepsilon\left\|\frac{[\bar{K},\sigma_t]}{t}a\right\|$$

$$\geqq (1+\varepsilon k_1)\left\|\frac{(\sigma_t-1)}{t}a\right\| - \varepsilon k_1\|a\|_1.$$

But since $\bar{K}\mathscr{D}_1 \subseteqq \mathscr{B}_1$ one has $\bar{K}a \in \mathscr{B}_1$, and hence in the limit $t \to 0$,

$$\|H(1+\varepsilon(\bar{K}+k_1))a\| \geqq (1+\varepsilon k_1)\|Ha\| - \varepsilon k_1\|a\|_1.$$

Therefore combining this with (2.4), one concludes that

$$\|(1+\varepsilon(\bar{K}+k_1))a\|_1 \geqq (1+\varepsilon k_1)\|a\|_1 - \varepsilon k_1\|a\|_1 = \|a\|_1$$

for all $a \in \mathscr{D}_1$, that is $\bar{K}+k_1$ is $\|\cdot\|_1$-dissipative. Next we prove that $(1+\varepsilon(\bar{K}+k_1))\mathscr{D}_1$ is dense in $\mathscr{B}_1$.

Suppose this is not the case. Then there is an $\omega \in \mathscr{B}_1^*$ such that

$$\omega((1+\varepsilon(\bar{K}+k_1))a) = 0$$

for all $a \in \mathscr{D}_1$. Since $\mathscr{D}_1$ is σ-invariant, it then follows from standard semigroup approximation theory, using Condition 1, that

$$\omega((1+\varepsilon(\bar{K}+k_1))Ra) = 0 \tag{2.5}$$

for all $a \in \mathscr{D}_1$, where $R = (1+H)^{-1}$. But R is a bounded map from $\mathscr{B}$ into $\mathscr{B}_1$ with norm less than or equal to 2, and hence $R^*\omega$ defines a bounded linear functional on $\mathscr{B}$. Now to prove $\omega = 0$ it suffices to prove that $R^*\omega = 0$ because the range of R is equal to $\mathscr{B}_1$. But from (2.5), one has

$$\begin{aligned}\omega(R(1+\varepsilon\bar{K})a) &= -\varepsilon k_1\omega(Ra) + \varepsilon\omega([R,\bar{K}]a)\\ &\leqq \varepsilon k_1\|R^*\omega\|\|a\| + \varepsilon\omega(R[\bar{K},H]Ra)\\ &\leqq 3\varepsilon k_1\|R^*\omega\|\|a\|\\ &\leqq 3\varepsilon k_1\|R^*\omega\|\|(1+\varepsilon\bar{K})a\|, \quad a \in \mathscr{D}_1,\end{aligned}$$

where the last step uses dissipativity of $\bar{K}$. But since $(1+\varepsilon K)(\mathscr{D}_1)$ is dense in $\mathscr{B}$, this gives

$$\|R^*\omega\| \leqq 3\varepsilon k_1 \|R^*\omega\|$$

and choosing ε such that $3\varepsilon k_1 < 1$, one deduces that $R^*\omega = 0$, that is $\omega = 0$.

Finally it follows from dissipativity and the range condition, by the Hille-Yosida theorem, that $\bar{K}+k_1$ generates a C_0-semigroup of contractions on $(\mathscr{B}_1, \|\cdot\|_1)$. Thus $\bar{K}$ generates a C_0-semigroup on $\mathscr{B}_1$ satisfying

$$\|\tau_t a\|_1 \leqq e^{k_1 t}\|a\|_1.$$

COROLLARY 2.4. *Adopt the assumptions of Theorem 2.1 but further assume that $\mathscr{B}$ is reflexive. Then $\bar{K}\mathscr{B}_3 \subseteqq \mathscr{B}_1$, and hence the C_0-semigroup of contractions τ generated by $\bar{K}$ satisfies $\tau\mathscr{B}_1 \subseteqq \mathscr{B}_1$ and $\tau|_{\mathscr{B}_1}$ is a C_0-semigroup such that*

$$\|\tau_t a\|_1 \leqq e^{k_1 t}\|a\|_1, \quad t \geqq 0, a \in \mathscr{B}_1.$$

PROOF. Since $\mathscr{B}$ is reflexive, σ is a C_0^*-group, and hence $a \in \mathscr{B}_1$ if, and only if

$$\sup_{t \neq 0} \left\| \frac{(1-\sigma_t)}{t} a \right\| < +\infty,$$

i.e., if and only if $a \in \mathscr{B}_{1/2}$ (see, for example, [5, Proposition 3.1.23]). But if $a \in \mathscr{B}_3$ then

$$\sup_{t \neq 0} \left\| \frac{(1-\sigma_t)}{t} Ka \right\| \leqq \sup_{t \neq 0} \left\| \frac{[\sigma_t, \bar{K}]}{t} a \right\| + \sup_{t \neq 0} \left\| \frac{\bar{K}(1-\sigma_t)}{t} a \right\|$$
$$\leqq k_1 \|a\|_1 + \kappa \|a\|_3$$

where the first term in the last expression derives from use of Condition 2, and the second term from use of Condition 1, in Theorem 2.1. Thus $\bar{K}\mathscr{B}_3 \subseteqq \mathscr{B}_1$ and the statement concerning τ follows from the last statement of Theorem 2.1.

The next example shows that a condition of the type $K\mathscr{D} \subseteqq \mathscr{B}_1$ is essential for the conclusion $\tau\mathscr{B}_1 \subseteqq \mathscr{B}_1$.

EXAMPLE 2.5. Let $\mathscr{B} = C_0(\mathbf{R})$, the space of complex continuous functions on $\mathbf{R}$ vanishing at infinity, equipped with the supremum norm, and let

$$(\sigma_t f)(x) = f(x-t).$$

Then $\mathscr{B}_1 = \{f \in \mathscr{B}; f' \in \mathscr{B}\}$ and $Hf = f'$. Let $\mathscr{D} = \mathscr{B}_\infty$ and define K by

$$(Kf)(x) = (1-e^{-|x|})f'(x).$$

Then $Kf \in \mathscr{B}_1$ if and only if $f'(0) = 0$. Now

$$\|Kf\| \leqq \|f'\| \leqq \|f\|_1$$

and

$$\|[\sigma_t, K]f\| \leqq |t|\|f'\| \leqq |t|\|f\|_1,$$

i.e. the conditions of Theorem 2.1 are satisfied. Next one computes that $\bar{K}$ generates the C_0-group of isometries given by

$$\begin{aligned}(\tau_t f)(x) &= f(\log(1-e^{-t}+e^{x-t})), \quad x \geqq 0\\ &= f(-\log(1-e^{t}+e^{t-x})), \quad x < 0.\end{aligned}$$

Thus if $f \in \mathscr{B}_1$ then

$$\begin{aligned}((\tau_t f)(x)-(\tau_t f)(0))/x &\to e^{-t}f'(0), \quad x \to 0+\\ &\to e^{t}f'(0), \quad x \to 0-.\end{aligned}$$

Consequently $\tau_t f \notin \mathscr{B}_1$ unless $f'(0) = 0$.

To conclude this section, we note that Theorem 2.1 has a generalization to inductive limits of Banach spaces which has some use in applications.

Corollary 2.6. *Let the Banach space $\mathscr{B}$ be the closure of an increasing sequence*

$$\mathscr{B}^{(1)} \subseteqq \mathscr{B}^{(2)} \subseteqq \cdots \subseteqq \mathscr{B}^{(n)} \subseteqq \cdots$$

of σ-invariant Banach subspaces and let $\mathscr{D} \subseteqq \mathscr{B}$ denote a σ-invariant core of H such that $\mathscr{D}^{(n)} = \mathscr{D} \cap \mathscr{B}^{(n)}$ is a core of $H_n = H|_{\mathscr{B}^{(n)}}$.

If $K: \mathscr{D} \to \mathscr{B}$ is a dissipative operator with the property that $K\mathscr{D}^{(n)} \subseteqq \mathscr{B}^{(n)}$ and K satisfies Conditions 1 and 2 of Theorem 2.1 on each $\mathscr{D}^{(n)}$ (with κ_ε and k_1 varying with n), then the closure $\bar{K}$ of K generates a C_0-semigroup of contractions τ.

Proof. Let $K^{(n)}$ denote the restriction of K to $\mathscr{D}^{(n)}$. It follows from Theorem 2.1 that $\bar{K}^{(n)}$ generates a C_0-semigroup of contractions $\tau^{(n)}$ of $\mathscr{B}^{(n)}$. But if $n \leqq m$, then $\mathscr{B}^{(n)} \subseteqq \mathscr{B}^{(m)}$ and $\tau^{(n)} \subseteqq \tau^{(m)}$, because $K^{(n)} \subseteqq K^{(m)}$. Thus defining τ on $\bigcup_{n \geqq 1} \mathscr{B}^{(n)}$ by setting $\tau = \tau^{(n)}$ on $\mathscr{B}^{(n)}$, one can then extend τ to a C_0-semigroup of contractions of $\mathscr{B}$ by continuity. Let $\hat{K}$ denote the generator of τ, and note that $\hat{K}|_{\mathscr{B}^{(n)}} = \bar{K}^{(n)}$ by construction. Now if $a \in D(\hat{K})$ and $b = (1+\hat{K})a$, then there exists a sequence $b_n \in \mathscr{D}^{(n)}$ such that $\|b_n - b\| \to 0$, and since $\bar{K}^{(n)}$ is a generator on $\mathscr{B}^{(n)}$ there exists a sequence $a_n \in D(\bar{K}^{(n)})$ such that

$$b_n = (1+\bar{K}^{(n)})a_n = (1+\hat{K})a_n.$$

Then it follows that $a_n = (1+\hat{K})^{-1}b_n \to a$. But since $\mathscr{D}^{(n)}$ is a core of $\bar{K}^{(n)}$, one concludes that $\mathscr{D} = \bigcup_{n\geq 1}\mathscr{D}^{(n)}$ is a core of $\hat{K}$, that is $\hat{K} = \bar{K}$.

REMARKS 1. If $K:\mathscr{D} \to \mathscr{B}$ is a dissipative operator, where $\mathscr{D}$ is a σ-invariant core for H, satisfying the conditions

1. $\quad \|Ka\| \leqq k\|a\|_p, \quad a \in \mathscr{D},$

for some $k \geqq 0$, $p \geqq 0$,

2. $\quad [\sigma_t, K] = 0,$

then it is easily seen that $\bar{K}$ generates a C_0-semigroup of contractions. For, $\bar{K}$ restricts to a dissipative, everywhere defined, hence bounded, operator, generating a uniformly continuous semigroup of contractions, on each spectral subspace $\mathscr{B}^\sigma(\Omega)$ for compact subsets Ω of R. These semigroups are mutually consistent, and extend by continuity to a C_0-semigroup of contractions generated by $\bar{K}$.

2. There is an alternative proof of Theorem 2.1 in the case, when Condition 1 is replaced by the slightly stronger condition:

$$\|Ka\| \leqq \kappa_0\|a\|_1, \quad a \in \mathscr{D}.$$

Let $f \in L^1(\mathrm{R})$ be such that $\hat{f}$ has compact support Λ, and put

$$K_n = n\int ds f(ns)\sigma_s K\sigma_{-s}.$$

Since $\overline{K_n}\mathscr{B}^\sigma(\Omega) \subseteqq \mathscr{B}^\sigma(\Omega+n\Lambda)$, an easy estimation shows that every vector in $\mathscr{B}^\sigma(\Omega)$ is analytic for $\overline{K_n}$, for any compact Ω. Therefore, by some standard theory, $\overline{K_n}$ generates a C_0-contraction semigroup. Some estimates similar to those in the proof above show that the resolvents $(I+\varepsilon\overline{K_n})^{-1}$ converge in the required fashions as $n \to \infty$, and as $\varepsilon \to 0$, and it follows that K extends to a generator.

3. Higher-order estimates.

Theorem 2.1 can be extended in a variety of ways. One possible extension involves bounds on higher order commutators such as $[\sigma_t, [\sigma_t, K]]$. Such bounds can be expressed in various equivalent fashions, and it is of some significance that they automatically imply smoothing properties such as $K\mathscr{B}_\infty \subseteqq \mathscr{B}_1$.

PROPOSITION 3.1. *Let σ be a C_0-group of isometries of the Banach space $\mathscr{B}$ with generator H and set $\mathscr{B}_\infty = \bigcap_{n\geq 1} D(H^n)$. If $K:\mathscr{B}_\infty \to \mathscr{B}$ is a linear operator* *satisfying*

$$\|Ka\| \leqq k_0\|a\|_n$$

for some $k_0 > 0$, $n \geqq 1$, *and all* $\in \mathscr{B}_\infty$, *then the following conditions are equivalent for each* $p \geqq 1$, $k_1 \geqq 0$:

1. $\|[\sigma_t, [\sigma_t, K]]a\| \leqq k_1 t^2 \|a\|_p$, $\quad t \in \mathrm{R},\ a \in \mathscr{B}_\infty$,
2. $\|[\sigma_s, [\sigma_t, K]]a\| \leqq k_1 |st| \|a\|_p$, $\quad s,t \in \mathrm{R},\ a \in \mathscr{B}_\infty$,
3. $K\mathscr{B}_\infty \subseteqq \mathscr{B}_1$ *and* $\|[\sigma_t, [H, K]]a\| \leqq k_1 |t| \|a\|_p$, $\quad t \in \mathrm{R},\ a \in \mathscr{B}_\infty$.

PROOF. 2 ⇒ 1. This is evident.

3 ⇒ 2. By differentiation of $u \to [\sigma_s, \sigma_u K \sigma_{t-u}]$ and subsequent integration one obtains the relation

$$[\sigma_s, [\sigma_t, K]]a = \int_0^t du [\sigma_s, \sigma_u [K, H] \sigma_{t-u}]a$$

$$= \int_0^t du \sigma_u [\sigma_s, [K, H]] \sigma_{t-u} a.$$

But by Condition 3

$$\|\sigma_u [\sigma_s, [K, H]] \sigma_{t-u} a\| \leqq k_1 |s| \|\sigma_{t-u} a\|_p = k_1 |s| \|a\|_p$$

and Condition 2 follows immediately.

1 ⇒ 3. Fix $a \in \mathscr{B}_\infty$ and for $t \neq 0$ define

$$b_t = t^{-1}(\sigma_t K \sigma_{-t} a - Ka),$$

$$b'_t = t^{-1}(\sigma_{2t} K \sigma_{-2t} a - \sigma_t K \sigma_{-t} a).$$

It then follows from Condition 1, with a replaced by $\sigma_{-2t} a$, that

$$\|b_t - b'_t\| \leqq k_1 |t| \|a\|_p.$$

But $b_t = (b_{t/2} + b'_{t/2})/2$ and so

$$\|b_t - b_{t/2}\| = \|b_{t/2} - b'_{t/2}\|/2 \leqq k_1 |t| \|a\|_p/4.$$

Writing $t_m = 2^{-m} t$ and replacing t by t_m, one finds

$$\|b_{t_m} - b_{t_{m+1}}\| \leqq 2^{-m-2} k_1 |t| \|a\|_p.$$

It follows that b_{t_m} converges to a limit d_t as $m \to \infty$ and

$$\|b_t - d_t\| \leqq k_1 |t| \|a\|_p / 2. \tag{3.1}$$

Now

$$\left\| \frac{(\sigma_t - 1)}{t} Ka - b_t + \sigma_t KHa \right\| = \left\| \sigma_t K \left\{ \frac{(1 - \sigma_{-t})}{t} a + Ha \right\} \right\| \underset{t \to 0}{\longrightarrow} 0 \tag{3.2}$$

because $\|\sigma_t Kc\| \leqq k_0 \|c\|_n$ for all $c \in \mathscr{B}_\infty$ and

$$\lim_{t \to 0} \left\| \frac{(1 - \sigma_{-t})}{t} a + Ha \right\|_n = 0.$$

In particular for t fixed

$$\lim_{m \to \infty} \left\| \frac{(\sigma_{t_m} - 1)}{t_m} Ka - d_t + KHa \right\| = 0$$

since $\|b_{t_m} - d_t\| \to 0$ and σ is strongly continuous. But for $s \neq 0, -t$, one has the identity

$$\frac{(\sigma_{s_m + t_m} - 1)}{s_m + t_m} Ka = \frac{s}{s+t} \sigma_{t_m} \frac{(\sigma_{s_m} - 1)}{s_m} Ka + \frac{t}{s+t} \frac{(\sigma_{t_m} - 1)}{t_m} Ka.$$

Letting $m \to \infty$ one finds

$$d_{s+t} - KHa = \frac{s}{s+t}(d_s - KHa) + \frac{t}{s+t}(d_t - KHa)$$

or, equivalently,

$$(s+t) d_{s+t} = s d_s + t d_t.$$

Thus the function $t \to t d_t$ is additive. But it follows from (3.1) and the boundedness hypothesis on K that

$$\|t d_t\| \leqq k_1 t^2 \|a\|_p / 2 + 2 k_0 \|a\|_n.$$

Consequently $t \to d_t$ must be constant. Setting $d_t = d$, it then follows from (3.1) and (3.2) and strong continuity of σ, that

$$\lim_{t \to 0} \left\| \frac{(\sigma_t - 1)}{t} Ka - d + KHa \right\| = 0.$$

Thus $Ka \in \mathscr{B}_1$ and

$$HKa = KHa - d.$$

In view of the identity

$$[\sigma_t, [\sigma_{t_m}, K]] = \sum_{r=1}^{2^m} \sigma_{(r-1)t_m}[\sigma_{t_m}, [\sigma_{t_m}, K]]\sigma_{(2^m-r)t_m}$$

it follows from Condition 1 that

$$\|[\sigma_t, [\sigma_{t_m}, K]]a\| \leqq 2^m k_1 t_m^2 \|a\|_p = k_1 t t_m \|a\|_p.$$

Hence

$$\|[\sigma_t, [H, K]]a\| = \lim_{m\to\infty} \left\| \left[\sigma_t, \frac{[\sigma_{t_m} - I, K]}{t_m} \right] a \right\|$$
$$\leqq k_1 |t| \|a\|_p.$$

This establishes Condition 3 and completes the proof of the proposition.

The foregoing result, and its proof, are very similar to the equivalent characterizations of the subspace $\mathscr{B}_{3/2}$ given in [15]. We define this subspace by

$$\mathscr{B}_{3/2} = \{a \in \mathscr{B}_1 ; Ha \in \mathscr{B}_{1/2}\}$$

and set

$$\|a\|_{3/2} = \|a\|_{1/2} \vee \|Ha\|_{1/2}.$$

Proposition 4.2 of [15] gives other equivalent definitions and we will use the fact that if $a \in \mathscr{B}_{3/2}$ then

$$\sup_{t\neq 0} \left\| \frac{(1-\sigma_t)}{t} Ha \right\| = \sup_{t \neq 0} \left\| \frac{(1-\sigma_t)^2}{|t|^2} a \right\|.$$

THEOREM 3.2. *Adopt the assumptions of Theorem* 2.1 *but further assume that* K *satisfies the condition*

3. *there is a* $k_2 > 0$ *such that*

$$\|[\sigma_t, [\sigma_t, K]]a\| \leqq k_2 \|a\|_2 t^2, \quad a \in \mathscr{D}, t \in \mathbf{R}.$$

Let τ *denote the* C_0*-contraction semigroup on* $\mathscr{B}$ *generated by* $\bar{K}$.

It follows that $\tau\mathscr{B}_1 \subseteqq \mathscr{B}_1$ *and* $\tau|_{\mathscr{B}_1}$ *is a* C_0*-semigroup satisfying*

$$\|\tau_t a\|_1 \leqq e^{k_1 t}\|a\|_1, \quad t \geqq 0, a \in \mathscr{B}_1. \tag{3.3}$$

Moreover $\tau\mathscr{B}_{3/2} \subseteqq \mathscr{B}_{3/2}$ *and*

$$\|\tau_t a\|_{3/2} \leqq e^{(k_2+2k_1)t}\|a\|_{3/2}, \quad t \geqq 0, a \in \mathscr{B}_{3/2}.$$

Finally if $\bar{K}\mathscr{D} \subseteqq \mathscr{B}_2$ *for some* σ*-invariant core* $\mathscr{D}$ *of* H, *then* $\tau\mathscr{B}_2 \subseteqq \mathscr{B}_2$ *and*

$\tau|_{\mathscr{B}_2}$ *is a* C_0*-semigroup satisfying*

$$\|\tau_t a\|_2 \leqq e^{(2k_1+k_2)t}\|a\|_2, \quad t \geqq 0,\ a \in \mathscr{B}_2.$$

PROOF. It follows as in Theorem 2.1 that K extends to $\mathscr{B}_2$ and Conditions 1, 2 and 3 also extend to all $a \in \mathscr{B}_2$. Therefore, by Proposition 3.1, $K\mathscr{B}_\infty \subseteqq \mathscr{B}_1$ and the first statement of the theorem follows from the last statement of Theorem 2.1.

Now to prove $\tau\mathscr{B}_{3/2} \subseteqq \mathscr{B}_{3/2}$ we proceed as in the proof of $\tau\mathscr{B}_{1/2} \subseteqq \mathscr{B}_{1/2}$ in Theorem 2.1.

First by Lemma 2.2 one has

$$\|r_{\alpha,\beta}(\varepsilon)a\|_1 \leqq \|a\|_{3/2}(1-\varepsilon k_1)^{-1} \leqq \|a\|_{3/2}(1-2\varepsilon k_1-\varepsilon k_2)^{-1}$$

for $2\varepsilon k_1+\varepsilon k_2 < 1$ and all $a \in \mathscr{B}_{3/2}$. Second by Condition 3 and the estimate (2.2), one obtains

$$\left\|\frac{(1-\sigma_t)^2}{t^2} r_{\alpha,\beta}(\varepsilon)a\right\| \leqq \|a\|_{3/2} + 2\varepsilon k_1\|r_{\alpha,\beta}(\varepsilon)a\|_2 + \varepsilon k_2\|r_{\alpha,\beta}(\varepsilon)a\|_2$$

for $a \in \mathscr{B}_{3/2}$. Hence if $2\varepsilon k_1+\varepsilon k_2 < 1$ one deduces that

$$\|r_{\alpha,\beta}(\varepsilon)a\|_2 \leqq \|a\|_{3/2}(1-2\varepsilon k_1-\varepsilon k_2)^{-1}.$$

Hence

$$\left\|\frac{(1-\sigma_t)^2}{t^2} r_{\alpha,\beta}(\varepsilon)a\right\| \leqq \|a\|_{3/2}(1-2\varepsilon k_1-\varepsilon k_2)^{-1}.$$

Consequently in the limit $\beta \to 0$, then $\alpha \to 0$, one finds

$$\left\|\frac{(1-\sigma_t)^2}{t^2} r(\varepsilon)a\right\| \leqq \|a\|_{3/2}(1-2\varepsilon k_1-\varepsilon k_2)^{-1}.$$

This establishes that $r(\varepsilon)\mathscr{B}_{3/2} \subseteqq \mathscr{B}_{3/2}$ and

$$\|r(\varepsilon)a\|_{3/2} \leqq \|a\|_{3/2}(1-2\varepsilon k_1-\varepsilon k_2)^{-1}.$$

Therefore by iteration

$$\left\|\frac{(1-\sigma_t)^2}{t^2} r(s/n)^n a\right\| \leqq \|a\|_{3/2}(1-(s/n)(2k_1+k_2))^{-n}, \quad s \geqq 0,$$

and in the limit $n \to \infty$

$$\left\|\frac{(1-\sigma_t)^2}{t^2} \tau_s a\right\| \leqq \|a\|_{3/2}e^{(2k_1+k_2)s}.$$

It then follows from (3.3) and the remark preceding the theorem that

$\tau\mathscr{B}_{3/2} \subseteqq \mathscr{B}_{3/2}$ and

$$\|\tau_s a\|_{3/2} \leqq \|a\|_{3/2} e^{(2k_1+k_2)s}, \quad s \geqq 0.$$

The proof of the final statement of the theorem is similar to the proof of the corresponding statement for $\tau|_{\mathscr{B}_1}$ in Theorem 2.1 and we omit the details.

This type of reasoning can be extended; higher order commutator estimates lead to increased smoothness properties of τ. Another type of generalization of Theorem 2.1 arises by weakening Condition 1. This is relatively straightforward if γH^n generates a C_0-contraction semigroup for some $\gamma \in \mathbf{C}$ with $|\gamma| = 1$ and some large n, as is the case when $\mathscr{B}$ is a Hilbert space.

THEOREM 3.3. *Let σ be a C_0-group of isometries of the Banach space $\mathscr{B}$ with generator H, $\mathscr{D} \subseteqq \mathscr{B}_p$ a σ-invariant core of H, and K an operator from $\mathscr{D}$ into $\mathscr{B}$.*

Assume K is dissipative,

1. $$\|Ka\| \leqq k\|a\|_p \quad , \quad a \in \mathscr{D}$$
2. $$\|[\sigma_t, K]a\| \leqq k_1\|a\|_1|t|, \quad t \in \mathbf{R},\ a \in \mathscr{D},$$

and γH^n is the generator of a C_0-contraction semigroup for some $n > p$ and some $\gamma \in \mathbf{C}$ with $|\gamma| = 1$.

It follows that the closure $\bar{K}$ of K is the generator of a C_0-contraction semigroup τ.

PROOF. The proof follows the general line of reasoning used to prove Theorem 2.1. We will just sketch the necessary modifications.

First one defines $H_{\alpha,\beta}$ by

$$H_{\alpha,\beta} = K_\alpha^{(n)} + \beta\gamma H^n, \quad \alpha,\beta > 0$$

where $K_\alpha^{(n)}$ is the n-fold regularization of K,

$$K_\alpha^{(n)} = \frac{1}{\alpha^n}\int_0^\alpha dt_1 \ldots \int_0^\alpha dt_n \sigma_{t_1+\cdots+t_n} K \sigma_{-t_1-\cdots-t_n}.$$

Then the $H_{\alpha,\beta}$ generate C_0-contraction semigroups.

Next one examines convergence of the resolvents $r_{\alpha,\beta}(\varepsilon)$ for small positive ε as $\beta \to 0$ then $\alpha \to 0$. For this it is necessary to bound $\|r_{\alpha,\beta}(\varepsilon)a\|_1$ uniformly in α and β for all $a \in \mathscr{B}_n$ and all small positive ε, and then to bound $\|r_{\alpha,\beta}(\varepsilon)a\|_n$ uniformly in β. If $b = r_{\alpha,\beta}(\varepsilon)a$, then $b \in \mathscr{B}_n$ and

$$H^n b = -(\varepsilon\beta\gamma)^{-1}(b + \varepsilon K_\alpha^{(n)} b - a) \in \mathscr{B}_{n-1},$$

so $b \in \mathscr{B}_{2n-1}$. The first bound is obtained as in Lemma 2.2, but the second bound needs a slight extension of the previous technique. One proceeds by induction. First one has

$$\|r_{\alpha,\beta}(\varepsilon)a\|_1 \leqq \|a\|_1(1-\varepsilon k_1)^{-1}$$

for $\varepsilon k_1 < 1$. Next suppose

$$\|r_{\alpha,\beta}(\varepsilon)a\|_s \leqq \|a\|_s c_s(\alpha, \varepsilon)$$

for $\varepsilon k_1 s < 1$, with $c_s(\alpha, \varepsilon)$ independent of β, and for $1 \leqq s < m$. Now

$$\|H^m r_{\alpha,\beta}(\varepsilon)a\| \leqq \|H^m a\| + \varepsilon\|[H^m, K_\alpha^{(n)}]r_{\alpha,\beta}(\varepsilon)a\|.$$

But one has the combinatorial relation

$$[H^m, A] = \sum_{r=1}^{m} \binom{m}{r} [H, A]^{(r)} H^{m-r}$$

where $[H, A]^{(r)}$ denotes the multiple commutator, i.e.

$$[H, A]^{(1)} = [H, A] \quad \text{and} \quad [H, A]^{(s)} = [H, [H, A]^{(s-1)}].$$

But

$$\|[H, K_\alpha^{(n)}]H^{m-1}b\| \leqq \sup_{t \neq 0} \left\| \left[\frac{\sigma_t}{t}, K_\alpha^{(n)}\right] H^{m-1}b \right\|$$

$$\leqq k_1\|b\|_m$$

by use of Condition 2. Moreover

$$[H, K_\alpha^{(n)}] = (\sigma_\alpha K_\alpha^{(n-1)}\sigma_{-\alpha} - K_\alpha^{(n-1)})/\alpha$$

and so

$$\|[H, K_\alpha^{(n)}]^{(2)}H^{m-2}b\| \leqq \alpha^{-1}\|[H, K_\alpha^{(n-1)}]\sigma_{-\alpha}H^{m-2}b\| + \alpha^{-1}\|[H, K_\alpha^{(n-1)}]H^{m-2}b\|.$$

Therefore using the preceding estimation technique

$$\|[H, K_\alpha^{(n)}]^{(2)}H^{m-2}b\| \leqq (2\alpha^{-1})k_1\|b\|_{m-1}.$$

Similarly

$$\|[H, K_\alpha^{(n)}]^{(r)}H^{m-r}b\| \leqq (2\alpha^{-1})^{r-1}k_1\|b\|_{m-r+1}.$$

Combining these estimates one obtains

$$\|H^m r_{\alpha,\beta}(\varepsilon)a\| \leqq \|a\|_m + \varepsilon k_1 m\|r_{\alpha,\beta}(\varepsilon)a\|_m + \varepsilon \sum_{r=2}^{m} \binom{m}{r} (2\alpha^{-1})^{r-1}k_1\|r_{\alpha,\beta}(\varepsilon)a\|_{m-r+1}.$$

Therefore by the induction hypothesis, one has

$$\|H^m r_{\alpha,\beta}(\varepsilon)a\| \leqq \varepsilon k_1 m \|r_{\alpha,\beta}(\varepsilon)a\|_m + d_m(\alpha, \varepsilon)\|a\|_m$$

for a suitable $d_m(\alpha, \varepsilon)$ and $\varepsilon k_1(m-1) < 1$. This immediately yields the bound

$$\|r_{\alpha,\beta}(\varepsilon)a\|_m \leqq (d_m(\alpha, \varepsilon) \vee c_{m-1}(\alpha, \varepsilon))\|a\|_m(1-\varepsilon k_1 m)^{-1}$$

with the additional constraint $\varepsilon k_1 m < 1$. This procedure works for all $m \leqq n$ to give a bound independent of β.

The proof of convergence of the resolvents $r_{\alpha,\beta}(\varepsilon)$ as $\beta \to 0$, then $\alpha \to 0$, now proceeds as in proof of Theorem 2.1, as does the rest of the proof, and so we will omit further details.

4. Applications and related results.

In this final section we discuss some of the foregoing topics in special contexts.

A. *Hilbert space.*

Let $\mathscr{H}$ be a Hilbert space, H a self-adjoint operator on $\mathscr{H}$, and $\sigma_t = \exp\{iHt\}$ the unitary group generated by $-iH$. Next let K be a symmetric operator on $\mathscr{H}$ with domain $D(K) \supseteqq D(H^p)$ for some $p \geqq 1$. Assume

$$\|K(1+iH)^{-p}\| < +\infty$$

$$\sup_{t \neq 0} \|[\sigma_t, K](1+iH)^{-1}\|/|t| < +\infty.$$

Then it follows that K is essentially self-adjoint and

$$e^{it\bar{K}}D(H) \subseteqq D(H).$$

This statement is a direct consequence of Theorem 3.3 once one notes that $-iH^n$ generates a C_0-group of isometries for all $n \geqq 1$, and, since K is symmetric, $\pm K$ are both dissipative.

If $H \geqq 0$ and $p = 1$ the essential self-adjointness of K corresponds to Glimma and Jaffe's original theorem, Theorem 1.2 of [7]. If $H \geqq 0$ and $p > 1$, it is a result of Jaffe, [12, Theorem 2.2].

If $H \geqq 0$ and $p = 1$, Faris and Lavine [6] showed that the quadratic form domain of H, i.e. the domain $q(H) = D((1+H)^{1/2})$ is invariant under the group $\exp\{i\bar{K}t\}$.

Various applications of these results may be found in [6], [7], [12], [13], [8], [14].

B. *Abelian C*-algebras.*

Let $\mathscr{B} = C_0(X)$, where X is a locally compact Hausdorff space, and let S be a one-parameter group of homeomorphisms of X. Define

$$X_0 = \{\omega \in X\,; S_t\omega = \omega,\ t \in \mathrm{R}\}$$

and set $(\sigma_t f)(\omega) = f(S_t\omega)$. Then σ is a C_0-group of isometries of $\mathscr{B}$, and in fact *-automorphisms, and the generator H of σ is a *-derivation.

Next for $\omega \in X$ define the period $p(\omega)$ by

$$p(\omega) = \inf\{t > 0\,; S_t\omega = \omega\},$$

and the frequency $\nu(\omega)$ by $\nu(\omega) = 1/p(\omega)$. Let λ be a real continuous function on $X \setminus X_0$ and define

$$\begin{aligned}(Kf)(\omega) &= \lambda(\omega)(Hf)(\omega), \quad \omega \in X \setminus X_0,\\ &= 0 \qquad\qquad\quad\ ,\quad \omega \in X_0\ .\end{aligned}$$

Then K defines a linear mapping of $\mathscr{B}_\infty$ into $\mathscr{B}$ if, and only if,

$$|\lambda(\omega)| \leqq c(1 + \nu(\omega)^p), \quad \omega \in X \setminus X_0,$$

for some $c \geqq 0$, $p \geqq 0$, [4], [2]. Under these conditions K is a *-derivation and $\pm K$ are dissipative. Now Condition 1 of Theorem 2.1 is satisfied if, and only if,

$$|\lambda(\omega)(Hf)(\omega)| \leqq \varepsilon\|f\|_2 + \kappa_\varepsilon\|f\|_1$$

or equivalently

$$|\lambda(\omega)| \leqq \varepsilon(1 \vee 4\nu(\omega)) + \kappa_\varepsilon$$

(see [2, Section 5]), for all $\varepsilon > 0$. This in turn means that

$$(4.1) \qquad |\lambda(\omega)| \leqq g(\nu(\omega)), \quad \omega \in X \setminus X_0$$

for some increasing function g such that $g(\nu) = o(\nu)$ as $\nu \to \infty$.

Condition 2 of Theorem 2.1 gives

$$|(\lambda(S_t\omega) - \lambda(\omega))(Hf)(S_t\omega)| \leqq k_1|t|\|f\|_1$$

for all $f \in \mathscr{B}_\infty$ and this is equivalent to the Lipschitz condition

$$|\lambda(S_t\omega) - \lambda(\omega)| \leqq k_1|t|.$$

Thus in this case Theorem 2.1 gives weaker results than are already known (see [3], [15]). For example it is known that $\bar{K}$ generates a C_0-group of *-automorphisms of $\mathscr{B}$ if λ satisfies (4.1), for any increasing function g and

$$|\lambda(S_t\omega) - \lambda(\omega)| \leqq g_1(\nu(\omega))|t|$$

for any increasing function g_1. But this result may be recovered from

Corollary 2.6 by taking

$$\mathscr{B}^{(n)} = \{f \in \mathscr{B}; f(S_t\omega) = f(\omega) \quad \text{for all } t \text{ if } \nu(\omega) \geqq n\},$$

(see [15]).

Next we prove that in this context Condition 1 of Theorem 2.1 may be replaced by $\mathscr{D} \subseteqq \mathscr{B}_n$ and

$$\|Ka\| \leqq k\|a\|_n, \quad a \in \mathscr{D}.$$

for some $k \geqq 0$, $n \geqq 1$, provided that $\pm K$ are dissipative. Thus one obtains a generalization in the direction of Theorem 3.3 but without the unsatisfactory assumption that γH^n is a generator.

Suppose $K : \mathscr{B}_\infty \to \mathscr{B}$, and that $\pm K$ are dissipative. Further suppose $f \in \mathscr{B}_\infty$, $\omega \in X$, and $f = 0$ in a neighbourhood U of ω. There exists an $h \in \mathscr{B}_\infty$ such that $h(\omega) = \|h\| \geqq \|f\|$ and $\operatorname{supp} h \subseteqq U$. For any $\gamma \in \mathbf{C}$ with $|\gamma| = 1$,

$$(h + \gamma f)(\omega) = \|h + \gamma f\| = \|h\|$$

so dissipativity gives

$$\operatorname{Re}((Kh)(\omega) + \gamma(Kf)(\omega)) = 0.$$

It follows that $(Kf)(\omega) = 0$. Thus $\operatorname{supp} Kf \subseteqq \operatorname{supp} f$, and K is local in the sense of [4]. But it follows from locality [4], [2] that there exist an integer n and complex functions λ_m, $0 \leqq m \leqq n$, such that

$$(Kf)(\omega) = \sum_{m=0}^{n} \lambda_m(\omega)(H^m f)(\omega), \quad \omega \in X,\ f \in \mathscr{B}_\infty.$$

Furthermore λ_0 is bounded and continuous on X, and λ_m is zero on X_0 and polynomially bounded in the frequency and continuous on $X \diagdown X_0$, for $1 \leqq m \leqq n$.

Consider a point $\omega \in X \diagdown X_0$. For $k > 2$, $\alpha > 0$, $\beta \in \mathbf{R}$, there is a function $f \in \mathscr{B}_\infty$ such that $\|f\| = 1$ and

$$f(S_t\omega) = 1 - \alpha t^2 - \beta t^k$$

for small $|t|$, [4], [2]. Dissipativity then gives

$$0 = \operatorname{Re}(Kf)(\omega) = \operatorname{Re}(\lambda_0(\omega) - 2\alpha\lambda_2(\omega) - \beta k!\lambda_k(\omega)).$$

Since this holds for all $\alpha > 0$ and $\beta \in \mathbf{R}$ it follows that $\operatorname{Re}\lambda_m(\omega) = 0$ for $0 \leqq m \leqq n$, $m \neq 1$. Next let p be the largest integer for which $\operatorname{Im}\lambda_p(\omega) \neq 0$, and suppose $p > 0$. For any $\beta \in \mathbf{R}$ there is a complex function $g \in \mathscr{B}_\infty$ such that

$$g(S_t\omega) = e^{i\beta t^p}$$

for small $|t|$, and $\|g\| = 1$. Dissipativity gives

$$0 = \mathrm{Re}(Kg)(\omega) = \mathrm{Re}(\lambda_0(\omega) + ip!\beta\lambda_p(\omega)).$$

It follows that $\mathrm{Im}\,\lambda_p(\omega) = 0$ so

$$Kf = \lambda_0 f + \lambda_1 Hf, \quad f \in \mathscr{B}_\infty$$

where λ_0 is purely imaginary and λ_1 is real. Thus K is a bounded perturbation of $\lambda_1 H$. Hence if $\overline{\lambda_1 H}$ generates a C_0-group of isometries, then $\bar{K}$ also generates such a group. In particular this is the case if Condition 2 of Theorem 2.1 is satisfied.

We note that the foregoing argument combined with Theorem 1.2c of [4] gives the following statement. If $K: \mathscr{B}_\infty \to \mathscr{B}$ is real, i.e. $K\bar{f} = \overline{Kf}$ for all $f \in \mathscr{B}_\infty$, then the following conditions are equivalent;

1. $\pm K$ are dissipative,
2. K is a *-derivation,
3. there is a unique function λ which vanishes on X_0 and is polynomially bounded in the frequency and continuous on $X \setminus X_0$ such that

$$K = \lambda H|_{\mathscr{B}_\infty}.$$

Next we remark that there are examples of dissipative operators $K: \mathscr{B}_\infty \to \mathscr{B}$ which do not have the form $K = \sum \lambda_m H^m$, e.g. $X = \{1, 2\}$, $\sigma = 1$, $K(\alpha, \beta) = (\alpha + \beta, \alpha + \beta)$. But if it is assumed that K is dissipative and

$$K = \sum_{m=0}^{n} \lambda_m H^m$$

then the above argument shows that

$$\begin{aligned} &\lambda_k = 0, \quad k > 2, \quad \lambda_2 \leqq 0, \quad \mathrm{Re}\,\lambda_0 \geqq 0, \\ &0 \leqq \mathrm{Re}(\lambda_0(\omega) + i\beta\lambda_1(\omega) - \beta^2\lambda_2(\omega)), \quad \beta \in \mathbf{R}. \end{aligned} \tag{4.2}$$

Hence

$$(\mathrm{Im}\,\lambda_1(\omega))^2 + 4\lambda_2(\omega)\mathrm{Re}\,\lambda_0(\omega) \leqq 0. \tag{4.3}$$

Conversely, suppose that (4.2) and (4.3) are satisfied. If $f(\omega) = 1 = \|f\|$ with $f \in \mathscr{B}_\infty$, then $t \to (f\bar{f})(S_t\omega)$ has a minimum at $t = 0$, so

$$2\mathrm{Re}(Hf)(\omega) = 0$$

$$2\mathrm{Re}(H^2 f)(\omega) + 2|(Hf)(\omega)|^2 \geqq 0.$$

Hence

$$\begin{aligned}\mathrm{Re}(Kf)(\omega) &= \mathrm{Re}\,\lambda_0(\omega) - \mathrm{Im}\,\lambda_1(\omega)\mathrm{Im}(Hf)(\omega) + \lambda_2(\omega)\mathrm{Re}(H^2f)(\omega)\\ &\geqq \mathrm{Re}\,\lambda_0(\omega) - |\mathrm{Im}\,\lambda_1(\omega)|(-\mathrm{Re}(H^2f)(\omega))^{1/2} - \\ &\quad - \lambda_2(\omega)(-\mathrm{Re}(H^2f)(\omega))\\ &\geqq 0\end{aligned}$$

by (4.2) and (4.3), so that K is dissipative. Consequently if $K; \mathscr{B}_\infty \to \mathscr{B}$ is real and local then from the foregoing and Theorem 1.2B of [4] the following conditions are equivalent:

1. K is dissipative,
2. K is a *-dissipation, i.e.

$$K f\bar{f} \leqq (K\bar{f})f + \bar{f}(Kf), \quad f \in \mathscr{B}_\infty.$$

C. *Non-Abelian C*-algebras.*

Let $\mathscr{B}$ be a C*-algebra and σ a C_0-group of *-automorphisms, so that H is a *-derivation. Suppose further that K is *-derivation. If K satisfies the stronger form of Condition 1 of Theorem 2.1,

$$(4.4) \qquad \|Ka\| \leqq k\|a\|_1, \quad a \in \mathscr{D},$$

then K is automatically dissipative (see [1]), Furthermore, if K extends to a *-derivation of $\mathscr{B}_1$ into $\mathscr{B}$, then a result of Longo establishes that (4.4) is automatically satisfied for some k (see [11]). Thus we can deduce the following result.

THEOREM. *Let $(\mathscr{B}, \mathsf{R}, \sigma)$ be a C*-dynamical system and denote the generator of σ by δ_0. If $\delta\colon \mathscr{D}(\delta_0) \to \mathscr{B}$ is a *-derivation such that*

$$\|[\sigma_t, \delta](a)\| \leqq k_1|t|\|a\|_1, \quad t \in \mathsf{R},\ a \in \mathscr{D}(\delta_0)$$

*for some $k_1 \geqq 0$, then δ is closable and its closure $\bar{\delta}$ generates a C_0-group of *-automorphisms of $\mathscr{B}$.*

This is a direct consequence of Theorem 2.1.

An example was given in [10] where $\mathscr{B}$ is a simple C*-algebra, and $\mathscr{B} \neq \mathscr{D}(\bar{\delta}) \supsetneqq \mathscr{D}(\delta_0)$, so that $\delta - \lambda\delta_0$ is not δ_0-bounded for any λ. However, in this example, $[\sigma_t, \delta] = 0$. Indeed, there does not appear to be any known example, where $\mathscr{B}$ is simple and the conditions of the Theorem above are satisfied, but δ is not a bounded perturbation of a derivation which commutes with σ.

Conversely suppose K satisfies the conditions of Theorem 2.1 and $\pm K$

are dissipative. Then K generates a C_0-group of isometries. If $\mathscr{B}$ has an identity 1 and $K1 = 0$ then τ is a group of *-automorphisms (see, for example [5, Section 3.2]) and therefore K is a *-derivation.

ADDED NOTE. Since this paper was written, the second author has shown that Theorem 3.3 is valid without the assumption that γH^n generates a C_0-contraction semigroup. The proof of the general result relies on singular perturbation theory of holomorphic semigroups. In addition to the estimates of Theorem 3.3 it requires new estimates on growth properties of perturbed holomorphic semigroups. The proof will be published in a separate article.

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