MATH. SCAND. 63 (1988), 43–50

SOME CHARACTERIZATIONS OF TILTED ALGEBRAS

ØYVIND BAKKE

Introduction.

The main aim of this paper is to give new characterizations of tilted algebras. (For definitions, see below.) In particular, we shall show that an artinian algebra A is a tilted algebra if and only if there is a sincere A-module M with the property that there is no chain

$$M'' \to \ldots \to X \to \ldots \to \operatorname{Tr} D\, X \to \ldots \to M'$$

of nonzero maps between indecomposable A-modules with M' and M'' in $\operatorname{add} M$. As an immediate corollary we have the following (obtained by Ringel [8, p. 376] using different arguments): If A has a sincere directing indecomposable module, then A is a tilted algebra.

Some places the references are not the original ones, although these are listed at the end.

Tilting theory and the theorem.

Let A be an artinian algebra over a commutative artinian ring. Only finitely generated right modules will be considered.

We recall that an A-module T is a *tilting module* if $\operatorname{pdim} T \leqq 1$, $\operatorname{Ext}^1(T, T) = 0$ and there is a short exact sequence

$$0 \to A \to T' \to T'' \to 0$$

of A-modules, with T' and T'' in $\operatorname{add} T$. The third condition can be replaced by T having the same number of types of (direct) summands as there are types of simple modules (see [3]).

A *torsion pair* in $\operatorname{mod} A$ is a pair $(\mathscr{T}, \mathscr{F})$ of full subcategories of $\operatorname{mod} A$, such that X is in $\mathscr{T}$ if and only if $\operatorname{Hom}(X, Y) = 0$ for all Y in $\mathscr{F}$, and Y is in $\mathscr{F}$ if and only if $\operatorname{Hom}(X, Y) = 0$ for all X in $\mathscr{T}$. $\mathscr{T}$ is closed with

Received March 2, 1987; in revised form June 25, 1987.

respect to factor modules, and $\mathscr{F}$ with respect to submodules, and both with respect to extensions. A torsion pair $(\mathscr{T}, \mathscr{F})$ is *split* if each indecomposable module is either in $\mathscr{T}$ or in $\mathscr{F}$, which is equivalent to the condition that $\mathrm{Ext}^1(Y, X) = 0$ for all X in $\mathscr{T}$ and Y in $\mathscr{F}$.

We will use the following facts due to Brenner, Butler, Happel and Ringel [4], [5], see also [3], freely or refer to them as "tilting theory";

Let B be the endomorphism ring of a tilting A-module T. Let

$$F = \mathrm{Hom}_A(T, -), \quad F' = \mathrm{Ext}^1_A(T, -),$$

$$G = - \otimes_B T \quad \text{and} \quad G' = \mathrm{Tor}^B_1(-, T);$$

then F and F' are functors from $\mathrm{mod}\, A$ to $\mathrm{mod}\, B$, and G and G' from $\mathrm{mod}\, B$ to $\mathrm{mod}\, A$. Let

$$\mathscr{T} = \mathscr{T}(T) = \mathrm{Ker}\, F' = \mathrm{Im}\, G$$

and

$$\mathscr{F} = \mathscr{F}(T) = \mathrm{Ker}\, F = \mathrm{Im}\, G'$$

be full subcategories of $\mathrm{mod} A$, and

$$\mathscr{X} = \mathscr{X}(T) = \mathrm{Ker}\, G = \mathrm{Im}\, F'$$

and

$$\mathscr{Y} = \mathscr{Y}(T) = \mathrm{Ker}\, G' = \mathrm{Im}\, F$$

full subcategories of $\mathrm{mod}\, B$. Then $(\mathscr{T}, \mathscr{F})$ and $(\mathscr{X}, \mathscr{Y})$ are torsion pairs in $\mathrm{mod}\, A$ and $\mathrm{mod}\, B$, respectively, and F induces an equivalence between $\mathscr{T}$ and $\mathscr{Y}$, and F' between $\mathscr{F}$ and $\mathscr{X}$, their inverses being the restrictions of G and G', respectively. A module X is in $\mathrm{add}\, T$ if and only if X is Ext-*projective* in $\mathscr{T}$, that is

$$\mathrm{Ext}^1(X, \mathscr{T}) = \mathrm{Ext}^1(X, -)|\mathscr{T} = 0.$$

Furthermore $T_{B^{\mathrm{op}}}$ is a tilting module in $\mathrm{mod}\, B^{\mathrm{op}}$ with End $T_{B^{\mathrm{op}}} \simeq A^{\mathrm{op}}$, and $T_{B^{\mathrm{op}}} \simeq DFD(A_{A^{\mathrm{op}}})$, $\mathscr{T}(T_{B^{\mathrm{op}}}) = D\mathscr{Y}(T_A)$, and $\mathscr{F}(T_{B^{\mathrm{op}}}) = \mathrm{D}\mathscr{X}(T_A)$.

A is a *tilted algebra* if it is the endomorphism ring of a tilting module of a hereditary algebra, or, equivalently (from the above), it has a tilting module with the endomorphism ring hereditary.

Now, we are ready to state the Theorem.

THEOREM. *Let A be an artinian algebra over a commutative artinian ring. The following are equivalent:*

(1) *A is a tilted algebra.*

(2) *There is a sincere A-module M with the property that there is no chain*

$$M'' \to \ldots \to X \to \ldots \to \operatorname{Tr} D X \to \ldots \to M'$$

of nonzero maps between indecomposable A-modules, with M' and M'' in add M.

(3) *There is a tilting A-module T inducing a torsion pair $(\mathcal{T}, \mathcal{F})$ satisfying any of the following equivalent conditions, where $\mathcal{T}' = \operatorname{add}(\operatorname{ind} \mathcal{T} \setminus \operatorname{ind} T)$ and $\mathcal{F}' = \operatorname{add}(\mathcal{F} \cup \{T\})$:*

(a) End T *is hereditary.*
(b) $\operatorname{Hom}(\mathcal{T}', T) = 0$.
(c) $(\mathcal{T}', \mathcal{F}')$ *is a torsion pair.*
(c′) $(\mathcal{T}', \mathcal{F}')$ *is a split torsion pair.*
(d) Either of the conditions (α) or (β), which are equivalent, together with any of the conditions (i)-(iii), which are equivalent under the assumption of (α):
 (α) $(\mathcal{T}, \mathcal{F})$ is split.
 (β) $\operatorname{pdim} \mathcal{X} \leqq 1$ (that is, $\operatorname{pdim} X \leqq 1$ for all X in $\mathcal{X}$), where $(\mathcal{X}, \mathcal{Y})$ is the torsion pair induced in mod End T.
 (i) $\operatorname{Hom}(\mathcal{T}', P) = 0$ for all projective modules P in add T.
 (ii) $\operatorname{Hom}(\mathcal{T}', A) = 0$.
 (iii) $\operatorname{idim} \mathcal{T} \leqq 1$.

Preliminaries.

We need the following two facts, which are direct consequences of a result of Auslander and Smalø [2]:

LEMMA 1. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair.*

(1) *A module X in $\mathcal{T}$ is* Ext-*projective in $\mathcal{T}$ if and only if $D\operatorname{Tr} X$ is in $\mathcal{F}$.*

(2) *$(\mathcal{T}, \mathcal{F})$ is split if and only if $\mathcal{F}$ is closed under $D\operatorname{Tr}$ and if and only if $\mathcal{T}$ is closed under $\operatorname{Tr} D$.*

The next lemma, which is a straight-forward consequence of the Harada-Sai Lemma (see [6]) and resembles Nakayama's Lemma, expresses a much used technique:

LEMMA 2. *Let $\mathcal{C}$ be a full subcategory of* mod A *and Y a module not in $\mathcal{C}$. If every map $Y \to C$ with C in* ind $\mathcal{C}$ *factors through a module C' in $\mathcal{C}$, such that all the components (relative to an indecomposable decomposition of the modules) of the induced map $C' \to C$ are nonisomorphisms, and the lengths of the modules in* ind $\mathcal{C}$ *are bounded, then* $\operatorname{Hom}(Y, \mathcal{C}) = 0$.

One last lemma will be needed:

LEMMA 3. *Let* $(\mathcal{T}, \mathcal{F})$ *be a torsion pair and* T *a tilting module which is* Ext-*projective in* $\mathcal{T}$. *Then* $(\mathcal{T}, \mathcal{F}) = (\mathcal{T}(T), \mathcal{F}(T))$.

PROOF. Assume that X is in $\mathcal{T}$. Then $\mathrm{Ext}^1(T, X) = 0$, so that X is in $\mathcal{T}(T)$. Conversely, assume that X is in $\mathcal{T}(T)$. Then X is a quotient of a direct sum of copies of T [see 3], and since $\mathcal{T}$ is closed with respect to direct sums and quotients, X is in $\mathcal{T}$.

Corollaries and comments.

Ringel [8, p. 180] has shown that A is tilted if and only if it has a *slice module*, that is, a module T such that:

(1) T is *sincere*, that is, there are nonzero maps from all the projective modules to T.

(2) If there is a chain

$$T'' \to \dots \to X \to \dots \to T'$$

of nonzero maps between indecomposable modules with T'' and T' in add T, then X is in add T.

(3) If X is noninjective, then at most one of X and $\mathrm{Tr}\, DX$ is in add T.

(4) If $X \to T'$ is an irreducible map between indecomposable modules with T' in add T, then either X is in add T or X is noninjective and $\mathrm{Tr}\, DX$ is in add T.

A slice module obviously satisfies the condition (2) of the Theorem. Conversely, it is easy to show that a tilting module T satisfying the conditions (3)(b) and (3)(d)(α) of the Theorem is a slice module. Thus we will in particular provide an alternative proof that A is tilted if and only if it has a slice module.

Note also that we will prove a little more than the Theorem states: Any M satisfying the condition (2) can be extended, by adding summands, to a T satisfying the condition (3); and, conversely, if T satisfies (3), $M = T$ satisfies (2).

We recall that an indecomposable module M is *directing* if there is no chain of nonzero nonisomorphisms between indecomposable modules from M to M.

COROLLARY 1 (Ringel). *If* A *has a sincere directing indecomposable module, then* A *is tilted* [8, p. 376].

PROOF. This follows from "(2) implies (1)" in the Theorem.

COROLLARY 2. *Let T be a tilting A-module not having any nonzero projective summands and such that the induced torsion pair is split. Then A is tilted.*

PROOF. This follows from "(3)(d)(α) and (3)(d)(i) imply (1)" in the Theorem.

This is essentially proved by Hoshino [7], whose result ("if T has no nonzero projective summands and $(\mathscr{X}(T), \mathscr{Y}(T))$ is split, then A is hereditary") after transforming by tilting theory states that if T has no nonzero projective summands as a B^{op}-module and $(\mathscr{T}(T), \mathscr{F}(T))$ is split, then A is tilted. Using a consequence of Happel's and Ringel's Connecting Lemma (see [5], and also [8, p. 171]), namely, that there is a nonzero injective A-module I such that FI is injective if and only if T_A has a nonzero projective summand, the equivalence of Hoshino's result and Corollary 2 is established.

Note also that if T is a tilting A-module, then A is hereditary if and only if $(\mathscr{X}(T), \mathscr{Y}(T))$ is split and $\operatorname{pdim} \mathscr{Y}(T) \leqq 1$. This follows from "(3)(a) is equivalent to (3)(d)(α) and (3)(d)(iii)" in the Theorem and tilting theory. The "only if" part is proved by Happel and Ringle [5] before.

We also have the following by-product:

COROLLARY 3. *Assume that the tilting module T induces the torsion pair $(\mathscr{T}, \mathscr{F})$. If $(\mathscr{T}, \mathscr{F})$ is nonsplit, there is an X in $\operatorname{ind} \mathscr{T} \setminus \operatorname{ind} T$ such that* $\operatorname{Hom}(X, T) \neq 0$.

PROOF. This follows from "(3)(b) implies (3)(d)(α)" in the Theorem.

Proof of the Theorem.

We start by showing the equivalence of (3)(d)(α) and (β). It is essentially proved by Hoshino [7], but for the convenience of the reader, a proof is included here:

Assume that

$$0 \to K \to Q \to X' \to 0$$

is exact in $\operatorname{mod} B$, with X' in $\mathscr{X}$ and Q projective, so that K and Q are in $\mathscr{Y}$. Using the functors G and G', an exact sequence

$$G'Q \to G'X' \to GK \to GQ \to GX'$$

is induced in $\operatorname{mod} A$, with the end terms equal to zero. Renaming, we get the exact sequence

$$0 \to Y \to X \to T' \to 0$$

in mod A, with Y in $\mathscr{F}$, X *in* $\mathscr{T}$ and T' in add T. Applying $\operatorname{Ext}^1(-, X'')$ with

X'' in $\mathscr{T}$, the exact sequence

$$\mathrm{Ext}^1(T', X'') \to \mathrm{Ext}^1(X, X'') \to \mathrm{Ext}^1(Y, X'') \to \mathrm{Ext}^2(T', X'')$$

is obtained. The first and last terms are zero. If $(\mathscr{T}, \mathscr{F})$ is split, $\mathrm{Ext}^1(Y, X'') = 0$, implying $\mathrm{Ext}^1(X, X'') = 0$, so that X is in add T and K is projective. If K is projective, X is in add T, so that $\mathrm{Ext}^1(X, X'') = 0$, whence $\mathrm{Ext}^1(Y, X'') = 0$. Having picked Y arbitrarily and chosen $X' = F'Y$, it follows that $(\mathscr{T}, \mathscr{F})$ is split.

Next, we show the equivalence of (3)(d)(i)-(iii) under the assumption of (α). It follows easily by tilting theory and results of Auslander and Reiten [1], first noting that in this case, by Lemma 1, a module is in $\mathscr{T}'$ if and only if $D\mathrm{Tr}$ of it is in $\mathscr{T}$.

(i) implies (ii). Note that $\mathscr{T}' \subseteqq \mathscr{T}$ by Lemma 1. If $\mathrm{Hom}(\mathscr{T}', P) \neq 0$ for an indecomposable projective module P, then P is in $\mathscr{T}$ since $(\mathscr{T}, \mathscr{F})$ is split. P is clearly Ext-projective in $\mathscr{T}$, so that P is in add T.

(ii) implies (i). This is obvious.

(ii) is equivalent to (iii) [see 8, p. 74].

Then we show the equivalence of (3)(a)-(d) of the Theorem.

(a) is equivalent to (b). There is a nonzero map $X \to T$ with X in ind $\mathscr{T}$ if and only if there is a nonzero map $FX \to \mathrm{End}\, T$ with FX in ind $\mathscr{Y}$, and X is in add T if and only if FX is projective.

(a) and (b) imply (d). (a) implies condition (d)(β). (b) implies (d)(i) trivially.

(b) and (d) imply (c'), By (d)(α), $(\mathscr{T}, \mathscr{F})$ is a split torsion pair. Now it is easy to check that (b) implies (c').

(c') implies (c). This is trivial.

(c) implies (b). This is obvious.

(d) implies (b). Let X' be in $\mathscr{T}'$. From (d)(α), $(\mathscr{T}, \mathscr{F})$ is split; thus, by Lemma 1, $X' = \mathrm{Tr}\, D X$, where X is in $\mathscr{T}$. Then, by (d)(iii),

$$\mathrm{Hom}(\mathrm{Tr}\, D\, X, T) \simeq D\, \mathrm{Ext}^1(T, X) = 0$$

[see 8, p. 76].

As already mentioned, (1) is equivalent to (3)(a) by tilting theory. Thus, proving the equivalence of (2) and (3) will finish the proof of the Theorem.

(3) *implies* (2). Let T be a tilting module satisfying the conditions (3)(b) and (3)(d)(α). Then $M = T$ obviously satisfies (2).

(2) *implies* (3). First, we define a split torsion pair $(\mathcal{T}, \mathcal{F})$, by $\mathcal{T}$ being the full additive subcategory of mod A generated by modules X such that there is a chain

$$M'' \to \ldots \to X$$

of nonzero maps between indecomposable modules, with M'' in add M, and $\mathcal{F} = \mathrm{add}(\mathrm{ind}\, A \setminus \mathrm{ind}\, \mathcal{T})$. Since X is in $\mathcal{T}$, by (2), $\mathrm{Hom}(\mathrm{Tr}D\,X, M) = 0$, so that

$$0 = D\,\underline{\mathrm{Hom}}(\mathrm{Tr}\,D\,X, M) \simeq \mathrm{Ext}^1(M, X)$$

[see 8, p. 75], showing that M is Ext-projective in $\mathcal{T}$.

Next, let T be an Ext-projective module in $\mathcal{T}$ such that all the Ext-projective modules in $\mathcal{T}$ are in add T. We shall show that T is a tilting module.

By Lemma 1, $D\mathrm{Tr}\,T$ is in $\mathcal{F}$, and there is no nonzero map from any injective module to $D\mathrm{Tr}\,T$, since the injective modules are obviously in $\mathcal{T}$. Thus pdim $T \leqq 1$ [see 8, p. 74].

By construction of T, $\mathrm{Ext}^1(T, T) = 0$.

According to Bongartz [3] and by splitness of $(\mathcal{T}, \mathcal{F})$, there is a short exact sequence

$$0 \to A \to X \oplus Y \to T^n \to 0$$

with X in $\mathcal{T}$ and Y in $\mathcal{F}$ such that $T \oplus X \oplus Y$ is a tilting module. (Note that this means that we have kept our promise of only considering finitely generated modules, since the number of types of summands of a tilting module is finite.) Applying $\mathrm{Ext}^1(-, X'')$ with X'' in $\mathcal{T}$, again by splitness of $(\mathcal{T}, \mathcal{F})$ we see that X is Ext-projective in $\mathcal{T}$, so that $T \oplus Y$ is a tilting module, too.

Assume that $Y \neq 0$. Then $\mathrm{Hom}(Y, T) \neq 0$ (if Y is mapped onto zero in the above sequence, it is projective, and hence has a nonzero map to the sincere module M). But, since pdim $T \leqq 1$,

$$\mathrm{Hom}(Y, D\mathrm{Tr}\,T) \simeq D\,\mathrm{Ext}^1(T, Y) = 0$$

[see 8, p. 76].

Let T' be in ind T, and

$$X' \oplus Y' \to T'$$

a minimal right almost split map in mod A, with X' in $\mathcal{T}$ and Y' in $\mathcal{F}$. Then $\mathrm{Tr}\,D\,Y'$ is in $\mathcal{T}$. If T' is nonprojective, $D\mathrm{Tr}\,X'$ is in $\mathcal{F}$, and if T' is projective, then by sincerity of M, a nonzero map $T' \to M'$ with M' in ind M is obtained, whence (2) implies that $D\mathrm{Tr}\,X'$ is in $\mathcal{F}$, By Lemma 1, X' is in add T. Thus, since Y' is $D\mathrm{Tr}$ of a module in add T and $\mathrm{Hom}(Y, D\,\mathrm{Tr}\,T) = 0$, every map $Y \to T'$ factors through the module X' in add T. Using Lemma 2 (with $\mathscr{C} = \operatorname{add} T$), we get $\operatorname{Hom}(Y, T) = 0$, which is a contradiction. Hence $Y = 0$, and T is a tilting module.

Applying Lemma 3, we see that $(\mathscr{T}, \mathscr{F})$ is actually the torsion pair induced by T, and $\operatorname{Hom}(\mathscr{T}', A) = 0$. This concludes the proof, since (3)(d)(α) and (ii) are now satisfied.

ACKNOWLEDGEMENT. I would like to give my thanks to S. O. Smalø for his assistance in the preparation of this paper.

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UNIVERSITY OF TRONDHEIM
DEPARTMENT OF MATHEMATICS AND STATISTICS
N-7055 DRAGVOLL
NORWAY