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MULTIPLICITIES OF SOLUTIONS TO SOME ENUMERATIVE CONTACT PROBLEMS

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Abstract.

Let G be a scheme parametrizing a family of hypersurfaces of degree d in P^N, $N \geqq 2$. We define and study a subscheme of G parametrizing those hypersurfaces that touch fixed nonsingular curves $C_1, \ldots, C_k$ simultaneously. We give equations cutting out this subscheme in some cases, and we show how such equations can in principle be found in any case.

When $k = \dim G$, we expect the subscheme to have isolated points. We show how the equations determine the multiplicities of these isolated solutions to the contact problem. Thereby we find the local contributions to the total number of isolated solutions, as determined e.g. by Fulton's refined intersection products.

Instead of working with conormal varieties, we use the bundles of principal parts of first order associated to the divisors in question. Hence a hypersurface with a singularity at a point of the curve C is said to touch C using our set-up.

We give some results, some of which are essentially well-known already. At last we use our results to study particular examples of plane curves, and of planes touching space curves.

1. Introduction.

In enumerative geometry a typical problem is to find how many varieties in a given p-parameter family that are simultaneously touching p fixed varieties. A classical example is to determine the number of reduced plane conics that are tangent to 5 fixed conics. One finds that when the 5 fixed conics are in general position, the number is 3264.

For a problem like this, denote by s the number of solutions when the fixed varieties are in general position with respect to the given family of varieties.

When the p fixed varieties are not in general position, the following may occur:

1) The set of solutions is infinite.
2) The set of solutions is finite, but there are less than s solutions set-theoretically.

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An example of 1) is the problem with the conics in the case where 4 of the 5 fixed ones possess a common tangent line. Then the union of this line and any tangent line to the fifth conic is a solution.

If there is a conic touching all the 5 fixed conics, and two of the contact points coincide, we have an example of 2) (provided the set of solutions is finite). When the set of solutions is finite, but there are less than s solutions, one would like to count the solutions with multiplicity in such a way that the total weighted number is s. The problem is: Is this possible, and how should one count? This is the topic of our paper.

In principle the question is answered by W. Fulton and others, see e.g. [1, p. 187–193]. One studies the parameter space associated to a p-dimensional family of varieties, and represents the sub-family of varieties tangent to one of the fixed varieties as a divisor or hypersurface in the parameter space. Then one uses the so-called refined intersection product of the p hypersurfaces in the parameter space.

In particular one associates intersection numbers to isolated solutions, that is: isolated points in the intersection of the p hypersurfaces. Hence the precise meaning of the phrase "how to count the solutions" will be to find the intersection numbers in the sense of Fulton. Again we refer to [1, p. 187–193] for details. See also [2] and [3] and [5].

In this paper we will work over an algebraically closed field of characteristic zero.

We will restrict ourselves to a situation where all the fixed varieties are curves in P^N for some N, and our family of varieties will be a family of hypersurfaces in P^N. Hence our parameter space can be regarded as a subvariety of $P^{\binom{N+D}{d}-1}$, where d is the degree of the hypersurfaces.

We introduce a general technique for how to determine the multiplicities (or intersection numbers) of the solutions in practice. This is done in a constructive computational way. We do not take up the question of how the global (total) number of solutions is determined.

Fix a point in the parameter variety representing a solution of our problem. The p conditions will normally represent p divisors in this parameter variety, giving rise to p power series in the parameters, locally at the point.

Below we give a result only concerning the leading forms of these power series (Proposition 1.1.). This result is essentially well-known. See [3] and [5].

We reproduce the conditions for when the multiplicity of the solution can be found simply as the product of the degrees of the p leading forms, that is: the divisors meet transversally.

Let P^M denote the linear system of hypersurfaces of degree d in P^N. For a nonsingular curve C and a hypersurface D in P^N consider the following measure of tangency between C and D. Put

$$t(D,C) = \sum_{P\in C} (I(P, C \cap D) - 1)_+,$$

where $I(P, C \cap D)$ is the usual intersection number of C and D at P. Possibly $t(D,C) = \infty$. The function $D \to t(D,C)$ is upper semicontinuous on P^M. For a given curve C and a positive integer t, let $H(C;t) \subset P^M$ be those D for which $t(D,C) \geqq t$. The $H(C;t)$ are cones with $H(C;\infty)$ in their vertex sets. For simplicity we denote $H(C;1)$ by $H(C)$. Assume we are not in the case where $d = 1$ and C contains a line component. Then $H(C)$ is a hypersurface in P^M; give it the reduced structure. For a point $P \in P^N$, let $H(P) \subset P^M$ be the set of hypersurfaces containing P.

Denote by $T_S(s)$ the tangent space of a scheme S at a point s. Denote by $\mathcal{O}_{S,s}$ the local ring of S at s, and by $\hat{R}$ the completion of a local ring R. For a hypersurface D denote by $g(D)$ or g the corresponding point in P^M. We now have:

PROPOSITION 1.1. *Let $G \subset P^M$ be a nonsingular locally closed subvariety of dimension p, and let $C_1, \ldots, C_p$ be nonsingular curves in P^N. Put*

$$X = G \cap H(C_1) \cap \ldots \cap H(C_p),$$

and let $g(D)$ be an isolated point of X. Set

$$M_D = \prod_{i=1}^{p} (\deg D \cdot \deg C_i - \operatorname{card}(C_i \cap D)).$$

Then

$$\operatorname{length}(\hat{\mathcal{O}}_{X,g(D)}) \geqq M_D,$$

with equality if and only if: For all $(P_1, \ldots, P_p)$ such that $I(P_j, C_j \cap D) \geqq 2$ for all j, we have

$$T_G(g(D)) \cap H(P_1) \cap \ldots \cap H(P_p) = \{g(D)\}.$$

In particular, length $\hat{\mathcal{O}}_{X,g(D)} = 1$ *if and only if $M_D = 1$, and*

$$T_G(g(D)) \cap H(P_1) \cap \ldots \cap H(P_p) = \{g(D)\}$$

for the unique points $P_1, \ldots, P_p$, such that $I(P_j, C_j \cap D) = 2$, for $j = 1, \ldots, p$.

In section 2 we will define the $H(C;t)$ scheme-theoretically (at least outside $H(C;\infty)$) for any positive integer t. This will not necessarily give a reduced structure on these varieties. We give the following result which will be made precise in Definition 2.1:

PROPOSITION 1.2. *Let t be an integer, let C be a nonsingular curve, and let $g(D)$ be a point of $H(C;t)$ with $t(D, C) = t$. Then the tangent space of $H(C;t)$ at $g(D)$ parametrizes the set of hypersurfaces E such that for all points P on C we have*

$$I(P, C \cap E) \geqq I(P, C \cap D) - 1.$$

In our main result (Theorem 2.2) we give a complete description of the power series expansions of $H(C)$ at a point $g(D)$ in the case where there is no point P with $I(P, C \cap D) \geqq 3$. We do however allow several simple tangencies. Since we need some detailed definitions, we wait until section 2 with the exact formulation of this theorem. In Proposition 1.3, we give the following application of Theorem 2.2:

PROPOSITION 1.3. *As in Proposition 1.1, let $G \subset P^M$ be a nonsingular locally closed subvariety of dimension p, and let $C_1, \ldots, C_p$ be nonsingular curves in P^N. Put*

$$X = G \cap H(C_1) \cap \ldots \cap H(C_p),$$

and let $g(D)$ be an isolated point of X. Assume that $M_D = 1$, or equivalently, that there exist unique points $P_1, \ldots, P_p$ with $I(P_i, C_i \cap D) = 2$, for $i = 1, \ldots, p$, and no other tangencies. Let $r = \dim_K T_X(g(D))$. Then

$$\mathrm{length}(\hat{\mathcal{O}}_{X,g(D)}) \geqq 2^r.$$

Furthermore, assume that G is a linear subspace. Let f_0 be a homogeneous polynomial defining D in P^N. Then the inequality above is strict if and only if there exist homogeneous ponynomials f and h of degree d, f not a scalar multiple of f_0, f and h corresponding to elements of G, such that the hypersurface S with equation $hf_0 + f^2 = 0$ satisfies $I(P_i, C_i \cap D) \geqq 3$, for all i.

In section 2 we introduce some necessary technical devices. In section 3 we prove our results. In Remark 3.4, we clarify the connection between our results and the intersection problem already mentioned. Section 4 is devoted to some applications of Theorem 2.2 for some explicit curves and families of hypersurfaces.

2. Definitions and the main theorem.

We work over a algebraically closed field of characteristic zero. Let C be a fixed nonsungular curve in P^N, and let $G \subset P^M$, where $M = \binom{N+d}{d} - 1$, be an equidimensional, locally closed, nonsingular scheme parametrizing an algebraic family of hypersurfaces of degree d in P^N. We exclude the possibility: $d = 1$, and C contains a line component.

We denote by CG the affine cone over G. The cone CG parametrizes the corresponding family of homogeneous polynomials of degree d in $N+1$ variables.

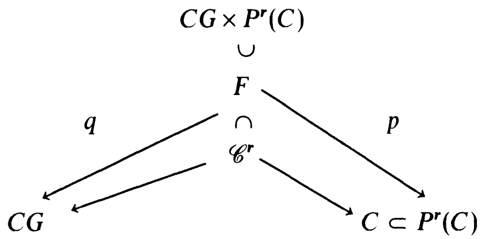
Let $P^r(C)$ be the bundle of rth order principal parts of the divisor class of C, corresponding to hypersurfaces of degree d. $P^r(C)$ is an affine rank $r+1$ bundle on C, for $r = 0, 1, 2, \ldots$ Each member of CG gives rise to a section of $P^r(C)$. Locally around a point P, this section can be described as follows: Choose a parameter t of C at P. The fiber of $P^r(C)$ over P is a vector space where the coordinates correspond to coefficients of trunkated Taylor series modulo (t^{r+1}). Each member of CG gives rise to a Taylor series at P. The section of $P^r(C)$ in question cuts the vector space over P in a point corresponding to the first $r+1$ coefficients of this Taylor series.

We now define the incidence correspondence

$$F \subset CG \times P^r(C)$$

set-theoreticaly: $(cg, p^{(r)}) \in CG \times P^r(C)$ is contained in F, iff $p^{(r)}$ is contained in the section induced by cg.

We study the following diagram:



where p and q are the natural projections from F to $P^r(C)$ and CG respectively, and C is embedded in $P^r(C)$ as the zero section, and $\mathscr{C}^r$ is defined as $p^{-1}(C)$.

For local equations cutting out F and $\mathscr{C}^r$ in $CG \times p^r(C)$, see the paragraph "A local description in $CG \times P^r(C)$" in section 3.

We can view $p^{-1}(C)$ as the affine cone over a variety that parametrizes those pairs of hypersurfaces and points, such that the hypersurface makes an r-fold contact with C at the point.

Denote by $F^m(q_*\mathcal{O}_{\mathscr{C}}r)$ the mth Fitting ideal of the $\mathcal{O}_{CG}$-module $q_*\mathcal{O}_{\mathscr{C}}r$.

For a variety X over K, and an $\mathcal{O}_X$-module $\mathscr{F}$, we recall that

$$V(F^r(\mathscr{F})) = \{x \in X \mid \dim_{K(x)}(\mathscr{F} \otimes K(x)) > r\},$$

and that $X \setminus V(F^r(\mathscr{F}))$ is the largest open subscheme of X where $\mathscr{F}$ can be generated locally by r elements.

Let $V_r^s \subset CG$ be defined by the ideal $F^{s-1}(q_*\mathcal{O}_{\mathscr{C}}r)$. We see that V_r^s is an affine cone. From now on we will regard PV_r^s as the subscheme of G parametrizing hypersurfaces in our family that make at least s r-fold contacts with C, counted with multiplicity. For a hypersurface D and a curve C, denote by $I(P, C \cap D)$ the usual intersection number of C and D at a point P in P^N. From the definition of the bundle $P^r(C)$ it follows that $D \in PV_r^1$, that is D makes an r-fold contact with C, iff $I(P, C \cap D) \geqq r+1$ for some point P. In particular D makes a 1-fold or simple contact with C at P, iff D touches C at P, or is singular at P. In the last case the conormal varieties of D and C do not necessarily meet.

For any homogeneous ideal $\mathfrak{A}$ in $\mathcal{O}_{CG}$ denote by $V_{\mathfrak{A}}$ and $PV_{\mathfrak{A}}$ the corresponding subschemes of CG and G, respectively.

To make Proposition 1.2 precise we make the following definition:

DEFINITION 2.1. Assume $G = P^M \setminus H(C;\infty)$ and let t be a positive integer. Then $H(C;t)$ is defined as the closure of PV_1^t in P^M. In particular $H(C) = \overline{PV_1^1}$.

REMARK. The fact that $H(C)$ is reduced will be shown in Remark 3.3.

A local description in $CG \times P^N$.

Assume $\dim G = p$, and let cg be a point of CG in the fibre over a non-singular point g of G. Let $\{b_0, \dots, b_p\}$ be a set of regular parameters of CG at cg. Since CG is the affine cone over G, we can take one parameter, say b_0, that corresponds to the direction of the cone generatrix through cg. The other parameters, $b_1, \dots, b_p$ can be identified with local parameters of G at g.

Let $P \in P^N$. We have

$$\hat{\mathcal{O}}_{CG \times P^N, (cg, P)} \simeq K[[b_0, \dots, b_p, X_1, \dots, X_N]]$$

where X_i, $i = 1, \dots, N$ are coordinates of some affine space containing P.

Let $CI \subseteqq CG \times P^N$ be the incidence variety consisting of those (cg, P) such that P is contained in the hypersurface determined by cg. We have

$$\hat{\mathcal{O}}_{CI, (cg, P)} \simeq K[[b_0, \dots, b_p, X_1, \dots, X_N]]/M(b_0, \dots, b_p, X_1, \dots, X_N)$$

where $M(b_0, \dots, b_p, X_1, \dots, X_N)$ is "the general polynomial parametrized by a point in an infinitesimal neighbourhood of cg". We set

$$M(b_0, \dots, b_p, X_1, \dots, X_N) = (1+b_0)M_{\text{const}} + \boldsymbol{R}(b_1, \dots, b_p, X_1, \dots, X_N) \tag{2.1}$$

where M_{const} is the polynomial corresponding to cg, and

$$\boldsymbol{R}(b_1, \dots, b_p, X_1, \dots, X_N) \in K[X_1, \dots, X_N][[b_1, \dots, b_p]].$$

When G is a linear subspace of $P^{\binom{N+D}{d}-1}$, we can take $\boldsymbol{R}(b_1, \dots, b_p, X_1, \dots, X_N)$ to be linear in $b_1, \dots, b_p$.

Choose $X_i = \sum_{j\geq 0}\beta_{i,j}\cdot t^j$, $i = 1,\ldots, N$, as local parametrizations of C at P. We define

$$\mathcal{N}(b_0,\ldots,b_p,t) = M\left(b_0,\ldots,b_p, \sum_{j\geq 0}\beta_{1,j}t^j,\ldots, \sum_{j\geq 0}\beta_{N,j}t^j\right).$$

We also write this as

$$\mathcal{N}(b_0,\ldots,b_p,t) = \sum_{j\geq 0}(A_j(b_0,\ldots,b_p)+\alpha_j)\cdot t^j \tag{2.2}$$

where the $\alpha_j \in K$, and the $A_j(b_0,\ldots,b_p)$ are power series without "constant terms". When G is a linear subspace of P^M, the $A_j(b_0,\ldots,b_p)$ will be homogeneous linear in $b_0,\ldots,b_p$ whenever $\boldsymbol{R}(b_0,\ldots,b_p,X_1,\ldots,X_N)$ is linear in $b_0,\ldots,b_p$.

When the hypersurface in question touches C at P, we have $\alpha_0 = \alpha_1 = 0$.

The main result.

We now are in a position where we can formulate our main result. Let $M(\boldsymbol{b},\boldsymbol{X}) = 0$ be the equation of the "general hypersurface around $g(D)$", and let

$$\mathcal{N}(\boldsymbol{b},t) = \sum_{j\geq 0}(A_j(\boldsymbol{b})+\alpha_j)\cdot t^j$$

be the local parametrization of the "general hypersurface around $g(D)$" as above.

THEOREM 2.2. *Let C be a nonsingular curve in P^N, and let D be a hypersurface with $I(P, C\cap D) \leqq 2$ for all P, and let $G \subset P^M$ be a nonsingular, locally closed subvariety of dimension p containing $g(D)$. Let $b_1,\ldots,b_p$ be a regular system of parameters of $\mathcal{O}_{G,g(D)}$, so that the completion is $K[[b_1,\ldots,b_p]]$. Set*

$$A_j = A_j(0,b_1,\ldots,b_p).$$

Now $H(C)\cap G$ has codimension at most 1 in G, and the local equation F of $H(C)\cap G$ has the following power series expansion at $g(D)$:

$$F = \prod \boldsymbol{F}_P(b_1,\ldots,b_p)$$

where the product is over those P such that $I(P, C\cap D) = 2$. Each $\boldsymbol{F}_P(b_1,\ldots,b_p)$ is of the following form:

$$\boldsymbol{F}_P(b_1,\ldots,b_p) = A_0 - SA_1 + \sum_{j\geq 2}(-S)^j(A_j+\alpha_j).$$

Here

$$S = -\sum_{j\geq 0}(-A_1)^{j+1}\cdot B_0\cdot Q_j(B_0,\ldots,B_j),$$

and

$$B_0 = \sum_{j \geqq 0} \frac{(-A_2)^j}{2\alpha_2^{j+1}}, \quad B_1 = -3(A_3+\alpha_3)B_0^2,$$

$$B_l = (-1)^l \cdot B_0^{l+1} \cdot \begin{bmatrix} 3(A_3+\alpha_3) & 4(A_4+\alpha_4) & \cdots & (l+2)(A_{l+2}+\alpha_{l+2}) \\ 1 & 3(A_3+\alpha_3) & \cdots & (l+1)(A_{l+1}+\alpha_{l+1}) \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 4(A_4+\alpha_4) \\ 0 & \cdots\ 0 & 1 & 3(A_3+\alpha_3) \end{bmatrix}$$

for $l \geqq 2$.

$Q_j(B_0,\ldots,B_j)$ is homogeneous of degree j in $B_0,\ldots,B_j$, and the coefficient corresponding to the monomial $B_0^{i_0}\ldots B_j^{i_j}$ is defined the following way:

Let $\mathscr{D}$ be the set of sequences $r_0, r_1, \ldots, r_j$ satisfying the conditions $r_0 = 0 \leqq r_1 \leqq \ldots \leqq r_{j-1} \leqq r_j = j$, and $r_k \geqq k$ (all k). Then for a given exponent vector $\boldsymbol{i} = (i_0,\ldots,i_j)$ with $\sum i_k = j$, say that a sequence $\boldsymbol{r}$ belongs to $\boldsymbol{i}$ if $\operatorname{card}\{k \mid r_k - r_{k-1} = s\} = i_s$ *for all s. Then the coefficient of $\boldsymbol{B}^{\boldsymbol{i}}$ is the number of $\boldsymbol{r}$ in $\mathscr{D}$ belonging to $\boldsymbol{i}$.*

The first few Q_j are:

$$Q_0 = 1, \quad Q_1 = B_1, \quad Q_2 = B_0B_2 + B_1^2, \quad Q_3 = B_0^2B_3 + 3B_0B_1B_2 + B_1^3,$$

$$Q_4 = B_0^3B_4 + 4B_0^2B_1B_3 + 2B_0^2B_2^2 + 6B_0B_1^2B_2 + B_1^4.$$

The first few terms of $\boldsymbol{F}_P(b_1,\ldots,b_p)$ are:

$$A_0 - \frac{A_1^2}{4\alpha_2} + \frac{1}{4\alpha_2^2}A_1^2A_2 - \frac{\alpha_3}{8\alpha_2^4}A_1^3 - \frac{A_1^2A_2^2}{2\alpha_2^3} + \frac{3}{16}\frac{A_1^3A_2}{\alpha_2^4} - \frac{A_1^3A_3}{8\alpha_2^2} - $$

$$- \frac{9}{64}\frac{\alpha_3^2}{\alpha_2^5}A_1^4 + \frac{\alpha_4}{16\alpha_2^4}A_1^4.$$

REMARK. I am indepted to Stein Arild Strømme for the present expression of the coefficients of the $B^{\boldsymbol{i}}$, which is less clumsy than our original expression.

3. Proofs of the results.

We keep the notation from section 2 and consider a nonsingular curve C in P^N.

A local description on $CG \times P^r(C)$.

We now regard C as the zero-section of $P^r(C)$. We use the terms P and t

for the point on C and the local parameter for C at the point, also when $C \subseteqq P^r(C)$.

Consider the point $\boldsymbol{P} = (cq, P)$ in $\mathscr{C}^r \subseteqq F \subseteqq CG \times P^r(C)$. A set of local parameters for $CG \times P^r(C)$ at $\boldsymbol{P}$ is

$$\{b_0, \ldots, b_p, t, v_0, \ldots, v_r\},$$

where $b_0, \ldots, b_p, t$ are as before, and $v_0, \ldots, v_r$ are the coordinates corresponding to the $r+1$ terms of trunkated Taylor series of order r.

We have

$$\hat{\mathcal{O}}_{\mathscr{C}^r, \boldsymbol{P}} \simeq K[[b_0, \ldots, b_p, t, v_0, \ldots, v_r]]/\boldsymbol{J}_r$$

where

$$\boldsymbol{J}_r = (v_0, \ldots, v_r, v_0 - \mathcal{N}(b_0, \ldots, b_p, t), v_1 - \frac{\partial}{\partial t}\mathcal{N}(b_0, \ldots, b_p, t),$$

$$2!v_2 - \frac{\partial^2}{\partial t^2}\mathcal{N}(b_0, \ldots, b_p, t), \ldots, r!v_r \frac{\partial^r}{\partial t^r} - \mathcal{N}(b_0, \ldots, b_p, t).$$

The $r+1$ first generators arise from the fact that $\mathscr{C}^r$ is the inverse image of the zero-section. The $r+1$ last generators are due to the incidence describing F. We easily conclude:

$$\hat{\mathcal{O}}_{\mathscr{C}^r, \boldsymbol{P}} \simeq K[[b_0, \ldots, b_p, t]]/(\mathcal{N}, \ldots, \partial^r \mathcal{N}/\partial t^r), \tag{3.1}$$

where $\mathcal{N} = \mathcal{N}(b_0, \ldots, b_p, t)$.

Recall that D is the hypersurface corresponding to $g \in G$. $\mathcal{N}(0, \ldots, 0, t)$ is of order at least $r+1$, iff $I(P, C \cap D) \geqq r+1$. Hence $\hat{\mathcal{O}}_{\mathscr{C}^r.\boldsymbol{P}} \neq 0$, iff $I(P, C \cap D) \geqq r+1$. Assume that $I(P_i, C \cap D) \geqq r+1$, for $i = 1, \ldots, k$, where k is finite. Denote by R_i the ring $\hat{\mathcal{O}}_{\mathscr{C}^r.\boldsymbol{P}_i}$ where $\boldsymbol{P}_i$ corresponds to P_i, for $i = 1, \ldots, k$.

We have

$$\hat{\mathcal{O}}V_{F^{s-1}(q_* \mathcal{O}_{\mathscr{C}^r}), cg} \simeq \hat{\mathcal{O}}_{CG.cg}/F^{s-1}\left(\bigoplus_{i=1}^{k} R_i\right).$$

A useful identity is:

$$F^{s-1}\left(\bigoplus_{i=1}^{k} R_i\right) = \sum_{j_1 + \ldots + j_k = s-1} F^{j_1}(R_1) \times \ldots \times F^{j_k}(R_k), \tag{3.2}$$

(see [4, p. 16]).

Denote by n_i the multiplicity of the ring R_i with respect to the maximal ideal of $\hat{\mathcal{O}}_{CG.cg}$.

When $\sum_{i=1}^{k} n_i = s$, formula (3.2) reduces to

$$F^{s-1}\left(\bigoplus_{i=1}^{k} R_i\right) = \sum_{i=1}^{k} F^{n_i-1}(R_i). \tag{3.3}$$

We always have

$$F^{0}\left(\bigoplus_{i=1}^{k} R_i\right) = \prod_{i=1}^{k} F^{0}(R_i). \tag{3.4}$$

We now proceed to find the $F^0(R_i)$. Fix the contact point P.

We will work with the case $r = 1$, and we will find a $K[[b_0, \ldots, b_p]]$-free resolution of $\hat{\mathcal{O}}_{\mathscr{C}^1,P}$. We will use this resolution to find explicit descriptions of the $K[[b_0, \ldots, b_p]]$-ideals $F^{s-1}(\hat{\mathcal{O}}_{\mathscr{C}^1,P})$, for $s = 1, 2, \ldots$.

In order to do so we will substitute $\mathcal{N}$ and $\partial\mathcal{N}/\partial t$ with power series T and S, which are in fact polynomials in t with coefficients in $K[[b_0, \ldots, b_p]]$. This will give a finite matrix description of the "crucial" map of the resolution.

From now on we drop the index $r\,(=1)$ in $J_r, \mathscr{C}^r$, and we denote by R the ring $K[[b_0, \ldots, b_p]]$. Hence $\hat{\mathcal{O}}_{\mathscr{C},P} = R[[t]]/J$. Assume $I(P, C \cap D) = n+1$. Then $\mathcal{N}(0, \ldots, 0, t)$ is of order $n+1$ in t, and $\partial\mathcal{N}/\partial t(0, 0, \ldots, 0, t)$ is of order n in t. We use Weierstrass' Preparation Theorem to find a polynomial S in t, of the form

$$S = t^n + S_{n-1}t^{n-1} + \ldots + S_0,$$

where $S_{n-1}, \ldots, S_0$ are power series in $b_0, \ldots, b_p$, and S generates the same ideal as $\partial\mathcal{N}/\partial t(b_0, \ldots, b_p, t)$ in $R[[t]]$.

Furthermore we use Weierstrass' Preparation Theorem to find a polynomial T in t of the form

$$T = T_{n-1} \cdot t^{n-1} + \ldots + T_0,$$

where $T_{n-1}, \ldots, T_0$ are power series in $(b_0, \ldots, b_p)$, and T generates the same ideal as $\mathcal{N}(b_0, \ldots, b_p, t)$ in $R[[t]]$ modulo S. We remark that the constructive proof of Weierstrass' Preparation Theorem in [8, p. 140–141, 145] gives an explicit algorithm for constructing S and T, using formula (2.2) in section 2.

We also remark that the power series $S_{n-1}, \ldots, S_0, T_{n-1}, \ldots, T_0$ contain no constant terms.

It is now clear that

$$\hat{\mathcal{O}}_{\mathscr{C},P} \simeq R[[t]]/(S, T)$$

and we have the following R-free resolution

$$R[[t]]/S \xrightarrow{\Phi} R[[t]]/S \xrightarrow{\text{I}} R[[t]]/(S, T) \to 0, \tag{3.5}$$

ϕ is multiplication by T, and τ is the natural map. We find the matrix representation of ϕ with respect to the basis $\{1, t, \ldots, t^{n-1}\}$.

REMARK. So far in section 3 we have assumed that $r = 1$. We will continue assuming that in the rest of the paper. Still we would like to remark that for arbitrary r, we can substitute $\partial^r \mathcal{N}/\partial t^r$ by a monic polynomial S in t. Furtnermore we can substitute $\partial^i \mathcal{N}/\partial t^i$ by polynomials ${}_iT$ in t, for $i = 0, \ldots, r-1$. Then we use the resolution

$$(R[[t]]/S)^r \xrightarrow{\phi_r} R[[t]]/S \xrightarrow{\tau_r} R[[t]]/(S, {}_0T, \ldots, {}_{r-1}T) \to 0.$$

Here ϕ_r maps an r-tuple $(P_0, \ldots, P_{r-1})$ to ${}_0TP_0 + \ldots + {}_{r-1}TP_{r-1}$ modulo S. All computations that we do for $r = 1$ can be copied for general r.

The following is easy to verify:

OBSERVATION 3.1.

(i) The entries in the first column of the matrix are $T_0, \ldots, T_{n-1}$.

(ii) The entries in the other columns of the matrix are contained in the R-ideal $(T_0, \ldots, T_{n-1})$.

(iii) Modulo the R-ideal $(b_0, \ldots, b_p)^2$ the matrix is

$$\begin{bmatrix} T_0 & 0 & \cdots & & 0 \\ T_1 & T_0 & & & \vdots \\ \vdots & \vdots & \ddots & & \vdots \\ & & & T_0 & 0 \\ T_{n-1} & T_{n-2} & & T_1 & T_0 \end{bmatrix}$$

Observation 3.1 gives rise to the following:

LEMMA 3.2.

a) $F^{n-1}(\hat{\mathcal{O}}_{\mathscr{C},P}) = (T_0, \ldots, T_{n-1})$.
In particular, when $n = 1$,
$F^0(\hat{\mathcal{O}}_{\mathscr{C},P}) = (T_0) = (T)$.

b) $F^0(\hat{\mathcal{O}}_{\mathscr{C},P})$ *is generated by an element, which is congruent to* T_0^n *modulo* $(b_0, \ldots, b_p)^{n+1}$.

Lemma 3.2 reduces the proofs of our results to explicit computations of $T_0, \ldots, T_{n-1}$ for each point P such that $I(P, C_i \cap D) \geqq 2$, for each fixed curve C_i.

PROOF OF PROPOSITION 1.1. We will study the intersection scheme

$$X = G \cap H(C_1) \cap \ldots \cap H(C_p).$$

Fix a curve C. We claim that $G \cap H(C)$ is the same scheme as the one cut out on G by the sheaf of homogeneous ideals $F^0(q_*\mathcal{O}_{\mathscr{C}})$. Denote by J the sheaf of ideals cut out by all the corresponding Fitting ideals for the fixed curves $C_1,\ldots,C_p$. In other words we claim that $X = PV_J$. Recall that $H(C_i)$ are by definition given a reduced structure. Since the formation of Fitting ideals commutes with base change, the claim is true if we can show that the $F^0(q_*\mathcal{O}_{\mathscr{C}})$ cut out reduced schemes when $G = P^M \setminus H(C; \infty)$. This will be shown in Remark 3.3. Hence we will compute $F^0(q_*\mathcal{O}_{\mathscr{C}})$ locally at cg for each fixed curve C. We study one such curve. Let $P_1,\ldots,P_k$ be the points where D touches C. Let $n_i = I(P_i, C \cap D) - 1 \geqq 1$, for all i. Let R_i be the ring $\mathcal{O}_{\mathscr{C},P_i}$ as in section 2. We must compute $F^0(\bigoplus_{i=1}^k R_i)$. By formula (3.4) this is equal to

$$\prod_{i=1}^{k} F^0(R_i).$$

For each P_i denote by

$$T^{(i)} = T^{(i)}_{n_i-1}t^{n_i-1} + \ldots + T^{(i)}_0$$

the polynomial defined by Weierstrass' Preparation Theorem as above.

By Lemma 3.2, part b), a generator of $F^0(R_i)$ is congruent to $(T^{(i)}_0)^{n_i}$ modulo $(b_0,\ldots,b_p)^{n_i+1}$. Hence a generator of $\prod_{i=1}^{k} F^0(R_i)$ is congruent to

$$\prod_{i=1}^{k} (T^{(i)}_0)^{n_i}$$

modulo $(b_0,\ldots,b_p)^{n_1+\ldots+n_k+1}$.

For each point P^i we rewrite formula (2.2) as

$$\mathcal{N}_i(b_0,\ldots,b_p,t) = \sum_{j\geq 0} (A_{j,i}(b_0,\ldots,b_p) + \alpha_{j,i}) \cdot t^j$$

where the $A_{j,i}$ are functions in $b_0,\ldots,b_p$. For a power series $f(b_0,\ldots,b_p)$ denote by $f^{\text{lin}}(b_0,\ldots,b_p)$ the linear part of $f(b_0,\ldots,b_p)$. We use the algorithm in the proof of Weierstrass Preparation Theorem in [8, p. 140–141] to find

$$T^{(i)}_0 \equiv A_{0,i}(b_0,\ldots,b_p) \text{ modulo } (b_0,\ldots,b_p)^2.$$

Hence we obtain that a generator of $\prod_{i=1}^k F^0(R_i)$ is congruent to

$$\prod_{i=1}^{k} [A^{\text{lin}}_{0,i}(b_0,\ldots,b_p)]^{n_i} \tag{3.6}$$

modulo $(b_0,\ldots,b_p)^{n_1+\ldots n_k+1}$.

This means that the leading form has degree at least

$$\sum_{i=1}^{k} n_i = \sum_{i=1}^{k} I(P_i, C \cap D) - k = \deg D \cdot \deg C - \operatorname{card}(C \cap D).$$

The isolated point g of PV_J is cut out by p equations in a p-dimensional nonsingular ambient space G. Then the multiplicity of PV_J at g is at least the product of the degrees of the p leading forms of these equations, that is

$$\prod_{j=1}^{p} (\deg D \cdot \deg C_j - \operatorname{card}(C_j \cap D)).$$

Furthermore, the multiplicity is exactly this number if, and only if g is set-theoretically the only point cut out from the embedded tangent space of G at g by the p leadings forms. This follows from standard local theory, see for instance Example 7.1.10. and Example 12.4.9. in [1]. Each leading form (corresponding to a fixed curve C)

$$\prod_{i=1}^{k} A_{0,i}^{\mathrm{lin}}(b_0, \ldots, b_p)^{n_i},$$

cuts out those hypersurfaces, parametrized by some point of $T_G(g)$, which pass through one of the points $P_1, \ldots, P_k$, where D touches C.

This implies the conclusion of Proposition 1.1.

Proof of Proposition 1.2. Let $\{P_1, \ldots, P_k\}$ be the set of points where D touches C. Let the rings R_i be as before. We will compute

$$F^{t-1}\left(\bigoplus_{i=1}^{k} R_i\right),$$

where

$$t = \sum_{i=1}^{k} (n_i - 1),$$

and $n_i = I(P_i, C \cap D)$, $i = 1, \ldots, k$.

By Formula (3.3) this is the same as $\sum_{i=1}^{k} F^{n_i}(R_i)$, and by Lemma 3.2, part a), it is the same as

$$(T_0^{(1)}, \ldots, T_{n_1-1}^{(1)}, \ldots, T_0^{(k)}, \ldots, T_{n_k-1}^{(k)}).$$

Temporarily we fix $P \in \{P_1, \ldots, P_k\}$, and study the corresponding power series $T_0, \ldots, T_{n-1}$, where $n = I(P, C \cap D)$.

We use Weierstrass' Preparation Theorem and find that the linear part of T_0 is $A_0^{\mathrm{lin}}(b_0, \ldots, b_p)$, using the notation of Formula (2.2).

In the same way we see that the linear part of T_j is

$$\frac{n+1-j}{n+1} A_j^{\text{lin}}(b_0, \ldots, b_p)$$

modulo the linear parts of $(T_0, \ldots, T_{j-1})$, for $j = 1, \ldots, n-1$. Hence the linear parts of $(T_0, \ldots, T_{j-1})$ generate the ideal

$$(A_0^{\text{lin}}(b_0, \ldots, b_p), \ldots, A_{n-1}^{\text{lin}}(b_0, \ldots, b_p)).$$

This ideal cuts out those hypersurfaces E, parametrized by some point of $T_G(g)$, such that $I(P, C \cap E) \geqq n$.

This is true because these hypersurfaces E correspond precisely to those choices of values $b_1, \ldots, b_p$ that force the power series

$$\sum_{j \geqq 0} (A_j^{\text{lin}}(b_0, \ldots, b_p) + \alpha_j) \cdot t^j$$

to be of order at least n in t. Treating all points $P_1, \ldots, P_k$ simultaneously, we obtain the conclusion of Proposition 1.2.

About the proof of Theorem 2.2. As usual let $\{P_1, \ldots, P_k\}$ be the set of points where D touches C. By combining formula (3.4), and Lemma 3.2, part a), we see that

$$F^0\left(\bigoplus_{i=1}^{k} R_i\right) = \prod_{i=1}^{k} T_0^{(i)} = \prod_{i=1}^{k} T^{(i)}.$$

Hence it is enough to prove that each $T^{(i)}$ is of the form $\boldsymbol{F}_p(b_1, \ldots, b_p)$, described in Theorem 2.2. (We once again forget the irrelevant variable b_0). This follows from the constructive proof of Weierstrass' Preparation Theorem on pp. 140–141, 145 of [8,]. We skip the easy, but tedious calculations here. The assumption $I(P, C \cap D) \leqq 2$ for all P, is only included to make the calculations more tractable. In principle one can use the same sort of calculations in any case, using formula (3.4), the resolution (3.5), and the constructive algorithm in [8].

Proof of Proposition 1.3. As in the proof of Proposition 1.1 we observe that

$$X = (G \cap H(C_1)) \cap \ldots \cap (G \cap H(C_p)) = H_1 \cap \ldots \cap H_p.$$

Here H_i is cut out in G by an ideal of the form $F^0(q_*\mathcal{O}_{\mathcal{C}})$ for each fixed curve C_i. In Remark 3.4, we will show that the multiplicity of X at g is $i(g, H_1 \cdot \ldots \cdot H_p, G)$, using the notation of [1].

Since $\dim(T_X(g)) = r$, we can pick r of the H_i, say $H_1, \ldots, H_r$ such that if

$$W = H_1 \cap \ldots \cap H_r, \quad Y = H_{r+1} \cap \ldots \cap H_p,$$

then

$$T_W(g) \subset T_Y(g).$$

Then $i(g, H_1 \cdot \ldots \cdot H_p, G) = i(g, W \cdot Y, G) \geqq 2^r$ by Example 12.4.10 in [1].

This conclusion can also be derived from the following discussion.

Assume that G is a linear subspace of P^M. Then all the $A_j(b_0, \ldots, b_p)$ described in Formula (2.2) can be taken to be linear.

Since $M_D = 1$, we know that D touches $C_1, \ldots, C_p$ in one point each, say $P_1, \ldots, P_p$, and that $I(P_i, C_i \cap D) = 2$, $i = 1, \ldots, p$. Let

$$\mathcal{N}_i(b_0, b_1, \ldots, b_p, t_i) = \sum_{j \geqq 0} (A_{j,i}(b_0, \ldots, b_p) + \alpha_{j,i}) \cdot t_i^j$$

be the expressions corresponding to Formula (2.2) for each pair (C_i, P_i), $i = 1, \ldots, p$.

Put $A_{j,i} = A_{j,i}(b_0, \ldots, b_p)$. (When $j = 0,1$, the $A_{j,i}$ do not involve b_0, for $i = 1, \ldots, p$.) Let J be the ideal of X in G.

We use Theorem 2.2 and find that $J = (T^{(1)}, \ldots, T^{(p)})$, where

$$T^{(i)} \equiv A_{0,i} - \frac{A_{1,i}^2}{4\alpha_{2,i}} \text{ modulo } (b_1, \ldots, b_p)^3 \quad \text{for } i = 1, \ldots, p.$$

The p linear equations $A_{i,0} = 0$, $i = 1, \ldots, p$, in the p variables $b_1, \ldots, b_p$ give rise to a $p \times p$-coefficient matrix

$$\mathcal{M} = (m_{ij})_{\substack{i = 1, \ldots, p \\ j = 1, \ldots, p}}.$$

By assumption there are exactly r independent relations between the rows of $\mathcal{M}$. We may assume that $p-r$ last rows of $\mathcal{M}$ generate the vector space generated by all p rows, and we can find constants $\lambda_{1,r+1}, \ldots, \lambda_{r+1}$ such that

$$\begin{aligned} A_{0,1} &= \lambda_{1,r+1} A_{0,r+1} + \ldots + \lambda_{1,p} A_{0,p} \\ &\vdots \\ A_{0,r} &= \lambda_{r,r+1} A_{0,r+1} + \ldots + \lambda_{r,p} A_{0,p} \end{aligned}$$

This means that $J/(b_0, \ldots, b_p)^3$ is generated by

$$\begin{gathered} \frac{A_{1,1}^2}{\alpha_{2,1}^2} - \lambda_{1,r+1} \frac{A_{1,r+1}^2}{\alpha_{2,r+1}} - \ldots - \lambda_{1,p} \frac{A_{1,p}^2}{\alpha_{2,p}} \\ \vdots \\ \frac{A_{1,r}^2}{\alpha_{2,r}} - \lambda_{r,r+1} \frac{A_{1,r+1}^2}{\alpha_{2,r+1}} - \ldots - \lambda_{r,p} \frac{A_{1,p}^2}{\alpha_{2,p}} \end{gathered}$$

and

$$\begin{aligned}&A_{0,r+1} + \text{quadratic term}\\ &\vdots\\ &A_{0,p} \quad + \text{quadratic term.}\end{aligned}$$

Now it is clear that the multiplicity of PV_J at g is at least 2^r.

By making a linear change of parameters we may assume $A_{0,j} = b_j$, $j = r+1, \ldots, p$. Then it is clear that the multiplicity of PV_J at g is 2^r if and only if the only r-tuple $(b_1, \ldots, b_r)$ satisfying the equations

$$\frac{A_{1,j}^2(b_1, \ldots, b_r, 0, \ldots, 0)}{\alpha_{2,j}} - \lambda_{j,r+1} \frac{A_{1,r+1}^2(b_1, \ldots, b_r, 0, \ldots, 0)}{\alpha_{2,r+1}} - \\ - \ldots - \frac{\lambda_{j,p} A_{1,p}(b_j, \ldots, b_r, 0, \ldots, 0)}{\alpha_{2,p}}$$

simultaneously for $j = 1, \ldots, r$, is the zero-tuple.

On the other hand, let $f_0, \ldots, f_p$ be such that f_0 defines $D, f_0, \ldots, f_r$ span $T_{V_J}(cg)$, and $f_0, \ldots, f_p$ span $T_{CG}(cg)$. We think of f_i as the polynomial corresponding to the parameter b_i, for all i.

The condition that a linear combination

$$c_1 f_1 + \ldots + c_p f_p$$

should define a hypersurface passing through $P_1, \ldots, P_p$, gives p equations in $c_1, \ldots, c_p$ with coefficient matrix $\mathcal{M}$.

The assumption $A_{0,j} = b_j$, $j = r+1, \ldots, p$ means that the subspace of $\mathrm{Span}(f_1, \ldots, f_p)$ consisting of polynomials defining hypersurface passing through $P_1, \ldots, P_p$ is $\mathrm{Span}(f_1, \ldots, f_r)$.

Assume there are polynomials f and h satisfying the conditions of the proposition, that is: f is not a scalar multiple of f_0, and $hf_0 + f^2 = 0$ is the equation of a hypersurface S, such that $I(P_i, C \cap S) \geqq 3$ for all i.

It is clear that f must be in Span $(f_0, \ldots, f_r)$, and it is easy to see that f can be chosen as an element of $\mathrm{Span}(f_1, \ldots, f_r)$. Furthermore h can be chosen as an element of $\mathrm{Span}(f_1, \ldots, f_p)$, but not of $\mathrm{Span}(f_1, \ldots, f_r)$.

Let f be a fixed polynomial in $\mathrm{Span}(f_1, \ldots, f_r)$. Consider polynomials of the form

$$(*) \qquad d_1 f_1 f_0 + \ldots + d_p f_p f_0 + d_{p+1} f^2,$$

where the d_i are constants. We know that there are at least r independent polynomials of this form that intersect C_i at least 3 times at P_i, for $i = 1, \ldots, p$.

These are the ones with

$$d_{r+1} = \ldots = d_p = d_{p+1} = 0.$$

The condition in the proposition is equivalent to the existence of an f such that there are at least $r+1$ hypersurfaces of the form (*) that have these intersection properties.

Let us divide by f_0 and study expressions of the form

$$d_1 f_1 + \ldots + d_p f_p + d_{p+1} \frac{f^2}{f_0}.$$

We are interested in the expressions that are of order at least one at P_i regarded as functions on C_i for all i. This gives p equations in $d_1, \ldots, d_{p+1}$ with a coefficient matrix which we denote by $\mathcal{M}_f$. The p first columns of $\mathcal{M}_f$ are the same as those of $\mathcal{M}$. The condition the proposition is equivalent to the existence of an f such that

$$\operatorname{rk} \mathcal{M}_f = \operatorname{rk} \mathcal{M} = p - r.$$

This means that all relations between the rows of $\mathcal{M}$ lift to relations between the rows of $\mathcal{M}_f$. Put $f = c_1 f_1 + \ldots + c_r f_r$.

We have the following power series expansion of f^2/f_0 locally at P_i

$$\frac{(A_{0,i}(c_1, \ldots, c_r) + A_{1,i}(c_1, \ldots, c_r) \cdot t_i + \ldots)^2}{\alpha_{2,i} t_i^2 + \alpha_{3,i} t_i^3 + \ldots} = \frac{A_{1,i}^2(c_1, \ldots, c_r)}{\alpha_{2,i}} + t_i \cdot \text{something},$$

since $A_{0,i}(c_1, \ldots, c_r) = 0$.

Hence the entries of the last column of $\mathcal{M}_f$ are

$$\frac{A_{1,i}^2(c_1, \ldots, c_r)}{\alpha_{2,i}}.$$

Comparing with the multiplicity condition already found, this gives our desired result.

REMARK 3.3. Let C and G be as usual, that is: C is a fixed nonsingular curve in P^N, for $N \geqq 2$, and G is a nonsingular, locally closed scheme parametrizing hypersurfaces of degree d in P^N that do not contain C. Set

$$H = PV_{F^0(q^* \mathcal{O}_{\mathcal{C}})}.$$

Let $g \in G$ correspond to a hypersurface D. From Formula (3.6) we see that the local equation of H at g has a power series expansion whose leading form is $\prod_{i=1}^k [A_{0,i}(\boldsymbol{b})]^{n_i}$, or a form of higher degree. This implies that H is regular of dimension $\dim G - 1$ at g if and only if a) and b) hold

a) $\sum_{P \in C \cap D} (I(P, C \cap D) - 1) = 1.$

b) Some hypersurface in P^N, corresponding to a point in the embedded tangent space to G in P^M at g, does not pass through the unique contact point P.

Hence H is reduced if and only if a) and b) hold on a dense set in H.

Assume that $G = P^M \setminus H(C;\infty)$ and that $H \neq \phi$. Then condition b) is irrelevant. As usual we exclude the possibility that $d = 1$, and C contains a line.

We will show that in this case H is a reduced divisor in G, and also $\bar{H} = H(C)$, that is: $\bar{H}$ is a reduced hypersurface in P^M. First we assume that C is irreducible. $\bar{H}$ is clearly a cone over $H(C;\infty)$, so we may assume that $H(C;\infty) = \emptyset$. We then study the image of C under the d-uple embedding. This embeds C as a nonsingular nondegenerate curve in some P^s, for $s \geqq 2$.

If $s = 2$, then C has only a finite number of bitangents and flex tangents in P^s. This shows our assertion in this case. Assume $s \geqq 3$. It follows from Theorem 11 in [7] that almost all tangent hyperplanes of C at a general point P of C do not intersect C more than 2 times at P. Moreover, by Bertini's Theorem in characteristic zero, almost all hyperplanes that touch C at P, do not touch C elsewhere.

This implies that $\bar{H}$ is a reduced divisor in P^M. By biduality $\bar{H}$ is even irreducible.

We now drop the assumption that C is irreducible. Then the various components of $H(C)$ must intersect in codimension at least 2 in P^M (biduality), and a general point of $H(C)$ is therefore on only one component.

This shows that when $G = P^M \setminus H(C;\infty)$, then $\bar{H}$ gives a reduced scheme structure on the hypersurface $H(C)$ in P^M.

REMARK 3.4. Assume that $\dim G = p$, and that we have p fixed curves $C_1, \ldots, C_p$. Let H_i, for $i = 1, \ldots, p$ be defined as H above, and asume that $H_1, \ldots, H_p$ are reduced divisors in G.

We are interested in the sum

$$s = \sum i(g, H_1 \cdot \ldots \cdot H_p, G)$$

in the sense of [1, see e.g. Example 7.1.10, p. 123]. The sum s is taken over all isolated points g of $H_1 \cap \ldots \cap H_p$.)

We will study each local contribution $i(g, H_1 \cdot \ldots \cdot H_p, G)$ at isolated points. Locally at g each H_i is represented by a power series f_i in the completion $\hat{\mathcal{O}}_{G,g}$. Since $\hat{\mathcal{O}}_{G,g}$ is nonsingular and therefore Cohen-Macauley, it follows from Example 7.1.10 in [1] that $i(g, H_1 \ldots H_p, G)$ is equal to the length of $\hat{\mathcal{O}}_{G,g}/(f_1, \ldots, f_p)$. In Propositions 1.1 and 1.3 we computed such lengths.

We conclude that these results can used to determine the local contributions $i(g, H_i \cdot \ldots \cdot H_n, G)$ to the global number s, wherever this number is defined in terms of the intersection products in [1].

4. Some applications of theorem 2.2.

We will use the formula of Theorem 2.2 in two example of families of plane curves. Moreover we will apply Proposition 1.3 (which is itself a consequence of Theorem 2.2) in the study of those planes in P^3 that are touching 3 fixed curves.

EXAMPLE 4.1. We let the points (a_0, a_1, a_2) in $G = P^2$ parametrize the members of the family of plane curves with equations

$$a_0X^2 + a_0Y^2 + a_1YZ + a_2Z^2 = 0.$$

The real picture is the family of circles with centers on the line $X = 0$.

Assume we have two fixed curves C_1 and C_2, both passing through the point $P = (-1, 0, 1)$. Let D be the "variable curve" parameterized by $(a_0, a_1, a_2) = (1, 0, -1) = g$, that is D is the curve with equation $X^2 + Y^2 = Z^2$. We also assume that

$$I(P, C_1 \cap D) = I(P, C_2 \cap D) = 2,$$

and that D touches neither C_1 nor C_2 in any point but P.

Set $J = (F^0(q_*\mathcal{O}_{\mathscr{C}_1}), F^0(q_*\mathcal{O}_{\mathscr{C}_2}))$. We will show that the multiplicity of PV_J at g is equal to $I(P, C_1 \cap C_2)$. The polynomial $M(b_0, b_1, X, Y, Z)$ defined in section 2 may be taken to be

$$(1+b_0)(X^2+Y^2-Z^2)+b_1YZ+b_2Z^2.$$

From now on we set $b_0 = 0$.

We parametrize C_1 and C_2 at P as follows:

$$C_1 : X = -1 + \sum_{j \geq 2} \gamma_j Y^j, \quad Y = Y, Z = 1$$

$$C_2 : X = -1 + \sum_{j \geq 2} \eta_j Y^j, \quad Y = Y, Z = 1.$$

Referring to formula (2.2), we obtain:

$$\mathcal{N}_i(b_1, b_2, t) = A_{0,i} + A_{1,i} + \sum_{j \geq 2} \alpha_{j,i} Y^j, \quad i = 1, 2$$

where $A_{0,1} = A_{0,2} = b_2$, and $A_{1,1} = A_{1,2} = b_1$, and $\alpha_{j,1} = \sum_{k=0}^{j} \gamma_k \gamma_{j-k}$, where $\gamma_0 = -1$, $\gamma_1 = 0$, $\alpha_{j,2} = \sum_{k=0}^{j} \eta_k \eta_{j-k}$, where $\eta_0 = -1$, $\eta_1 = 0$. We now use Theorem 2.2, and we obtain that

$$\hat{\mathcal{O}}_{PV_{J,g}} \simeq K[[b_1, b_2]]/(T^{(1)}, T^{(2)}),$$

where

$$T^{(i)} = b_2 - \frac{b_i^2}{4\alpha_{2,i}} - \frac{\alpha_{3,i} b_i^3}{8\alpha_{2,i}^3} + \sum_{j \geqq 4} R_j(\alpha_{2,i}, \ldots, \alpha_{j,i}) b^j{}_1, \quad \text{i} = 1, 2,$$

where the $R_j(\alpha_{2,i}, \ldots, \alpha_{j,i})$ are rational functions in $\alpha_{2,i}, \ldots, \alpha_{j,i}$, such that

$$R_j(\alpha_{2,i}, \ldots, \alpha_{j,i}) = \left(\frac{-1}{2\alpha_{2,i}}\right)^j \cdot \alpha_{j,i} +$$

$$+ \text{ terms only involving } (\alpha_{2,i}, \ldots, \alpha_{j-1,i}).$$

Hence the power series $T^{(1)}$ and $T^{(2)}$ are congruent modulo (b_1^m), but not modulo (b_1^{m+1}), where $m = \min_j\{\alpha_{j,1} \neq \alpha_{j,2}\}$, and the multiplicity of PV_J at g is m.

We also see that m is equal to m', where

$$m' = \min_j\{\gamma_j \neq \eta_j\}.$$

But m' is clearly equal to $I(P, C_1 \cap C_2)$. Hence the multiplicity of PV_J at g is equal to $I(P, C_1 \cap C_2)$ under the assumptions given.

EXAMPLE 4.2. We work with the same family of curves as in Example 4.1. Let C_1 and C_2 be two fixed curves that pass through $Q = (0, 0, 1)$, and that are nonsingular at Q. Let D be the "variable curve" with equation

$$X^2 + Y^2 = 0, \quad \text{i.e. a union of two lines.}$$

We assume $I(Q, C_i \cap D) = 2$, $i = 1, 2$, and that D touches neither C_1 nor C_2 at any point but Q. Let $g \in G$ be the point $(1, 0, 0)$, in other words g corresponds to D. Set $J = \sum_{i=1}^{2} F^0(q_* \mathcal{O}_{\mathscr{C}_i})$.

Let C'_1 be the "mirror image" of C_1 with respect to the line $X = 0$. We will show that the multiplicity of PV_J at g is equal to $I(Q, (C_1 \cup C'_1) \cap C_2)$. We work in 3 steps:

STEP 1. Let t_i be a parameter for C_i at Q, $i = 1, 2$. We proceed as in Example 4.1 and find two power series $T^{(1)}$ and $T^{(2)}$. Then it is easy to see that if C_2 touches neither C_1 nor C'_1 at Q, then the multiplicity of PV_J at g is 2.

STEP 2. Assume C_2 touches C_1 or C'_1 at Q, and that the common tangent is not the line with equation $X = 0$. Then we use X as a common parameter for C_1 and C_2 at Q, we proceed as in Example 4.1, and we find that the multiplicity of PV_J at g is

$$\max[I(Q, C_1 \cap C_2), I(Q, C'_1 \cap C_2)] + 1.$$

STEP 3. Assume the common tangent of C_1 and C'_1 and C_2 is the line $X = 0$. Then we use Y as a common parameter for C_1 and C_2 at Q. We find that the multiplicity of PV_J at g is

$$I(Q, (C_1 \cup C'_1) \cap C_2).$$

Summing up, we find that in any case the multiplicity of PV_J at g is

$$I(Q, (C_1 \cup C'_1) \cap C_2).$$

EXAMPLE 4.3. We will give an application of Proposition 1.3. Let C_1, C_2, C_3 be 3 curves in P^3, and let $G = \check{P}^3$, that is: G parametrizes the planes in P^3. We study $X = H(C_1) \cap H(C_2) \cap H(C_3)$ at $g = g(D)$, where D is a plane which touches C_i at P_i, for $i = 1, 2, 3$, and all tangencies are simple. Then the length of $\hat{\mathcal{O}}_{X,g}$ is at least 2 iff P_1, P_2, P_3 are on a line L, by Proposition 1.1. The tangent space of X at g in G corresponds to the pencil of planes containing L.

Set $P^3 = \operatorname{Proj} K[X, Y, Z, W]$. Let the equation of L be $X = Y = 0$, and let the equation of D be $Y = 0$. Then Proposition 1.3 gives:

Length $\hat{\mathcal{O}}_{X,g} \geqq 3$ if an only if a quadratic surface, with an equation of the following form, intersects C_i at least 3 times at P_i, for all i:

$$(*) \qquad Y(aZ+bW)+cX^2 = 0, \quad \text{where } ac \text{ or } bc \text{ is nonzero.}$$

In [4], one studies trisecant lines to space curves. Consider the (reducible) curve $C = C_1 \cup C_2 \cup C_3$. Denote by $\mathcal{G}$ the Grassmannian $\{l(L);\ L \text{ a line in } P^3\}$. In [4], one defines a scheme (curve) $T \subset \mathcal{G}$ that parametrizes the trisecant lines to C. Let L be a trisecant line. Then T is singular at $l(L)$, iff there exists a plane D that intersects C_i at least 2 times at P_i, for all i (we assume that L is not a 4-secant). If this happens, then T has an ordinary node with 2 distinct curve tangents at $l(L)$, provided that no "additional unexpected phenomenon occurs". In [6], we showed that the "additional phenomenon" is exactly the existence of a quadric cone with an equation of the form (*), such that the cone intersects C_i at least 3 times at P_i, for all i. Hence we have in the situation described above:

$$\dim_K \hat{\mathcal{O}}_{X,g} \geqq 3 \Leftrightarrow T \text{ has a cusp or tacnode at } l(L),$$
$$\text{or } T \text{ is nonreduced at } l(L).$$

Moreover: T is nonreduced at $l(L) \Rightarrow \dim_K \hat{\mathcal{O}}_{X,g} = \infty$.

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