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ON REGULAR FROBENIUS BASES

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1. Definition of regular bases.

In a recent paper [6] in Math. Scand., Marstrander considers the Frobenius number $g(A_k)$ of a basis of $k+1$ positive integers

$$A_k = \{a_0, a_1, a_2, \ldots, a_k\}; \quad a_0 > 1, (a_0, a_1) = 1.$$

He does not comment the fact that the condition $(a_0, a_1) = 1$ is really a *restriction* for $k > 2$. It is well-known in Frobenius theory (cf. Rödseth [7]) that we may remove a common factor of *all but one* of the basis elements, but not for a smaller number of elements.

If necessary by reindexing $a_2, a_3, \ldots, a_k$, Marstrander gives A_k in *ordered form* by

$$(1.1) \qquad \begin{cases} a_i = a_1 b_i - a_0 c_i, & i = 1, 2, \ldots, k+1 \quad (a_{k+1} = 0) \\ 1 = b_1 < b_2 < \cdots < b_k < b_{k+1} = a_0 \\ 0 = c_1 < c_2 < \cdots < c_k < c_{k+1} = a_1. \end{cases}$$

To obtain this, some *dependent* bases are excluded (but there may still be dependencies in a basis in ordered form). We put

$$B_k = \{1, b_2, \ldots, b_{k+1}\}, \quad C_k = \{0, c_2, \ldots, c_{k+1}\}.$$

Many of Marstrander's results bear a certain resemblance to earlier results of Hofmeister. In most cases, however, this resemblance is mainly formal. In one instance (Remark to Theorem 3), Marstrander makes a direct reference to results in Hofmeister's lecture notes [3]. These results have later been *published* in [4] (not easily accessible). Marstrander's reference is to Theorem 5 of [4].

For later use, we must quote some more definitions from Marstrander. Since $b_1 = 1$, any positive integer may be expressed by the basis $\{1, b_2, \ldots, b_j\}$, $j \leqq k+1$, as $n = \sum_1^j x_i b_i$ with $x_i \geqq 0$ (in many ways). Following Hofmeister, we denote the (unique) *regular* representation by $n = \sum_1^j e_i b_i$. We then

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introduce

$$R(n, j) = \sum_1^j e_i c_i, \quad R(n) = R(n, k)$$
$$M(n, j) = \max \{\sum_1^j x_i c_i \mid n = \sum_1^j x_i b_i\}.$$

Marstrander now defines the (ordered) basis A_k to be *regular* if

$$R(n, k+1) = M(n, k+1), \quad \forall n \in \mathbf{N}. \tag{1.2}$$

This property clearly depends on the choice of the (coprime) basis elements a_0 and a_1.

The definition (1.2) may seem artificial, but it turns out to be highly useful. A good illustration of its usefulness is Marstrander's striking Lemma 2.

Incidentally, this Lemma also has a function which is not pointed out by Marstrander. The condition (1.2) is apparently "infinite", since it shall hold for all natural numbers n. However, regularity is equivalent to the condition

$$l = \sum_1^k e_i b_i \Rightarrow t_l = \sum_1^k e_i a_i, \quad l = 1, 2, \ldots, a_0 - 1, \tag{1.3}$$

where the minimal system $\{t_l\}$ can always be determined by a "finite work".

2. On the conditions for regularity.

Marstrander's main use of regular bases lies in the determination of the Frobenius number $g(A_k)$. Our interest has been a study of regular bases *as such*, regardless of applications. We shall treat some aspects which were not considered by Marstrander, but first quote one more result from his paper: Let $\langle x \rangle$ denote the smallest integer $\geqq x$, and put

$$b_{i+1} = q_i b_i - s_i, \quad q_i = \left\langle \frac{b_{i+1}}{b_i} \right\rangle, \quad \text{hence } 0 \leqq s_i < b_i. \tag{2.1}$$

Let further $j < k+1$, and assume that

$$R(n, j) = M(n, j), \quad \forall n \in \mathbf{N}$$

(always satisfied for $j = 2$). Then Marstrander's Lemma 4 says that

$$R(n, j+1) = M(n, j+1), \quad \forall n \in \mathbf{N}$$

if and only if

$$c_{j+1} \geqq q_j c_j - R(s_j). \tag{2.2}$$

If this condition is satisfied for *all* $j = 2, 3, \ldots, k$, we get (1.2) and hence regularity. We shall then call A_k *completely* regular (our term).

For later use, let us write out explicitly the conditions (2.2) for $j = 2$ and $j = 3$:

(2.3) $$c_3 \geqq q_2 c_2 = \left\langle \frac{b_3}{b_2} \right\rangle c_2$$

(note that $R(s_2) = 0$ since $s_2 < b_2$ and $c_1 = 0$),

(2.4) $$c_4 \geqq q_3 c_3 - R(s_3) = \left\langle \frac{b_4}{b_3} \right\rangle c_3 - \left[\frac{s_3}{b_2} \right] c_2$$

(since $b_3 > s_3 = e_1 + e_2 b_2$, with $e_2 = [s_3/b_2]$).

We make three observations regarding the conditions (2.2):

(i) Analysing Marstrander's proof of his Lemma 4, it is easily seen that the condition (2.2) is *necessary* for regularity of A_k when $j = k$, and for all $j < k$ such that

$$b_{j+1} + s_j < b_{j+2}$$

(since then all representations of $b_{j+1} + s_j$ by $\{1, b_2, \ldots, b_{j+1}\}$ are the same as by the full basis B_k).

(ii) Since $s_j < b_j$, we can replace $R(s_j)$ of (2.2) by $R(s_j, j-1)$, or just as well by $R(s_j, j)$:

(2.5) $$c_{j+1} \geqq q_j c_j - R(s_j, j).$$

It is easily seen that this condition is *equivalent* to (2.2), even if $s_i < b_i$ is *deleted* in (2.1):

(2.6) $$b_{i+1} = q_i b_i - s_i, \quad s_i \geqq 0$$

(the remainder $s_i - [s_i/b_i] b_i$ then corresponds to the s_i of (2.1)). – This observation, though perhaps trivial, will be very useful in section 4 below.

(iii) For $k = 2$, the one condition (2.3) is necessary and sufficient for regularity of A_2. Already for $k = 3$, there are *regular* bases A_3 which satisfy (2.4) but not (2.3), and which are consequently *not completely* regular. A simple example is given by

(2.7) $$A_3 = \{4, 5, 2, 3\}, \quad B_3 = \{1, 2, 3, 4\}, \quad C_3 = \{0, 2, 3, 5\}.$$

The regularity of A_3 is easily established from (1.3). By Marstrander's Lemma 4, there must exist an $n \in \mathrm{N}$ such that $R(n, 3) < M(n, 3)$. We can use $n = 4 = b_3 + b_1 = 2b_2$, where $R(n, 3) = c_3 = 3$, $M(n, 3) = 2c_2 = 4$.

3. The case $k = 2$.

In the simplest case $k = 2$, we have

$$A_2 = \{a_0, a_1, a_2\}, \quad a_2 = a_1 b_2 - a_0 c_2$$
$$B_2 = \{1, b_2, a_0\}, \quad C_2 = \{0, c_2, a_1\}.$$

As already noted, A_2 is then regular if and only if the condition (2.3) is satisfied:

(3.1) $$a_1 \geqq \left\langle \frac{a_0}{b_2} \right\rangle c_2.$$

In this case, we shall see that regularity corresponds to a well-known property in Frobenius theory.

The Frobenius number $g(A_2)$ was determined by Rödseth [7]. He puts

$$a_2 \equiv a_1 s_0 \pmod{a_0}, \quad 0 \leqq s_0 < a_0; \text{ hence } s_0 = b_2,$$

and then performs the Euclidean division algorithm with *negative* remainders on the ratio $a_0 : s_0$ The algorithm stops after a certain number v of division steps, determined by a condition of the form $R_{v+1} \leqq 0 < R_v$. In particular,

$$R_0 = c_2, \quad R_1 = \left\langle \frac{a_0}{b_2} \right\rangle c_2 - a_1,$$

and a comparison with (3.1) shows that A_2 *is regular just when* $v = 0$ *in Rödseth's algorithm.*

In this case, Rödseth's general formula for $g(A_2)$ shows that

(3.2) $$g(A_2) = -a_0 + a_1(b_2 - 1) + a_2(q_2 - 1) - \min\{a_1 s_2, a_2\},$$

where $a_0 = b_3 = q_2 b_2 - s_2$ of (2.1).

The same result, under a condition equivalent to (3.1), was already given by Hofmeister [2], as a special case of a rather complicated theorem. A direct and simple proof of (3.2) was presented by the author [8] (before Rödseth [7] appeared).

4. Regular partial bases.

If $\kappa < k$, and A_k is regular, we may ask under what conditions a "*partial basis*"

$$A_\kappa = \{a_0, a_1, \ldots, a_\kappa\}$$

is also regular. Even if all conditions (2.2) should be satisfied, the question is far from trivial, since now

$$B_\kappa = \{1, b_2, \ldots, b_\kappa, a_0\}, \quad C_\kappa = \{0, c_2, \ldots, c_\kappa, a_1\}$$

are *not partial bases* of B_k and C_k.

We can, however, prove the following

THEOREM 1. *Let* $k \geqq 3$, *and* $2 \leqq \kappa < k$. *If* A_k *satisfies the conditions*

$$R(n, \kappa) = M(n, \kappa), \quad \forall n \in \mathrm{N}$$

$$c_{j+1} \geqq q_j c_j - R(s_j), \quad j = \kappa, \kappa+1, \ldots, k$$

(hence A_k *regular), then the partial basis* A_κ *is regular. In particular, all* A_κ *are regular if* A_k *is completely regular.*

It will clearly suffice to prove Theorem 1 first for $\kappa = k-1$, and then use this result *repeatedly*. We thus assume that

$$R(n, k-1) = M(n, k-1), \quad \forall n \in \mathrm{N} \tag{4.1}$$

$$\begin{cases} c_k \geqq q_{k-1} c_{k-1} - R(s_{k-1}, k-1) \\ c_{k+1} \geqq q_k c_k - R(s_k, k-1). \end{cases} \tag{4.2}$$

Since $s_{k-1} < b_k$ and $s_k < b_k$, we may insert a second argument $k-1$ in the R-functions. From

$$b_k = q_{k-1} b_{k-1} - s_{k-1}, \quad b_{k+1} = q_k b_k - s_k,$$

we get

$$b_{k+1} = q_k q_{k-1} b_{k-1} - (q_k s_{k-1} + s_k)$$

for use in the "reduced" basis

$$B_\kappa = B_{k-1} = \{1, b_2, \ldots, b_{k-1}, b_{k+1} = a_0\}.$$

Departing from (4.1), and using the condition (2.2) in the form (2.5–2.6), we see that $A_\kappa = A_{k-1}$ is regular if and only if

$$c_{k+1} \geqq q_k q_{k-1} c_{k-1} - R(q_k s_{k-1} + s_k, k-1).$$

The two inequalities (4.2) give

$$c_{k+1} \geqq q_k q_{k-1} c_{k-1} - \{q_k R(s_{k-1}, k-1) + R(s_k, k-1)\},$$

so we are through if we can show that

$$R(q_k s_{k-1} + s_k, k-1) \geqq q_k R(s_{k-1}, k-1) + R(s_k, k-1).$$

And this is an immediate consequence of (4.1) and Marstrander's Lemma 3.

The many $\geqq$ in the proof indicate that all the conditions of Theorem 1 are not always necessary. As an example in the simplest case $k = 3$, $\kappa = 2$, consider the basis A_3 of (2.7). Even if this fails to satisfy (2.3), the partial basis

$$A_2 = \{4, 5, 2\}, \quad \text{with} \quad B_2 = \{1, 2, 4\}, \quad C_2 = \{0, 2, 5\},$$

satisfies (3.1) and is thus regular.

It may be useful with one remark on the *repeated* use of the above proof, hence the next step if $\kappa < k-1$: In the "reduced" basis B_{k-1}, we must now form

$$a_0 = b'_k = q'_{k-1}b_{k-1} - s'_{k-1}, \quad 0 \leqq s'_{k-1} < b_{k-1},$$

and utilize the condition

$$a_1 = c'_k \geqq q'_{k-1}c_{k-1} - R(s'_{k-1}).$$

This is *not* the same as the original condition (2.2) for $j = k-1$ in Theorem 1. However, it follows from observation (i) of section 2 that this new condition must also be satisfied, since we have already shown that A_{k-1} is regular.

5. The connection with pleasant h-bases.

We assume knowledge of the "postage stamp problem", see for instance [9]. A comprehensive treatment of this problem is contained in the author's research monograph [10] (freely available on request).

A "stamp" basis (an h-basis)

$$\mathscr{A}_k = \{\alpha_0, \alpha_1, \ldots, \alpha_k\}, \quad 1 = \alpha_0 < \alpha_1 < \cdots < \alpha_k,$$

is *pleasant* if and only if the regular representation $n = \sum_0^k e_i\alpha_i$ has a minimal coefficient sum among all possible representations $n = \sum_0^k x_i\alpha_i$, for all natural numbers n. Then the h-range $n_h(\mathscr{A}_k)$ equals the *regular* h-range $g_h(\mathscr{A}_k)$, which is easily determined (see for instance [5]).

For an arbitrary (not necessarily pleasant) $\mathscr{A}_k$, we form the "complementary basis"

$$\bar{\mathscr{A}}_k = \{\alpha_k - \alpha_{k-1}, \alpha_k - \alpha_{k-2}, \ldots, \alpha_k - \alpha_1, \alpha_k - 1, \alpha_k\}, \tag{5.1}$$

and consider this as a Frobenius basis. By Meures' theorem, we then have

$$n_h(\mathscr{A}_k) = h\alpha_k - g(\bar{\mathscr{A}}_k) - 1, \quad h \geqq h_1. \tag{5.2}$$

The bound h_1 is usually difficult to determine. When $\mathscr{A}_k$ is *pleasant*, however, both h_1 and $n_h(\mathscr{A}_k)$ are known, and the Frobenius number $g(\bar{\mathscr{A}}_k)$ then follows directly from (5.2).

It is natural to ask when $\mathscr{A}_k$ of (5.1) can be organized as a *regular* basis. We now have two coprime elements, $\alpha_k - 1$ and α_k, which can be used as a_0 and a_1 (in any order). The most interesting choice turns out to be $a_0 = \alpha_k$, $a_1 = \alpha_k - 1$. It is easily seen that this leads to the ordered form (1.1):

$$\text{(5.3)} \qquad \begin{cases} A_k = \{\alpha_k, \alpha_k - 1, \alpha_k - \alpha_1, \alpha_k - \alpha_2, \ldots, \alpha_k - \alpha_{k-1}\} \\ B_k = \mathscr{A}_k; \quad c_i = b_i - 1, \ i = 1, 2, \ldots, k+1. \end{cases}$$

The regularity condition (1.2) says that for all representations $n = \sum_1^{k+1} x_i b_i$, the regular one should give the maximal

$$\sum_1^{k+1} x_i c_i = \sum_1^{k+1} x_i (b_i - 1) = n - \sum_1^{k+1} x_i.$$

In other words, the coefficient sum must be minimal for the regular representation of any n by $B_k = \mathscr{A}_k$. We have thus proved

THEOREM 2. *A_k of (5.3) regular $\Leftrightarrow$ $\mathscr{A}_k$ pleasant.*

This has the very interesting consequence that we may consider regularity of Frobenius bases as a *generalization* of pleasantness for h-bases. Properties in the former case then carry over into similar properties in the latter.

As an example, let us study the analogue of Marstrander's condition (2.2) in the case (5.3). Now $b_i = \alpha_{i-1}$, $c_i = b_i - 1$, and a straightforward calculation gives the following result: Put

$$\alpha_j = \left\langle \frac{\alpha_j}{\alpha_{j-1}} \right\rangle \alpha_{j-1} - \sum_0^{j-2} e_i \alpha_i,$$

where the sum is *regular* by $\mathscr{A}_{j-2}$. The condition (2.2) then takes the form

$$\left\langle \frac{\alpha_j}{\alpha_{j-1}} \right\rangle > \sum_0^{j-2} e_i.$$

Assuming $\mathscr{A}_{j-1}$ pleasant, this is the necessary and sufficient condition for $\mathscr{A}_j$ to be pleasant.

This is a well-known result of Djawadi [1] in the theory of h-bases.

As another example, Theorem 1 above corresponds to an earlier result by the author [11]: If $k \geqq 3$, $1 \leqq \kappa \leqq k-2$, and $\mathscr{A}_i$ is pleasant for $i = \kappa, \kappa+1, \ldots, k$, then the "sub-basis"

$$\mathscr{A}_k^{(\kappa)} = \{1, \alpha_1, \ldots, \alpha_\kappa, \alpha_k\}$$

is also pleasant.

Conversely, however, we can *not* always draw conclusions from pleasant

h-bases to regular Frobenius bases. As an example, Djawadi [1] showed that if $\mathscr{A}_3$ is pleasant, then so is $\mathscr{A}_2 = \{1, \alpha_1, \alpha_2\}$, and Zöllner [12] could replace $\mathscr{A}_3$ by $\mathscr{A}_k$ in this statement. Hence, the condition (2.3) is *necessary* for regularity of A_k in (5.3). On the other hand, we gave in (2.7) an example of a regular Frobenius basis which does not satisfy (2.3).

We mentioned above the alternative choice $a_0 = \alpha_k - 1$, $a_1 = \alpha_k$ in $\bar{\mathscr{A}}_k$ of (5.1). The resulting ordered basis A_k is easily constructed, in analogy with (5.3). However, nothing as interesting as Theorem 2 comes out of this choice. We only mention that if the resulting A_k is *completely* regular, it is "highly" dependent, and reduces to one of the two cases

$$\{\alpha_k - \alpha_{k-1}, \alpha_k - 1\} \quad \text{or} \quad \{\alpha_k - \alpha_{k-1}, \alpha_k\}$$

(since $\alpha_k - \alpha_{k-1}$ divides all the other basis elements of $\bar{\mathscr{A}}_k$). In either case, the determination of $g(A_k)$ is of course trivial.

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