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POLAR SETS FOR SUPERSOLUTIONS OF DEGENERATE ELLIPTIC EQUATIONS

JUHA HEINONEN* and TERO KILPELÄINEN

1. Introduction.

Let us consider the homogeneous second order differential equation

$$\nabla \cdot A(x, \nabla u) = 0, \tag{1.1}$$

where $A: \mathsf{R}^n \times \mathsf{R}^n \to \mathsf{R}^n$ is an elliptic nonlinear operator with $A(x,h)\cdot h \approx |h|^p$, $1 < p < \infty$; see Section 2 for the precise assumptions. Weak solutions of (1.1) are always continuous and we call them *A-harmonic*. A prototype of the operators considered here is the p-harmonic operator $A(x,h) = |h|^{p-2}h$, and solutions of the equation

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0$$

are customarily called *p-harmonic*.

A function u in an open set $G \subset \mathsf{R}^n$ is *A-superharmonic* if

(1.2) u is lower semicontinuous (l.s.c.),

(1.3) $-\infty < u \leqq +\infty$, and

(1.4) for each domain $D \subset\subset G$ and for each A-harmonic function $h \in C(\bar{D})$, $h \leqq u$ in ∂D, implies $h \leqq u$ is D.

The authors have shown in [5] that weak supersolutions of (1.1) are A-superharmonic when properly pointwise redefined, see Proposition 2.8 below, and therefore the above definition allows us to study also supersolutions as pointwise defined functions. On the other hand, we obtain a larger class of functions since an A-superharmonic function need not be in the Sobolev space loc $W_p^1(G)$. Previously, S. Granlund, P. Lindqvist, and O. Martio have used this approach in the conformally invariant case $p = n$, and it has turned out that a nonlinear potential theory can be developed where A-superharmonic functions play a rôle similar to that of superharmonic functions in the

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classical theory, see [2], [5], [8], [9]. It is our hope that this approach will also shed some light on topics in nonlinear variational problems, especially in the regularity theory for obstacle problems, cf. [14], [15], and [6].

In this paper we study balayage and polar sets in a nonlinear situation. As in the classical potential theory we say that a set E in $\mathbf{R}^n$ is *A-polar* if there is an open neighborhood V of E and an A-superharmonic function $v: V \to \mathbf{R} \cup \{\infty\}$ such that $v = \infty$ in E and that $v \not\equiv \infty$ in each component of V. Also the balayage of an A-superharmonic function is defined analogously to the classical case; see Section 3 for this.

The efficient use of the obstacle method yields the following result which relates potential theoretical polar sets to other small sets ubiquitous in Sobolev space theory and in the theory of partial differential equations, cf. [11], [13], [17].

1.5. THEOREM. *Let E be a set in $\mathbf{R}^n$, $n \geqq 2$. Then the following are equivalent:*

(i) *E is A-polar;*

(ii) *there is an open neighborhood G of E such that if u is a positive A-superharmonic function in G, then the balayage $\hat{R}_E^u$ vanishes identically in G;*

(iii) *E is of (outer) p-capacity zero;*

(iv) *there is a nonnegative l.s.c. function w in $W_p^1(\mathbf{R}^n)$ such that $E \subset w^{-1}(\infty)$.*

1.6. COROLLARY. *A-polarity depends only on p, not on the operator A.*

1.7. COROLLARY. *A countable union of A-polar sets is A-polar.*

Observe that Corollary 1.7 is not immediate since the sum of two A-superharmonic functions is not A-superharmonic in general.

We are also able to show that if $1 < p < n$, then there is a positive A-superharmonic function u in $\mathbf{R}^n$, $u \not\equiv \infty$, such that $u = \infty$ in a given polar set; if we insist on u being positive, this result is no longer true for $p = n$.

Theorem 1.5 for $1 < p < n$ is proved in Sections 3 and 4, and the somewhat different borderline case $p = n$ is studied in Section 5.

In the final section, Section 6, we prove the Fundamental Convergence Theorem in the nonlinear case, cf. [1, p. 70].

In the proof of Theorem 1.5, only variational methods are used. For example, the passage from (iv) to (i) derives from solving the obstacle problem with the function w as an obstacle and, consequently, in the well-known equivalence of (iii) and (iv) no reference to Bessel potentials is made. On the other hand, in higher order spaces our method is of no use since we strongly employ the lattice property of the space W_p^1, cf. [4].

Polar sets in a nolinear situation were first considered by P. Lindqvist and

O. Martio [10]. They have established the equivalence of (i) and (iv) when the equation (1.1) is the Euler equation of a convex variational integral in the borderline case $p = n$. We wish to thank both of them for inspiring discussions.

2. Preliminaries.

Our notation is standard. Throughout, G will be an open set in $\mathbf{R}^n$, $n \geqq 2$, and $D \subset\subset G$ means that $\bar{D}$, the closure of D, is compact in G. If $B = B(x_0, r)$ is an open n-ball and $\sigma > 0$, then $\sigma B = B(x_0, \sigma r)$. The complement of a set F is marked as $\complement F = \mathbf{R}^n \setminus F$.

Let A: $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ be an operator satisfying the following assumptions for some numbers $1 < p < \infty$ and $0 < \alpha \leqq \beta < \infty$:

(2.1) the function $x \mapsto A(x, h)$ is measurable for all $h \in \mathbf{R}^n$, and the function $h \mapsto A(x, h)$ is continuous for a.e. $x \in \mathbf{R}^n$; for all $h \in \mathbf{R}^n$ and a.e. $x \in \mathbf{R}^n$,

(2.2) $A(x, h) \cdot h \geqq \alpha |h|^p$,

(2.3) $|A(x, h)| \leqq \beta |h|^{p-1}$,

(2.4) $(A(x, h_1) - A(x, h_2)) \cdot (h_1 - h_2) > 0$
whenever $h_1 \neq h_2$, and

(2.5) $A(x, \lambda h) = |\lambda|^{p-2} \lambda A(x, h)$
for all $\lambda \in \mathbf{R}$, $\lambda \neq 0$.

A function u in the Sobolev space $\operatorname{loc} W_p^1(G)$ is a solution (a supersolution) of (1.1) if

$$\int_G A(x, \nabla u) \cdot \nabla \phi \, dx = 0 \quad (\geqq 0) \tag{2.6}$$

for all $\phi \in C_0^\infty(G)$ ($\phi \in C_0^\infty(G)$, $\phi \geqq 0$). It is well-known that solutions of (1.1) are locally Hölder continuous; for this and other properties we refer to Serrin's fundamental work [17]; for supersolutions see also Trudinger [18].

We recall some basic properties of A-superharmonic functions. For the proofs see [5].

First observe that if u is A-superharmonic, then so is $\lambda u + \mu$ whenever $\lambda \geqq 0$ and $\mu \in \mathbf{R}$; this property can be used in many places to compensate the lack of linearity. Further, if u and v are A-superharmonic, then so is $\min(u, v)$.

The class of A-superharmonic functions is closed under upper directed monotone convergence: if u_i is an increasing sequence of A-superharmonic functions, then $u = \lim u_i$ is A-superharmonic.

The *comparison principle* holds: if G is bounded, if u and $-v$ are A-superharmonic in G with

$$\limsup_{y \to x} v(y) \leqq \liminf_{y \to x} u(y)$$

for all $x \in \partial G$ and if the left and the right hand sides are not simultaneously ∞ or $-\infty$, then $v \leqq u$ in G.

If u is A-superharmonic in G, then

$$u(x) = \operatorname{ess} \liminf_{y \to x} u(y) \tag{2.7}$$

for all $x \in G$; this important property is well-known in the clasical theory.

The following result is nontrivial and essential in what follows.

2.8. PROPOSITION. [5, 3.13 and 3.17] *If u is locally bounded and A-superharmonic in G, then u belongs to* $\operatorname{loc} W^1_p(G)$ *and is a supersolution of* (1.1). *Conversely, if u is a supersolution of* (1.1) *in G, then there is a unique A-superharmonic representative of u, given by* (2.7).

It follows from Propositon 2.8 that a function u *is* A-superharmonic if it is locally A-superharmonic.

Observe that while every supersolution of (1.1) can be considered as an A-superharmonic function, the converse is not true unless $p > n$.

2.9. *Condensers and capacity.* If $E \subset G$, then the *inner* and the *outer* *p-capacity*, $1 < p < \infty$, of the *condenser* (E, G) is defined, respectively, by

$$_*\mathrm{cap}_p(E, G) = \sup_{C \subset E \text{ compact}} \mathrm{cap}_p(C, G)$$

and

$$^*\mathrm{cap}_p(E, G) = \inf_{U \supset E \text{ open}} {_*\mathrm{cap}_p}(U, G)$$

where

$$\mathrm{cap}_p(C, G) = \inf_{u \in W(C,G)} \int_G |\nabla u|^p \, dx.$$

$W(C, G) = \{u \in C_0^\infty(G) \colon u = 1 \text{ in } C\}$, is the usual variational p-capacity. The set function $E \mapsto {^*\mathrm{cap}_p}(E, G)$ defines a Choquet capacity, cf. [1, A.II].

A set E in R^n is of (*outer*) *p-capacity zero*, abbreviated $\mathrm{cap}_p E = 0$, if ${^*\mathrm{cap}_p}(E \cap G, G) = 0$ for all open $G \subset \mathsf{R}^n$. It is an easy task to show that if E is bounded, then $\mathrm{cap}_p E = 0$ as soon as ${^*\mathrm{cap}_p}(E, G) = 0$ for some bounded G, cf. [13, Chapter 9], [16].

For a thorough discussion of variational capacities we refer to [13], [16].

2.10. *Regular open sets.* Let G be bounded and $\theta \in W_p^1(G)$. Then there is a unique A-harmonic function u in G such that $u-\theta \in W_{p,0}^1(G)$, cf. [12]. A boundary point $x \in \partial G$ is said to be *regular* if

$$\lim_{y \to x} u(y) = \theta(x) \tag{2.11}$$

whenever u and θ are as above and, in addition, $\theta \in C(\bar{G})$. As is well-known, the boundary point x is regular if the Wiener criterion

$$\int_0^1 \left(\frac{\operatorname{cap}_p(\bar{B}(x,t) \cap \complement G, B(x,2t))}{\operatorname{cap}_p(\bar{B}(x,t), B(x,2t))} \right)^{1/(p-1)} \frac{dt}{t} = \infty \tag{2.12}$$

holds, see [12]. It is an open problem whether (2.12) is also necessary for (2.11) when $p \leqq n-1$; for $p > n-1$ the necessity has been proved in [9], see also [6]. If each boundary point of G is regular, then G is said to be *regular*.

L. I. Hedberg and Th. H. Wolff have shown that the Wiener criterion holds except on a set of zero p-capacity [4] (see [16] for the equivalence of the two capacities), that is, the *Kellogg property* holds.

2.13. *Obstacle problem.* Let G be bounded and let $\psi \in W_p^1(G)$. Then there is a unique function $u \in W_p^1(G)$ such that u is A-superharmonic in G, $u \geqq \psi$ a.e. in G, $u-\psi \in W_{p,0}^1(G)$, and

$$\int_G A(x, \nabla u) \cdot \nabla \phi \, dx \geqq 0$$

for all $\phi \in C_0^\infty(G)$ with $\phi \geqq \psi - u$ a.e. in G. The function u is *the solution to the obstacle problem with the obstacle* ψ. See [5] for this.

2.14. *Poisson modification.* Let u be A-superharmonic in G and let $D \subset\subset G$ be a regular open set such that $u \not\equiv \infty$ in each component of D. Then the *Poisson modification* of u in D, abbreviated $P(u,D)$, is constructed as follows. Let ϕ_i be a sequence of functions in $C^\infty(\mathbb{R}^n)$ such that $\phi_i \nearrow u$ in $\bar{D}$. Let w_i be the unique A-harmonic function in D with boundary values ϕ_i. Then $w_i \leqq w_{i+1} \leqq u$ and, by Harnack's principle, see [5, 3.3], $w = \lim w_i$ is A-harmonic in D. Define

$$P(u,D) = \begin{cases} w & \text{in } D \\ u & \text{in } G \setminus D. \end{cases}$$

Then $P(u,D)$ is l.s.c. in G and it is easy to verify that $P(u,D)$ is indeed A-superharmonic in G. Moreover, $P(u,D) \leqq u$ in G.

3. Balayage, capacity and polar sets.

In this section we study polar sets and the balayage of an A-superharmonic function. The balayage seems to be a natural tool, and (ii) a natural auxiliary, in establishing the implication (i) $\Rightarrow$ (iii) in Theorem 1.5.

Let $A: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ be an operator satisfying (2.1)–(2.5).

Let $u: G \to \mathbf{R} \cup \{\infty\}$ be nonnegative and A-superharmonic in G and let E be a subset of G. By Φ_E^u we denote the set of all nonnegative A-superharmonic functions v in G such that $v \geqq u$ in E. The function

$$R_E^u = R_E^u(G; A) = \inf \Phi_E^u$$

is called the *réduite* of u relative to E in G. If $u \equiv c =$ a constant, we write $R_E^c = R_E^u$. The lower regularization

$$\hat{R}_E^u(x) = \hat{R}_E^u(G; A)(x) = \liminf_{y \to x} R_E^u(y)$$

is the *balayage* of u relative to E in G. It is clear that $u = R_E^u$ in E, that $u = R_E^u = \hat{R}_E^u$ in the interior of E and that $E_1 \subset E_2 \subset G$ implies $\hat{R}_{E_1}^u \leqq \hat{R}_{E_2}^u$. For a more general approach to balayage see [7].

The following three propositions are well-known in the classical theory.

3.1. PROPOSITION. (i) *$\hat{R}_E^u$ is A-superharmonic in G.*
(ii) *$\hat{R}_E^u$ is A-harmonic or $\hat{R}_E^u \equiv \infty$ in each component of $G \setminus \bar{E}$.*

PROOF. Since $\hat{R}_E^u$ is the greatest lower semicontinuous minorant of R_E^u, (i) follows directly from the definitions.

To prove (ii), note first that if $v_1, v_2 \in \Phi_E^u$, then $\min(v_1, v_2) \in \Phi_E^u$, and it follows from Choquet's topological lemma [1, p. 792] that there is a decreasing sequence of functions $v_i \in \Phi_E^u$ with the limit $v = \lim v_i$ such that

$$\liminf_{y \to x} v(y) = \hat{v}(x) = \hat{R}_E^u(x).$$

Pick a ball $B \subset\subset G \setminus \bar{E}$ and let $w_i = P(v_i, B)$ be the Poisson modification of v_i in B. Then $w_i \in \Phi_E^u$, $w_i \geqq w_{i+1}$ and $w = \lim w_i \leqq v$. Since $\hat{w} = \hat{R}_E^u$ and since, by Harnack's principle, $\hat{w}$ is either A-harmonic or identically ∞ in B, the claim follows.

3.2. PROPOSITION. *Suppose that G is a bounded domain and that $E \subset\subset G$. Let $x \in \partial G$ be a regular boundary point of G. If $\hat{R}_E^u$ is not identically ∞ in G, then* $\lim_{y \to x} \hat{R}_E^u(x) = 0$.

PROOF. Let $v \in \Phi_E^u$, $v \not\equiv \infty$ in G. Choose a regular neighborhood D of ∂G such that $\bar{D} \cap \bar{E} = \varnothing$. Replacing v by its Poisson modification in an appropriate neighborhood U of $\partial D \cap G$, $U \subset G \setminus \bar{E}$, we may assume that $v < M < \infty$ in $\partial D \cap G$. Let h be the A-harmonic function in $D \cap G$ with the boundary values 0 in ∂G and M in $\partial D \cap G$. Then the function

$$w = \begin{cases} v & \text{in } G \setminus D \\ \min(v, h) & \text{in } D \end{cases}$$

belongs to Φ_E^u, and since x is a regular boundary point of $D \cap G$ (it follows from the barrier characterization that the regularity is a local property; see e.g. methods in [3]), $\lim_{y \to x} w(y) = 0$ as desired.

Next, we turn to study A-polar sets.

3.3. PROPOSITION. *Let u be positive and A-superharmonic in G and let $E \subset G$. Then*

(i) *if E is A-polar, then there is an open neighborhood V of E, $V \subset G$, such that $R_E^u(V; A)$ has a zero in each component of V;*
(ii) *if $R_E^u(G; A)(x) = 0$, then $\hat{R}_E^u \equiv 0$ in the x-component of G.*

PROOF. The claim (i) is a trivial consequence of the fact that if v is A-superharmonic and if D is a component of the open set $\{x : v(x) > 0\}$, then either $v \equiv \infty$ or the set $\{x \in D : v(x) < \infty\}$ is everywhere dense in D, cf. [10, 2.6], [7, 2.10]. The minimum principle yields (ii).

3.4. THEOREM. *Let u be positive and A-superharmonic in G and let $E \subset G$. Then $\hat{R}_E^u = \hat{R}_E^u(G; A) \equiv 0$ implies $\operatorname{cap}_p E = 0$.*

PROOF. Since it suffices to show that $^*\operatorname{cap}_p(E', B(x_0, r)) = 0$ whenever $B = B(x_0, r) \subset G$ and $E' = E \cap \frac{1}{2}B$, we may assume that $G = B$ is a ball and that $E \subset \frac{1}{2}B$. Since $u \geqq \delta > 0$ in E, for $w = \min(1, \delta^{-1}u)$ we have $\hat{R}_E^w \equiv 0$; hence we are also free to assume that $u \equiv 1$ in E.

Let $\lambda > 1$. We can find a function $v \in \Phi_E^u$, $v \leqq 1$, such that v is continuous in a neighborhood of $\partial \frac{3}{4}B$ and that $v < 1/\lambda$ in $\partial \frac{3}{4}B$, cf. the proof of Proposition 3.2. Then the function

$$u_\lambda = \begin{cases} \min(1, \lambda v) & \text{in } B \setminus \frac{3}{4}\bar{B} \\ \lambda v & \text{in } \frac{3}{4}\bar{B}, \end{cases}$$

is a locally bounded A-superharmonic function in B, and therefore a supersolution of (1.1) in B by Proposition 2.8. Using the estimate (2.25) in [5] yields

$$\int_{\frac{3}{4}B} |\nabla \log u_\lambda|^p \, dx \leqq c, \tag{3.5}$$

where c does not depend on λ.

To complete the proof, fix $\varepsilon > 0$. Choose $\lambda \geqq \exp\{(c/\varepsilon)^{1/p}\}+1$ and let

$$E_\lambda = \{x \in B : u_\lambda > \lambda - 1\}.$$

Then E_λ is open, $E \subset E_\lambda$ and

$$h = \max(0, (\varepsilon/c)^{1/p} \log u_\lambda) \geqq 1 \quad \text{in } E_\lambda.$$

Now h is admissible for each condenser (C, B), where $C \subset E_\lambda$ is compact, and, by (3.5),

$$\int_B |\nabla h|^p dx \leqq \frac{\varepsilon}{c} \int_{\frac{3}{4}B} |\nabla \log u_\lambda|^p dx \leqq \varepsilon.$$

This establishes the desired conclusion.

3.6. THEOREM. *Let E be a set in R^n with $\operatorname{cap}_p E = 0$. Then there is a nonnegative l.s.c. function w: $\mathsf{R}^n \to \mathsf{R} \cup \{\infty\}$ such that $w \in W^1_p(\mathsf{R}^n)$ and $w = \infty$ in E.*

PROOF. Fix a positive integer k and write $B = B(0, k)$, $E' = E \cap \frac{1}{2}B$. Choose for each $i = 1, 2, \ldots$ an open neighborhood U_i of E', $U_i \subset \frac{1}{2}B$, such that

$$\operatorname{cap}_p(C, B) < i^{-2p}$$

whenever $C \subset U_i$ is compact. Next, choose an increasing sequence of compact sets $C_{i,j} \subset C_{i,j+1} \subset U_i$ such that $B \setminus C_{i,j}$ is a regular open set and that $\bigcup_{j=1}^{\infty} C_{i,j} = U_i$. Let $\phi_{i,j} \in C(\bar{B})$ be the p-superharmonie function for which

$$\int_B |\nabla \phi_{i,j}|^p dx = \operatorname{cap}_p(C_{i,j}, B),$$

and $\phi_{i,j} = 1$ in $C_{i,j}$, $\phi_{i,j} = 0$ in ∂B. Since $\phi_{i,j}$ is p-harmonic in $B \setminus C_{i,j}$, the comparison principle yields $\phi_{i,j} \leqq \phi_{i,j+1}$ in B. Let

$$\phi_i = \lim_{j \to \infty} \phi_{i,j}.$$

We easily infer that ϕ_i belongs to $W^1_{p,0}(B)$, whence $\nabla \phi_{i,j} \to \nabla \phi_i$ weakly in $L^p(B)$ and, by the weak lower semicontinuity of norms,

$$\|\nabla \phi_i\|_{p,B} \leqq \liminf_{j \to \infty} \|\nabla \phi_{i,j}\|_{p,B} \leqq i^{-2}.$$

The function ϕ_i admits the zero extension in all of R^n, and this extension is a nonnegative l.s.c. function in R^n. By the Poincaré inequality

$$\|\phi_i\|_{p,\mathsf{R}^n} + \|\nabla \phi_i\|_{p,\mathsf{R}^n} \leqq c i^{-2},$$

where $c = c(n,k) < \infty$ is a constant. Hence the function $\psi_k = \sum_{i=1}^{\infty} \phi_i$ is nonnegative, l.s.c., and belongs to $W_p^1(\mathbf{R}^n)$. Moreover, $\psi_k = \infty$ in $E \cap \frac{1}{2}B$. Finally, choose positive numbers $\lambda_1, \lambda_2, \ldots$ such that

$$w = \sum_{k=1}^{\infty} \lambda_k \psi_k$$

is the desired function. The theorem is proved.

We are in the position that the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) of Theorem 1.5 are established. The rest of the paper is divided into two sections; the first deals with the case $1 < p < n$ and the second with the case $p = n$. These two cases differ in that there is no nonconstant positive A-superharmonic function in $\mathbf{R}^n$ if $p = n$, see Remark 5.4. At this point it is convenient to point out that the case $p > n$ is rather uninteresting, since then A-superharmonic functions are continuous, see [5, 3.20], and hence the only A-polar set is the empty set; thus Theorem 1.5 is trivial when $p > n$.

4. Polar sets when $1 < p < n$.

The main result of this section is Theorem 4.1; it completes the proof of Theorem 1.5 for $1 < p < n$.

4.1. THEOREM. *Let $1 < p < n$ and let $w \in W_p^1(\mathbf{R}^n)$ be a nonnegative l.s.c. function. Then there is a nonnegative A-superharmonic function u in $\mathbf{R}^n$, $u \not\equiv \infty$, such that $u = \infty$ in $E = \{x : w(x) = \infty\}$. In particular, the set E is A-polar.*

PROOF. Let $B_i = B(0,i)$, $i = 1, 2, \ldots$, and let v_i be the solution to the obstacle problem in B_i with the obstacle w. Then v_i is A-superharmonic, whence, together with (2.7),

$$v_i(x) = \operatorname*{ess\,lim\,inf}_{y \to x} v_i(y) \geqq \operatorname*{ess\,lim\,inf}_{y \to x} w(y) \geqq w(x)$$

for each x in B_i. The sequence v_i is increasing. Indeed, since v_{i+1} is A-superharmonic in B_i and since

$$\min(v_{i+1}, v_i) - w = \min(v_{i+1} - w, v_i - w) \in W_{p,0}^1(B_i),$$

then $v_{i+1} \geqq v_i$ a.e. in B_i, see [5, 2.8], whence

$$v_{i+1}(x) = \operatorname*{ess\,lim\,inf}_{y \to x} v_{i+1}(y) \geqq \operatorname*{ess\,lim\,inf}_{y \to x} v_i(y) = v_i(x)$$

for each x in B_i. Thus, $v = \lim v_i$ is A-superharmonic in $\mathbf{R}^n$ and $v \geqq w \geqq 0$; in particular, $v = \infty$ in E.

We proceed to show that $v \not\equiv \infty$. For this observe first that

$$0 \leqq \int_{B_i} A(x, \nabla v_i) \cdot \nabla(w - v_i) dx,$$

and hence using Hölder's inequality yields

$$\int_{B_i} |\nabla v_i|^p dx \leqq c \int_{B_i} |\nabla w|^p dx \leqq c \int_{\mathbf{R}^n} |\nabla w|^p dx, \tag{4.2}$$

where c depends only on n, p, α and β. Next, we employ a Poincaré type inequality: if $u \in W^1_{p,0}(B(0, R))$, then

$$\int_{B(0,R)} \frac{|u(x)|^p}{|x|^p} dx \leqq \left(\frac{p}{n-p}\right)^p \int_{B(0,R)} |\nabla u(x)|^p dx; \tag{4.3}$$

the proof of (4.3) is an easy integration by parts procedure. Since $w - v_i \in W^1_{p,0}(B_i)$, (4.3) and (4.2) yield

$$\int_{B_i} \frac{|v_i(x) - w(x)|^p}{|x|^p} dx \leqq c \int_{B_i} |\nabla v_i(x) - \nabla w(x)|^p dx \leqq c \int_{\mathbf{R}^n} |\nabla w(x)|^p dx,$$

where c depends only on n, p, α and β. Letting $i \to \infty$ yields

$$\int_{\mathbf{R}^n} \frac{|v(x) - w(x)|^p}{|x|^p} dx \leqq c \int_{\mathbf{R}^n} |\nabla w(x)|^p dx < \infty,$$

whence $v \not\equiv \infty$. The theorem is proved.

4.4. Corollary. *Let $1 < p < n$ and $E \subset \mathbf{R}^n$. Then the following are equivalent:*

(i) *E is A-polar;*
(ii) *if G is open in $\mathbf{R}^n$, then $R^{\infty}_{E \cap G}(G; A)$ has a zero in each component of G;*
(iii) *$\hat{R}^{\infty}_{E \cap G}(G; A) \equiv 0$ whenever G is open in $\mathbf{R}^n$.*

4.5. Remark. The A-superharmonic function v constructed in Theorem 4.1 belongs to $\operatorname{loc} W^1_p(\mathbf{R}^n)$. Thus v is a supersolution of (1.1) in $\mathbf{R}^n$, see [5, 3.14].

4.6. Remark. We mention the following result which is well-known in the classical theory: Let E be a closed set in $\mathbf{R}^n$. *There exist non-constant bounded A-harmonic functions in the complement of E if and only if E is not A-polar.*

Indeed, the necessity follows from Theorem 1.5 together with well-known

Liouville and removability theorems. If $p = n$, see Remark 5.4 for an explication.

To prove the converse, recall that if E is not A-polar, then there are at least two distinct regular points $x_0, x_1 \in \partial \complement E$ (this follows from the Kellogg property [4] and from Theorem 1.5, see 2.10). We may assume that $\complement E$ is connected and that $x_0, x_1 \in B(0,1)$. For each $j = 1, 2, \ldots$ choose functions $\phi_j \in C_0^\infty(B_j)$, $B_j = B(0,j)$, such that $0 \leqq \phi_j \leqq \phi_{j+1} \leqq 1$, and $\phi_j = \phi_{j+1} = i$ in a fixed neighborhood of x_i, $i = 0, 1$. Let u_j be the unique A-harmonic function in $B_j \setminus E$, $u_j - \phi_j \in W^1_{p,0}(B_j \setminus E)$. By the uniqueness, $u_j \leqq u_{j+1}$, whence $u = \lim u_j$ is A-harmonic in $\complement E$. Moreover, appealing to the strong boundary estimate [12, p. 51] yields that

$$\lim_{x \to x_i} u_j(x) = i, \quad i = 0, 1,$$

uniformly so that the function u is not constant.

Note that the above construction works for each p, $1 < p < \infty$.

5. The borderline case $p = n$.

In this section we establish the missing link (iv) $\Rightarrow$ (i) in Theorem 1.5 in the case $p = n$.

In effect, the implication (iv) $\Rightarrow$ (i) has been proved by P. Lindqvist and O. Martio if $p = n$ and if the equation (1.1) is the Euler equation of a convex variational integral, see [10, 3.13]. By appealing to the "ess lim inf-property" (2.7), we obtain this result in a more straightforward way.

5.1. Theorem. *Suppose that $w \colon \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$ is a nonnegative l.s.c. function in $W^1_n(\mathbf{R}^n)$. Then the set*

$$E = \{x \in \mathbf{R}^n : w(x) = \infty\}$$

is A-polar.

Proof. It is not difficult to verify that the Hausdorff dimension of E is zero; hence there are bounded domains $G_1, G_2, \ldots$ such that $E \subset \bigcup_{i=1}^\infty G_i$ and that $G_i \cap G_j = \emptyset$ whenever $i \neq j$, cf. [10, 3.11]. Thus, we may assume that G is a bounded, connected neighborhood of E. Let $u \in W^1_n(G)$ be the solution to the obstacle problem in G with the obstacle w. Now, by 2.13, u is A-superharmonic, and hence

$$u(x) = \operatorname*{ess\,lim\,inf}_{y \to x} u(y) \geqq \operatorname*{ess\,lim\,inf}_{y \to x} w(y) \geqq w(x)$$

for $x \in G$. In particular, $u = \infty$ in E. What is more, $u \not\equiv \infty$ in G since $u \in W_n^1(G)$. Thus E is A-polar as required.

5.2. REMARK. The above proof and Remark 4.5 yield (for each $p \in (1, \infty)$): *a set E is A-polar if and only if there is a neighborhood V of E and a l.s.c. supersolution u of* (1.1) *in V such that $u = \infty$ in E.*

Corollary 4.4 has the following counterpart when $p = n$.

5.3. COROLLARY. *Let $p = n$ and let $E \subset \mathbb{R}^n$. Then E is A-polar if and only if $\hat{R}^u_{E \cap G}(G; A) \equiv 0$ whenever G is a bounded open set and $u > 0$ is A-superharmonic in G.*

PROOF. The sufficiency follows from Theorem 1.5. To prove the converse, let E be A-polar and G a bounded open set. As in the proof of Theorem 5.1 above we construct a nonnegative A-superharmonic function v in G such that $v = \infty$ in $E \cap G$ and that the set $\{x \in G: v(x) < \infty\}$ is everywhere dense in G. This shows that $\hat{R}^\infty_{E \cap G}(G: A) \equiv 0$ in G and the claim follows since $\hat{R}^\infty_{E \cap G} \geqq \hat{R}^u_{E \cap G}$.

5.4. REMARK. Corollary 4.4 fails if $p = n$ since in this case there is no nonconstant lower bounded entire A-superharmonic function. To see this, let u be a bounded A-superharmonic function in $\mathbb{R}^n$. The estimate (2.25) in [5] yields

$$\int_{B(0,r)} |\nabla \log u|^n dx \leqq c \operatorname{cap}_n(\bar{B}(0,r), B(0,R)) = c\left(\log \frac{R}{r}\right)^{1-n}$$

for $0 < r < R$; the constant c depends only on n and β/α. Letting $R \to \infty$ establishes the desired conclusion.

Consequently, by the removability theorem [5, 4.7], we have the following generalization of a classical result in a nonlinear situation:

if $p = n$ and u is a lower bounded A-superharmonic function in a complement of a closed polar set, then u is a constant.

However, also when $p = n$ it should be possible to construct an A-superharmonic function u in $\mathbb{R}^n$, $u \not\equiv \infty$, such that $u = \infty$ in the given A-polar set E. The authors have not yet been able to do this.

5.5. QUESTION. Suppose that $E \subset \mathbb{R}^n$ is A-polar $(1 < p \leqq n)$ and that $x_0 \in \bar{E} \setminus E$. Is it true that there is an A-superharmonic function u in a neighborhood V of x_0 such that $u_{|V \cap E} = \infty$ and $u(x_0) < \infty$?

6. The fundamental convergence theorem.

Appealing to the Kellogg property [4] and to Theorem 1.5 we prove in this final section the following theorem, well-known in the classical potential theory [1, Theorem VI, 1].

6.1. THEOREM. *Let $\mathscr{F}$ be a family of A-superharmonic functions in G. Let $w = \inf \mathscr{F}$ and let $\hat{w}$ be its l.s.c. regularization,*

$$\hat{w}(x) = \liminf_{y \to x} w(y).$$

If $\mathscr{F}$ is locally uniformly lower bounded, then $\hat{w}$ is A-superharmonic and $\hat{w} = w$ except on an A-polar set.

For the proof, we require the following lemma.

6.2. LEMMA. *Let u be nonnegative and A-superharmonic in G and let $E \subset G$ be compact. Then the set*

$$S = \{x \in G : \hat{R}_E^u(x) < R_E^u(x)\}$$

is an A-polar subset of ∂E.

PROOF. It is clear that $S \subset \partial E$ and that we may assume $E = \partial E$. Then choose a regular open set $D \subset\subset G$ with $E \subset D$ and an increasing sequence of functions $\varphi_i \in C_0^\infty(D)$ with $\lim \varphi_i = u$ in E. Let h_i be the A-harmonic function in $D \setminus E$ with $h_i - \varphi_i \in W_{p,0}^1(D \setminus E)$. Then by 2.10,

$$\lim_{y \to x} h_i(y) = \varphi_i(x)$$

for each $x \in E \setminus T$, where

$$T = \left\{x \in E : \int_0^1 \left(\frac{\operatorname{cap}_p(\bar{B}(x,t) \cap E, B(x,2t))}{\operatorname{cap}_p(\bar{B}(x,t), B(x,2t))}\right)^{\frac{1}{p-1}} \frac{dt}{t} < \infty\right\}$$

By the Kellogg property [4, Theorem 2] and Theorem 1.5, T is A-polar. Thus it suffices to show that $S \subset T$.

To this end, note first that

$$R_E^u \geqq h_i$$

in $D \setminus E$. Indeed, let $v \in \Phi_E^u$ and write $v' = \min(v, \max h_i)$. Then $v' \in W_p^1(D)$ by Proposition 2.8, and since for each $\varepsilon > 0$,

$$\min(v' + \varepsilon, h_i) - \varphi_i \in W_{p,0}^1(D \setminus E)$$

implies $v'+\varepsilon \geqq h_i$ in $D\setminus E$ by [5, 2.8 and 3.15], we infer that $R_E^u \geqq h_i$ in $D\setminus E$.

Thus, we obtain for $x\in E\setminus T$

$$\begin{aligned}\hat{R}_E^u(x) &= \sup_{r>0}\ \inf_{B(x,r)} R_E^u \\ &= \min\left(\liminf_{y\to x,\, y\in D\setminus E} R_E^u(y), u(x)\right) \\ &\geqq \min\left(\lim_{y\to x,\, y\in D\setminus E}, u(x)\right) = \phi_i(x).\end{aligned}$$

Letting $i\to\infty$ yields $\hat{R}_E^u(x) \geqq u(x) = R_E^u(x)$ whenever $x\in E\setminus T$. Thus $S\subset T$ as desired.

Proof of Theorem 6.1. It is clear that $\hat{w}$ is A-superharmonic. Next, we may assume that $\mathcal{F}$ is downward directed, cf. [1, p. 37], and hence by applying Choquet's topological lemma as in Proposition 3.1 we may even assume that $\mathcal{F}$ consists of a decreasing sequence of A-superharmonic functions w_i with $w=\lim w_i$.

Now for each positive integer j write

$$S_j = \{x\in G : \hat{w}(x)+1/j < w(x)\}.$$

By Corollary 1.7, it suffices to show that S_j is A-polar for each j. For that, fix j and let $C\subset S_j$ be compact. Since S_j is a Borel set and since Borel sets are capacitable, it suffices to verify that C is A-polar. To this end, let $D\subset\subset G$ be an open neighborhood of C. If $v=\hat{w}+1/j$, then each w_i belongs to Φ_C^v in D (we may clearly assume that $\hat{w}$ is nonnegative), whence $\hat{R}_C^v(D;A)\leqq \hat{w}$ in D. Thus $\hat{R}_C^v(D;A) < \hat{w}+1/j = v$ in C and hence, by Lemma 6.2, C is A-polar as required.

Added in October 1988: It was later proved by the second author, that if E is A-polar $(1<p\leqslant n)$ and $x_0\notin E$, then there is an A-superharmonic function u in R^n such that $u|_E=\infty$ and $u(x_0)<\infty$.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF JYVÄSKYLÄ
SEMINAARINKATU 15
SF-40100 JYVÄSKYLÄ
FINLAND

INSTITUT FÜR ANGEWANDTE MATHEMATIK
DER UNIVERSITÄT BONN
W. GERMANY