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SETS OF MINIMAL POINTS IN $L_p([0, 1], dt)$

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0. Introduction.

The notion minimal point with respect to a subset M of a Banach space X was introduced in [1] and further studied in [2] and [3]. In this paper we study the geometric properties of this notion in L_p spaces. We start by giving some definitions and basic facts. Let X be a Banach space and assume that $M \subseteqq X$ is a subset.

Definition 0.1. We call $x \in X$ minimal with respect to M if

$$\|y - m\| \leqq \|x - m\|, \quad \forall m \in M \Rightarrow y = x.$$

The set of all minimal points with respect to M is denoted by min M.

Definition 0.2. A subset $M \subseteqq X$ is said to be optimal if $\min M = M$.

As a starting point we have the following two results from [2].

Theorem 0.1. *If X is reflexive, smooth and strictly convex and of dimension larger than two, then X is isometric to a Hilbert space if and only if the closed unit ball is optimal.*

Proposition 0.1. *There exists a function*

$$\varrho : [1, \infty] \to [1, 2]$$

such that

i) $\varrho(p)B_p^0 \subseteqq \min B_p \subseteqq \varrho(p)B_p$ *and*
ii) $\min B_1 = 2B_1^0$, *where B_p denotes the closed unit ball in $L_p([0, 1], dt)$.*

It is easy to see that B_∞ is optimal. For if $f \in L_\infty([0, 1], dt)$ and $\|f\| > 1$, then

$$\|g - m\| \leqq \|f - m\|, \quad \forall m \in B_\infty,$$

where $g(x) = \text{sgn}(f(x))\min(|f(x)|, 1)$. Hence $f \notin \min B_\infty$.

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This paper deals with the structure of $\min B_p$ and thus with the function ϱ. Section 1 contains some preliminary topological results which in particular show that $\min B_p = \varrho(p)B_p$, that is $\min B_p$ is closed, if $1 < p < \infty$. In Theorem 2.1 of section 2 we reduce the problem of determining $\varrho(p)$ to a finite dimensional problem. From this reduction we are able, in Theorem 3.1, to obtain estimates of ϱ and, in Theorem 3.2, to show that $\varrho: [1, \infty[\to [1, 2]$ is continuous. The estimates we get show however that ϱ is discontinuous at ∞, since

$$\varrho(\infty) = 1 \neq 2 = \lim_{p \to \infty} \varrho(p).$$

Finally, in section 4, we apply a theorem of Kakutani to get a characterization of Hilbert space in terms of the notion optimal set.

We mention a few basic properties of the notion minimal point.

$$\min(\lambda M) = \lambda \min M \quad \text{if} \quad \lambda \in \mathsf{R}_+.$$

If M is a subset of a ball of radius r, then $\min M$ is a subset of a ball of radius $2r$. And if X is a Hilbert space, then $\min M = \overline{\operatorname{cvx} M}$. Thus the operation of taking minimal points generalizes the one of taking the closed convex hull in Hilbert space. In general, however, there is only little knowledge of the geometrical properties of the set of all minimal points. For example it is remarked in [2] that it seems unknown if $\min M$ is convex whenever M is.

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1. Some topological properties.

If $(X, \|\cdot\|)$ is a Banach space, we denote the unit sphere $\{x \in X\,;\, \|x\| = 1\}$ by $S(X)$. Let $M \subseteqq X$ be a subset. We will show that under some boundedness assumptions on M and some convexity assumptions on the norm, $\min M$ is closed.

DEFINITION 1.1. The Banach space X is said to be strictly convex if

$$\|x+y\| = \|x\| + \|y\|, \quad x,y \in X,$$

implies that $x = ty$ for some $t \geqq 0$ or $y = 0$.

This is the same as to say that $S(X)$ does not contain any nontrivial line segments.

DEFINITION 1.2. The Banach space X is said to be uniformly convex whenever given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $x,y \in S(X)$ and $\|x-y\| \geqq \varepsilon$, then $\|x+y\| \leqq 2(1-\delta(\varepsilon))$.

We have the following easy consequence.

FACT. Let X be a uniformly convex Banach space. Then given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that whenever $\|y\| \leqq \|x\| = 1$ and $\|x-y\| \geqq \varepsilon$, then $\|x+y\| \leqq 2(1-\delta(\varepsilon))$.

PROPOSITION 1.1. *Let X be a strictly convex Banach space and let $M \subseteqq X$ be a compact subset. Then* $\min M$ *is closed.*

PROOF. Given $x \notin \min M$ there exists $y \neq x$ with $\|y-m\| \leqq \|x-m\|$, $\forall m \in M$. By strict convexity $\|(y+x)/2-m\| < \|x-m\|$. For each $m \in M$, put $a(m) = \|x-m\| - \|(y+x)/2-m\| > 0$. Each set

$$V_m = \{z \in X\,; \|x-z\| - \|(y+x)/2-z\| > a(m)/2\},$$

defined for $m \in M$, contains an open neighborhood of m. By compactness the covering $\{V_m\}_{m\in M}$ of M has a finite subcovering, say $\{V_{m_i}\}_{i=1}^N$. Put

$$\varepsilon = \min_{1 \leq i \leq N} \{a(m_i)/2\}.$$

Then

$$\|x-m\| - \|(y+x)/2-m\| > \varepsilon, \quad \forall m \in M.$$

Clearly the set

$$\{x' \in X\,; \|x'-m\| \geqq \|(y+x)/2-m\|, \quad \forall m \in M\}$$

contains an open neighborhood of x, which is disjoint from $\min M$. Hence $\min M$ is closed.

In case of uniform convexity it is sufficient that M is bounded to conclude that $\min M$ is closed. We use the notation $d(x, M)$ for the distance of a point x to a set M, that is

$$d(x, M) = \inf_{m\in M} \|x-m\|.$$

PROPOSITION 1.2. *Let X be a uniformly convex Banach space and let $\delta(\varepsilon)$ be its modulus of convexity. Suppose that $M \subseteqq X$ is a bounded subset and that $x \notin \min M$. Then*

$$B^0\left(x, \delta\left(\frac{\|y'-x\|}{\sup_{m\in M}\|x-m\|}\right)d(x, M)\right) \cap \min M = \emptyset, \tag{1.1}$$

for every $y' \in \{y \in X\,; \|y-m\| \leqq \|x-m\|, \forall m \in M\}$. In particular, $\min M$ *is closed.*

PROOF. Given $x \notin \min M$ there exists $y \neq x$ with $\|y-m\| \leqq \|x-m\|$, $\forall m \in M$. Fix $m_0 \in M$. The 'Fact' applied to the elements $(x-m_0)/\|x-m_0\|$,

$(y - m_0)/\|x - m_0\|$, and $\varepsilon = \varepsilon_{m_0} = \|y - x\|/\|x - m_0\|$ yields

$$\|(y + x)/2 - m_0\| \leqq (1 - \delta(\varepsilon_{m_0}))\|x - m_0\|.$$

If we set $y' = (y+x)/2$, then

$$\|y' - m\| \leqq \|x - m\| - \delta\left(\frac{\|y-x\|}{\sup\limits_{m \in M} \|x-m\|}\right) d(x, M), \quad \forall m \in M.$$

The proof is now easily completed.

COROLLARY 1.1. *If* $1 < p < \infty$, *then* $\min B_p = \varrho(p) B_p$.

PROOF. Combine Propositions 0.1 and 1.2.

Given a function f outside $\min B_p$, $1 < p < \infty$. Proposition 1.2 gives some information about the closed, convex set

$$U_f = \{g \in L_p([0,1], dt); \|g - m\| \leqq \|f - m\|, \forall m \in B_p\}$$

of all functions that are closer than f to each function in B_p. More precisely we have

COROLLARY 1.2. *There exists a constant* $k_p > 0$, $1 < p < \infty$, *depending only on p, such that if* $f \notin \min B_p$, *then* $U_f \subseteqq B(f, r)$ *where*

$$r = (1 + \|f\|)\left(\frac{\|f\| - \varrho(p)}{k_p(\|f\| - 1)}\right)^{1/(p)} \quad \text{and} \quad (p) = \max(2, p).$$

PROOF. Let $\delta_p(\varepsilon)$ be the modulus of convexity of $L_p([0,1], dt)$, $1 < p < \infty$, and assume that $g \in U_f$. By (1.1) we have

$$\|f\| - \varrho(p) \geqq \delta_p\left(\frac{\|g - f\|}{\sup\limits_{m \in B_p} \|f - m\|}\right) d(f, B_p) \geqq \delta_p\left(\frac{\|g - f\|}{(1 + \|f\|)}\right)(\|f\| - 1).$$

Since there exists, cf. [4], $k_p > 0$ such that $\delta_p(\varepsilon) \geqq k_p \varepsilon^{(p)}$ we get

$$\|f\| - \varrho(p) \geqq k_p(\|g - f\|/(1 + \|f\|))^{(p)}(\|f\| - 1)$$

or equivalently

$$\|g - f\| \leqq (1 + \|f\|)((\|f\| - \varrho(p))/k_p(\|f\| - 1))^{1/(p)}, \quad \forall g \in U_f.$$

PROPOSITIONS 1.1 and 1.2 are related to the following results of [2].

PROPOSITION 1.3. *Let X be a reflexive and strictly convex Banach space. If* $M \subseteqq X$ *is a compact subset, then* $\min M$ *is weakly compact (and hence closed).*

PROPOSITION 1.4. *Let X be a reflexive and locally uniformly convex Banach space. If $M \subseteqq X$ is a compact subset, then* $\min M$ *is compact.*

2. Minimal points with respect to B_p.

In this section we will study the function $\varrho: [1,\infty] \to [1,2]$ which was introduced in Proposition 0.1. By Theorem 0.1, Proposition 0.1, and Corollary 1.1, $\min B_p = \varrho(p)B_p$, $1 < p \neq 2 < \infty$, for some $\varrho(p) > 1$, while $\varrho(1) = 2$ and $\varrho(2) = \varrho(\infty) = 1$. Below we obtain some expressions for $\varrho(p)$, which in the next section will be applied to get estimates of $\varrho(p)$ and to prove that ϱ is continuous on $[1, \infty[$. It turns out that the simple functions which only take two, respectively three, values will be of special interest. We therefore introduce the notation $m = (f_1, f_2, f_3; \lambda_1, \lambda_2, \lambda_3)$ for the (right continuous) decreasing simple function that takes the values f_1, f_2 and, f_3 on intervals of lengths λ_1, λ_2, and λ_3, respectively, where $\sum_{i=1}^3 \lambda_i = 1$. The function $(f_1, f_2, f_3; \lambda_1, \lambda_2, 0)$ will be denoted by $(f_1, f_2; \lambda_1, \lambda_2)$.

The main effort will be to prove

THEOREM 2.1.

(i) *If $1 < p < \infty$, then*
$$\varrho(p) = \sup\{a \in \mathsf{R} : \exists m = (f_1, f_2; \lambda_1, \lambda_2) \in S(L_p) \text{ with } \sum_{i=1}^2 \lambda_i (f_i - a|f_i - a|^{p-2} \geqq 0\}.$$

(ii) *If $2 < p < \infty$, then*
$$\begin{aligned}\varrho(p) &= \sup\{a \in \mathsf{R} : \exists m = (f_1, f_2; \lambda_1, \lambda_2) \in S(L_p) \text{ with } \\ &\qquad |f_i - a|^{p-2} = \textstyle\int_0^1 |m(t) - a|^{p-2} m(t)dt \cdot f_i^{p-1},\ i = q, 2, \text{ and } \\ &\qquad \textstyle\sum_{i=1}^2 \lambda_i (f_i - a)|f_i - a|^{p-2} \geqq 0\} \\ &= \sup\{a \in \mathsf{R} : \exists m = (f_1, f_2; \lambda_1, \lambda_2) \in S(L_p) \text{ with } \\ &\qquad |f_i - a|^{p-2} = \textstyle\int_0^1 |m(t) - a|^{p-2} m(t)dt \cdot f_i^{p-1},\ i = 1, 2, \text{ and } \\ &\qquad a\|m\|_{p-1}^{p-1} \leqq 1\}.\end{aligned}$$

THEOREM 2.1 easily implies

THEOREM 2.2. *If $1 < p < \infty$, then*

$$\varrho(p) = \sup_{\substack{0 \leqq x \leqq 2^{p/(p-1)}(a-1)^{-p} \\ 0 \leqq y \leqq 1}} \left\{ \frac{y(1 + x^{p-1})^{1/p} + x(1 - y^{p-1})^{1/p}}{x + y} \right\}$$

PROOF OF THEOREM 2.2 ASSUMING THEOREM 2.1 (i). Let $(f_1, f_2; \lambda_1, \lambda_2) \in S(L_p)$, where $f_1 > f_2$. Then $\lambda_1 = (1 - f_2^p)/(f_1^p - f_2^p)$ and $\lambda_2 = (f_1^p - 1)/(f_1^p - f_2^p)$.

By inserting this one easily checks that

$$\sum_{i=1}^{2} \lambda_i(f_i-a)|f_i-a|^{p-2} \geqq 0 \quad \text{if and only if}$$

$$a \leqq (f_1+f_2A(f_1,f_2))/(1+A(f_1,f_2)), \quad \text{where } A(f_1,f_2) = \left(\frac{f_1^p-1}{1-f_2^p}\right)^{1/(p-1)}$$

Hence, by Theorem 2.1 (i),

$$\varrho(p) = \sup_{\substack{1<f_1 \\ 0\leqq f_2<1}} \{(f_1+f_2A(f_1,f_2))/(1+A(f_1,f_2))\}. \tag{2.1}$$

In the supremum it is, however, sufficient to consider f_1's with $f_1 < 2(a-1)^{1-p}$ To see this, assume that

$$\lambda_1(f_1-a)^{p-1} \geqq \lambda_2(a-f_2)^{p-1}, \quad \text{where } f_1 > 2.$$

Then

$$\tfrac{1}{2}(a-1)^{p-1} \leqq (1-f_1^{-p})(a-1)^{p-1} \leqq \lambda_2(a-f_2)^{p-1} \leqq \lambda_1(f_1-a)^{p-1} \leqq$$
$$\leqq f_1^{-p}f_1^{p-1} = f_1^{-1},$$

which shows that $f_1 < 2(a-1)^{1-p}$. The substitution $x = (f_1^p-1)^{1/(p-1)}$ $y = (1-f_2^p)^{1/(p-1)}$ now completes the proof of Theorem 2.2.

Let

$$A = \left\{a\in \mathbf{R}_+ : \sup_{m\in B_p} \{\|g-m\|-\|a-m\|\} > 0, \ \forall a \not\equiv g \in L_p\right\}$$

and

$$A' = \left\{a\in \mathbf{R}_+ : \sup_{m\in B_p} \{\|c-m\|-\|a-m\|\} > 0, \ \forall 0 \leqq c < a\right\}.$$

By Proposition 0.1

$$\varrho(p) = \sup\{a\,;a\in A\}. \tag{2.2}$$

Obviously $A' \supseteqq A$. We will, however, prove the reverse inequality, sot that in fact $A = A'$. Note that

$$A' = \left\{a\in \mathbf{R}_+ : \sup_{m\in S(L_p)} \{\|c-m\|^p-\|a-m\|^p\} > 0, \ \forall 0 \leqq c < a\right\}.$$

Given real numbers a and c with $0 \leqq c < a$, the functional $I_{a,c}: L_p \to \mathbf{R}$ defined by

$$I_{a,c}(m) = \|c-m\|^p-\|a-m\|^p,$$

will be of interest.

As a first step we have

LEMMA 2.1. *Let $1 < p < \infty$ and $0 \leqq c < a \leqq 2$ be given. Then $I_{a,c}: S(L_p) \to \mathsf{R}$ attains its supremum.*

PROOF. Let $s = \sup_{m \in S(L_p)} I_{a,c}(m)$. Assume that $(f_n)_{n=1}^{\infty} \subseteqq S(L_p)$ is a maximizing sequence, i.e. $\lim_{n\to\infty} I_{a,c}(f_n) = s$. The idea of the proof is to show that there exists a subsequence $(f_{n_k})_{k=1}^{\infty}$ which converges pointwise a.e. to an element $f \in S(L_p)$, and that $I_{a,c}$ attains its supremum at f.

We may assume that each f_n is positive and decreasing, so that $f_n(x) \leqq x^{-1/p}$. Let $(r_k)_{k=1}^{\infty}$ be an enumeration of $\mathsf{Q} \cap [0,1]$. By a diagonal procedure we can extract a subsequence of (f_n) which converges pointwise on $\mathsf{Q} \cap [0,1]$. For simplicity in notation, assume that (f_n) already has this property, i.e. assume that $\lim_{n\to\infty} f_n(r_k) = f_k$, $k = 1, 2, 3, \ldots$. Define a function $f_0: [0,1] \to \mathsf{R}$ by letting $f_0(r_k) = f_k$ for $k = 1, 2, 3, \ldots$. If $x \in [0,1] \setminus \mathsf{Q}$ we let $f_0(x) = \lim_{k\to\infty} f_0(q_k)$, where $(q_k)_{k=1}^{\infty}$ is any decreasing sequence in $\mathsf{Q} \cap [0,1]$ with $\lim_{k\to\infty} q_k = x$.

Let $x \in [0,1] \setminus \mathsf{Q}$. If $(s_j)_{j=1}^{\infty}$ is an increasing sequence and $(t_j)_{j=1}^{\infty}$ is a decreasing sequence in $\mathsf{Q} \cap [0,1]$, both convergent to x, then for every j

$$f_0(t_j) = \lim_{n\to\infty} f_n(t_j) \leqq \varliminf_{n\to\infty} f_n(x) \leqq \varlimsup_{n\to\infty} f_n(x) \leqq \lim_{n\to\infty} f_n(s_j) = f_0(s_j). \tag{2.3}$$

f_0 is positive, decreasing and, for each $\delta > 0$, bounded on $\mathsf{Q} \cap [\delta, 1]$. Hence f_0 can have at most countable many discontinuity points.

Consequently

$$\lim_{j\to\infty} f_0(t_j) = \lim_{j\to\infty} f_0(s_j) = f_0(x) \quad \text{for almost all } x.$$

By (2.3), $\lim_{n\to\infty} f_n(x) = f_0(x)$ a.e..

Fatou's lemma implies that $\|f_0\| \leqq 1$. Thus each maximizing sequence in $S(L_p)$ has a subsequence which is almost everywhere pointwise convergent to a function in B_p. We will show that, actually, $f_0 \in S(L^p)$ and that the supremum is attained at f_0.

Choose $0 < \varepsilon < 1$. The contribution to $I_{a,c}$ on $[0,\varepsilon]$ will be shown to be $0(\varepsilon)$ as $\varepsilon \to 0$. For simplicity, we suppress the dependence on a and c and let I_A denote the restriction of $I_{a,c}$ to the measurable subset $A \subseteqq [0,1]$, i.e.

$$I_A(f) = \int_A (|c - f(x)|^p - |a - f(x)|^p)dx, \quad f \in B_p.$$

Let $f \in B_p$ be positive and set

$$A_1 = \{x \in [0, \varepsilon]\,;\, f(x) > a+1\}$$

$$A_2 = \{x \in [0, \varepsilon]\,;\, (a+c)/2 < f(x) \leqq a+1\}$$

$$A_3 = \{x \in [0, \varepsilon]\,;\, f(x) \leqq (a+c)/2\}.$$

Then

$$|I_{A_2}(f)| \leqq \int_{A_2} |c-f(x)|^p dx \leqq (a+1-c)^p \varepsilon \leqq 3^p \varepsilon,$$

$$|I_{A_3}(f)| \leqq \int_{A_3} |a-f(x)|^p dx \leqq a^p \varepsilon \leqq 2^p \varepsilon$$

and

$$|I_{A_1}(f)| \leqq \int_{A_1} p(a-c)(f(x)-c)^{p-1} dx$$

$$\leqq 2p \int_{A_1} f(x)^{p-1} dx \leqq \text{(Hölder's inequality)}$$

$$\leqq 2p\left(\int_{A_1} f(x)^p dx\right)^{(p-\frac{1}{2})/p} \left(\int_{A_1} f(x)^{-p} dx\right)^{1/2p}$$

$$\leqq 2p(a+1)^{-1/2}\varepsilon^{1/2p} \leqq 2p\varepsilon^{1/2p}.$$

Thus

$$(2.4) \qquad |I_{[0,\varepsilon]}(f)| \leqq (3^p+2^p)\varepsilon + 2p\varepsilon^{1/2p}.$$

The proof is now easily completed. Choose n_1 so large that

$$(2.5) \qquad s - I_{a,c}(f_n) < \varepsilon \quad \text{if } n \geqq n_1$$

By the dominated convergence theorem, we can choose $n_2 \geqq n_1$ so large that

$$(2.6) \qquad |I_{[\varepsilon,1]}(f_n) - I_{[\varepsilon,1]}(f_0)| < \varepsilon \quad \text{if } n \geqq n_2.$$

For $n \geqq n_2$ we get

$$s - I_{[0,1]}(f_0) = s - I_{[0,1]}(f_n) + I_{[0,\varepsilon]}(f_n) - I_{[0,\varepsilon]}(f) + I_{[\varepsilon,1]}(f_n) - I_{[\varepsilon,1]}(f)$$

$$\leqq (3^p+2^p)\varepsilon + 2p\varepsilon^{1/2p} + 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary,

$$\sup_{m\in S(L_p)} I_{a,c}(m) = I_{a,c}(f_0).$$

By the definition of $I_{a,c}$, we obviously must have $\|f_0\| = 1$.

We derive a necessary condition in

LEMMA 2.2. *Let $1 < p < \infty$ and $0 \leqq c < a \leqq 2$ be given. If $I_{a,c}: S(L_p) \to \mathrm{R}$ attains its supremum at f, then f satisfies the equation*

$$(2.7)\quad |c-f(x)|^{p-1}\operatorname{sgn}(c-f(x))-|a-f(x)|^{p-1}\operatorname{sgn}(a-f(x))$$
$$= f(x)^{p-1}\int_0^1 (|c-f(t)|^{p-1}\operatorname{sgn}(c-f(t))-|a-f(t)|^{p-1}\operatorname{sgn}(a-f(t)))f(t)dt$$

almost everywhere.

The proof uses a lemma, cf. [5], in which we use the notation $E(p)$ for the integer part of p.

LEMMA. *Let $1 < p < \infty$. There exists a constant M_p, depending only on p such that*

$$(2.8)\qquad \left| |a+b|^p - \left(|a|^p + p|a|^{p-1}\operatorname{sgn}(a)b + \sum_{i=2}^{E(p)} \binom{p}{i} |a|^{p-1}\operatorname{sgn}(a)^i b^i\right)\right|$$
$$\leqq M_p|b|^p, \quad \forall a,b\in \mathrm{R}.$$

PROOF OF LEMMA 2.2. Suppose that $I_{a,c}: S(L_p) \to \mathrm{R}$ attains its supremum at f. A necessary condition is

$$\frac{d}{dt} I_{a,c}((f+tg)/\|f+tg\|)|_{t=0} = 0, \quad \forall g\in L_p,$$

provided that the derivative exists. To determine this condition, let $g \in S(L_p)$ be arbitrary. Integration in (2.8) and in the case $p > 2$, use the Hölder's inequality give (note that $f(x) \geqq 0$ a.e.)

$$\|f+tg\|^p = 1+tp\int_0^1 f(x)^{p-1}g(x)dx + O(t^\gamma), \quad \gamma = \min(p,2).$$

Putting $v = \int_0^1 f(x)^{p-1}g(x)dx$ we get

$$\frac{1}{c}(f+tg)/\|f+tg\| = \frac{1}{c}(f+tg)(1+tv+O(t^\gamma))^{-1} = \frac{1}{c}(f+tg)(1-tv+O(t^\gamma))$$

or

$$1-\frac{1}{c}(f+tg)/\|f+tg\| = (1-f/c)-(tg/c-tvf/c+O(t^{\gamma})). \tag{2.9}$$

Thus

$$\int_0^1 (|c-(f+tg)/\|f+tg\||^p-|c-f|^p)dx$$

$$= \int_0^1 c^p(|1-(1/c)(f+tg)/\|f+tg\||^p-|1-f/c|^p)dx = (2.8)-(2.9)$$

$$= \int_0^1 c^p p|1-f/c|^{p-1}\operatorname{sgn}(1-f/c)[-tg/c+tvf/c+O(t^{\gamma})]dx+$$

$$+\int_0^1 c^p O(|-tg/c+tvf/c+O(t^{\gamma})|^p)dx$$

and hence

$$\lim_{t\to 0}(\|c-(f+tg)/\|f+tg\|\|^p-\|c-f\|^p)/t$$

$$= p\int_0^1 c^p|1-f/c|^{p-1}\operatorname{sgn}(1-f/c)(-g/c+vf/c)dx$$

$$= p\int_0^1 |c-f|^{p-1}\operatorname{sgn}(c-f)(-g+vf)dx.$$

The condition $(d/dt)I_{a,c}((f+tg)/\|f+tg\|)|_{t=0} = 0$ implies that

$$\int_0^1 \{|c-f|^{p-1}\operatorname{sgn}(c-f)-|a-f|^{p-1}\operatorname{sgn}(a-f)\}$$

$$\left\{-g+f\int_0^1 f^{p-1}g\,dt\right\}dx = 0, \quad \forall g\in L_p.$$

We thus arrive at the necessary condition

$$|c-f(x)|^{p-1}\operatorname{sgn}(c-f(x))-|a-f(x)|^{p-1}\operatorname{sgn}(a-f(x))$$
$$=f(x)^{p-1}\int_0^1 (|c-f(t)|^{p-1}\operatorname{sgn}(c-f(t))-|a-f(t)|^{p-1}\operatorname{sgn}(a-f(t)))f(t)dt$$

almost everywhere. This is equation (2.7).

REMARK. The values $f(x)$ must, for almost all x, solve the equation

$$|c-y|^{p-1}\operatorname{sgn}(c-y)-|a-y|^{p-1}\operatorname{sgn}(a-y) = y^{p-1}\lambda \tag{2.10}$$

for some negative number λ. We observe that given a, c, and λ there are at most three solutions to (2.10).

The next step will be to show that if $a\in A'$, then there exists $m\in S(L_p)$, with $I_{a,c}(m)>0$ simultaneously for all $c\in[0,a[$. Let $a\in A'$ and choose an increasing sequence $(c_k)_{k=1}^{\infty}$ with $\lim_{k\to\infty}c_k = a$. Let $(m_k)_{k=1}^{\infty}\subseteqq S(L_p)$ be a sequence of decreasing simple functions satisfying (2.7) with $c=c_k$, respectively, and $I_{a,c_k}(m_k)>0$. Since $I_{a,a}(m_k)=0$ and $I_{a,c}$ is convex in c we have that $I_{a,c}(m_k)>0$ if $0\leqq c\leqq c_k$. As in the proof of Lemma 2.1 we may assume that $m_k(x)\to m(x)$ a.e. (note that the class of decreasing simple functions taking at most three values and of norm less than or equal to one is closed under pointwise limits, thus m is of this kind, too) where $\|m\|\leqq 1$ and $I_{a,c}(m)\geqq 0$ for every $0\leqq c<a$. If $\|m\|<1$ we can increase $I_{a,c}(m)$ further. Thus we can assume that $\|m\|=1$. It follows that

$$\frac{\partial}{\partial c} I_{a,c}(m)|_{c=a}\leqq 0. \tag{2.11}$$

Conversely, if (2.11) holds for some $m\in S(L_p)$, then $I_{a,c}(m)>0$ if $0\leqq c<a$, since $S(L_p)$ does not contain any line segments. We have proven

LEMMA 2.3. *Let $1<p<\infty$ and $0<a\leqq 2$ be given. Then*

$$\sup_{m\in S(L_p)} I_{a,c}(m)>0 \ \forall 0\leqq c<a \quad \textit{if and only if}$$

$$\exists m=(f_1,f_2,f_3;\lambda_1,\lambda_2,\lambda_3)\in S(L_p) \ \textit{with} \ \frac{\partial}{\partial c}I_{a,c}(m)|_{c=a}\leqq 0.$$

Let us now justify our interest in the functional $I_{a,c}$ by showing, as promised in the beginning of this section, that $A'\subseteqq A$.

LEMMA 2.4. *Let $1<p<\infty$ and $a\in \mathbf{R}_+$ be given. Then*

$$\sup_{m \in S(L_p)} \{\|c-m\| - \|a-m\|\} > 0 \quad \forall 0 \leqq c < a \quad \mathit{iff}$$

$$\sup_{m \in S(L_p)} \{\|g-m\| - \|a-m\|\} > 0 \quad \forall a \not\equiv g \in L_p.$$

PROOF. The sufficiency is trivial. To prove the necessity we first consider a decreasing function $g \in L_p$ with $0 \leqq g(x) \leqq a$, $x \in [0,1]$. By Lemma 2.3 there exists a positive, decreasing, simple function $m_0 = (f_1, f_2, f_3; \lambda_1, \lambda_2, \lambda_3)$ such that $I_{a,c}(m_0) > 0$ for each $0 \leqq c < a$.

Let $(g_k)_{k=1}^{\infty}$ be a pointwise decreasing sequence of decreasing simple functions with

$$\lim_{k \to \infty} g_k(x) = g(x), \quad x \in [0,1].$$

Suppose that g_k takes the values c_j^k on intervals I_j^k, $1 \leqq j \leqq N_k$. Let $(M_v^k)_{v=1}^{n_k}$ be an enumeration of all intersections of the form $\bigcap_{i=1}^k I_{j_i}^i$ where $1 \leqq j_i \leqq N_i$. Suppose that m_k is any rearrangment of m_0 such that, for all v, m_k takes the values f_1, f_2, and f_3 on subintervals of M_v^k of lengths $\lambda_1|M_v^k|$, $\lambda_2|M_v^k|$, and $\lambda_3|M_v^k|$, respectively. (Here $|\cdot|$ denotes Lebesque measure.) We then use the following properties

(i) $(g_k(x))_{k=1}^{\infty}$ is decreasing for every $x \in [0,1]$,
(ii) $\|(g_k - m_k)|_{M_v^k}\|^p - \|(a-m_k)|_{M_v^k}\|^p > 0$, $k = 1,2,3,\ldots$,
(iii) $\lambda \to \|(\lambda - m)|_{M_v^k}\|^p - \|(a-m)|_{M_v^k}\|^p$ is convex,
(iv) $(M_v^{k+1})_{v=1}^{n_{k+1}}$ is a refinment of $(M_v^k)_{v=1}^{n_k}$,

to conclude that

$$\|g_k - m_k\|^p - \|a-m_k\|^p \underset{\text{(iv)}}{=} \|g_k - m_{k+1}\|^p - \|a - m_{k+1}\|^p$$

$$\underset{\text{(i-iii)}}{\leqq} \|g_{k+1} - m_{k+1}\|^p - \|a - m_{k+1}\|^p.$$

But

$$\|a - m_k\| = \|a - m_{k+1}\|,$$

since m_k is just a rearrangement of m_0. Therefore $(\|g_k - m_k\| - \|a-m_k\|)_{k=1}^{\infty}$ is increasing. This sequence is also positive and bounded. Put

$$\lim_{k \to \infty} \|g_k - m_k\| - \|a - m_k\| = \sigma > 0.$$

We get

$$|(\|g-m_k\| - \|a-m_k\|) - (\|g_k - m_k\| - \|a-m_k\|)|$$

$$= |\|g-m_k\| - \|g_k - m_k\|| \leqq \|g - g_k\| \to 0 \quad \text{as } k \to \infty.$$

Hence

$$\sup_{m \in S(L_p)} \{\|g-m\| - \|a-m\|\} \geqq \sigma > 0.$$

Clearly

$$\sup_{m \in S(L_p)} \{\|g-m\| - \|a-m\|\} > 0$$

for every positive function $g \in L_p$. If g is an arbitrary function in L_p we can choose $m \in S(L_p)$ such that $m(x) \geqq 0$ a.e. and

$$\|g - m\| - \|a - m\| \geqq \||g| - m\| - \|a - m\| > 0.$$

We can now give the

Proof of Theorem 2.1.: (i) Calculation of $(\partial/\partial c) I_{a,c}(m)|_{c=a}$ for a function $m = (f_1, f_2, f_3; \lambda_1, \lambda_2, \lambda_3)$ and combination of (2.2). Lemma 2.3, and Lemma 2.4 proves that

$$\varrho(p) = \sup\{a \in \mathbb{R} : \exists m = (f_1, f_2, f_3; \lambda_1, \lambda_2, \lambda_3) \in S(L_p) \text{ with } \sum_{i=1}^{3} \lambda_i (f_i - a)|f_i - a|^{p-2} \geqq 0\}.$$

It remains to show that it is sufficient, in the supremum, to consider step functions that takes only two values. Assume that $(f_1, f_2, f_3; l_1, l_2, l_3) \in S(L_p)$ satisfies

$$\sum_{i=1}^{3} l_i (f_i - a)|f_i - a|^{p-2} \geqq 0,$$

and consider the function

$$F(\lambda_1, \lambda_2, \lambda_3) = \sum_{i=1}^{3} \lambda_i (f_i - a)|f_i - a|^{p-2}$$

on $D = \{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3_+ : \sum_{i=1}^{3} \lambda_i = 1$ and $\sum_{i=1}^{1} \lambda_i f_i^p = 1\}$. By convexity and since $F(l_1, l_2, l_3) \geqq 0$ we must have $F(\lambda_1^0, \lambda_2^0, \lambda_3^0) \geqq F(l_1, l_2, l_3) \geqq 0$ for some point $(\lambda_1^0, \lambda_2^0, \lambda_3^0)$ on the boundary of D. We conclude that precisely one of the λ_i^0's equals zero. Hence

$$\sum_{i=1}^{2} \lambda_i (f_i - a)|f_i - a|^{p-2} \geqq 0 \quad \text{for some } (f_1, f_2; \lambda_1, \lambda_2) \in S(L_p).$$

This proves (i).

We turn to the proof of (ii). Let $m = (f_1, f_2; \lambda_1, \lambda_2) \in S(L_p)$. Then

$$-\frac{1}{p} \frac{\partial}{\partial c} I_{a,c}(m) = \int_0^1 (m(x) - a)|m(x) - a|^{p-2} dx.$$

We define a functional $F : S(L_p) \to \mathsf{R}$ by

$$F(m) = \int_0^1 (m(x)-a)|m(x)-a|^{p-2}dx.$$

To complete the proof it is sufficient to show that

(a) F attains its suprema at some function $m_0 = (f_1, f_2; \lambda_1, \lambda_2)$,
(b) if F attains its supremum at m_0, then

$$|m(x)-a|^{p-2} = \int_0^1 |m(t)-a|^{p-2}m(t)dt \cdot m(x)^{p-1} \quad \text{a.e.,}$$

and finally,
(c) if $m = (f_1, f_2; \lambda_1, \lambda_2) \in S(L_p)$ satisfies (b), then

$$\sum_{i=1}^{2} \lambda_i(f_i-a)|f_i-a|^{p-2} \geqq 0 \quad \text{iff} \quad a\|m\|_{p-1}^{p-1} \leqq 1.$$

That F attains its supremum (at some function) is proven in the same way as in the proof of Lemma 2.1. Assume it is attained at $m \in S(L_p)$, and let $g \in S(L_p)$ be arbitrary. Clearly $m(x) \geqq 0$ a.e.. We have

$$F((m+tg)/\|m+tg\|) - F(m)$$

$$= \int_0^1 \left(\operatorname{sgn}\left(\frac{m+tg}{\|m+tg\|}-a\right)\left|\frac{m+tg}{\|m+tg\|}-a\right|^{p-1} - \operatorname{sgn}(m-a)|m-a|^{p-1}\right)dx$$

$$= \int_0^1 \left\{\operatorname{sgn}\left(\frac{m+tg}{\|m+tg\|}-a\right)\left[|m-a|^{p-1}+(p-1)|m-a|^{p-2}\operatorname{sgn}(m-a)\cdot\right.\right.$$

$$\left.\left.\cdot\left(-mt\int_0^1 m^{p-1}gdy+tg\right)+O(t^2)\right]-\operatorname{sgn}(m-a)|m-a|^{p-1}\right\}dx$$

$$= \int_0^1 \left(\operatorname{sgn}\left(\frac{m+tg}{\|m+tg\|}-a\right)-\operatorname{sgn}(m-a)\right)|m-a|^{p-1}+$$

$$+(p-1)t\int_0^1 \operatorname{sgn}\left(\frac{m+tg}{\|m+tg\|}-a\right)|m-a|^{p-2}\operatorname{sgn}(m-a)\cdot$$

$$\left(g-m\int_0^1 m^{p-1}gdy\right)dx+O(t^2).$$

This implies easily that

$$0 = \lim_{t\to 0}\left[F\left(\frac{m+tg}{\|m+tg\|}\right) - F(m)\right]/t$$

$$= (p-1)\int_0^1 \left(|m-a|^{p-2} - \left(\int_0^1 |m-a|^{p-2} m\,dy\right) m^{p-1}\right) g\,dx, \quad \text{for every } g \in L_p.$$

Thus, if F attains its supremum at $m \in S(L_p)$, then a necessary condition is that

$$(2.12) \qquad |m(x)-a|^{p-2} = \int_0^1 |m(t)-a|^{p-2} m(t)\,dt \cdot m(x)^{p-1} \text{ a.e.}$$

Since it is enough to consider decreasing functions (2.12) shows that F attains its supremum at some $m = (f_1, f_2, f_3; \lambda_1, \lambda_2, \lambda_3) \in S(L_p)$. But, by a previous argument, we see that the supremum is in fact attained at some $m_0 = (f_1, f_2; \lambda_1, \lambda_2)$. Finally, by (2.12) it follows that

$$-\frac{1}{p}\frac{\partial}{\partial c} I_{a,c}(m_0)|_{c=a} = \sum_{i=1}^{2} \lambda_i (f_i - a)|f_i - a|^{p-2}$$

$$= \sum_{i=1}^{2} \lambda_i f_i |f_i - a|^{p-2} - a \sum_{i=1}^{2} \lambda_i |f_i - a|^{p-2}$$

$$= \int_0^1 |m_0(t)-a|^{p-2} m_0(t)\,dt \left(1 - a\|m_0\|_{p-1}^{p-1}\right),$$

which is $\geqq 0$ if and only if $a\|m_0\|_{p-1}^{p-1} \leqq 1$. This completes the proof of (ii).

REMARK. The necessity of Lemma 2.4 follows from Lemma IV.2.1 of [2], but we think the proof here contributes to a better understanding.

3. Bounds and continuity of ρ.

As a first application of Theorem 2.1 we show that the closed unit ball B_p in $L_p([0,1],dt)$ is optimal only if $p = 2$ or ∞.

PROPOSITION 3.1. *If $1 \leqq p \neq 2 < \infty$, then the closed unit ball, B_p, in $L_p([0,1],dt)$ is not optimal.*

PROOF. That B_1 is not optimal follows from Proposition 0.1. Let $1 < p \neq 2 < \infty$. By (2.1),

$$\varrho(p) = \sup_{\substack{1 < f_1 \\ 0 \leq f_2 < 1}} \{(f+gA(f,g))/1+A(f,g))\},$$

where $A(f,g) = ((f^p-1)/(1-g^p))^{1/(p-1)}$. Put $f_\varepsilon = 1+\frac{1}{2}p\varepsilon$ and $g_\varepsilon = 1-\varepsilon$, where $0 < \varepsilon < 1$. Since $(f+gA(f,g))/(1+A(f,g)) > 1$, iff $A(f,g) < (f-1)/(1-g)$ and since

$$\lim_{\varepsilon \to 0} A(f_\varepsilon, g_\varepsilon) = \lim_{\varepsilon \to 0} ((1+\tfrac{1}{2}p\varepsilon)^p - 1)/(1-(1-\varepsilon)^p)^{1/(p-1)}$$

$$= \left(\frac{p}{2}\right)^{1/(p-1)} < \tfrac{1}{2}p = (f_\varepsilon - 1)/(1-g_\varepsilon),$$

it follows that $\varrho(p) > 1$.

THEOREM 3.1. *Let $1 < p < \infty$. Then*

(i) $$\varrho(p) \geqq \left[1+\left(\frac{p-1}{2}\right)^{\left(\frac{p-1}{2-p}\right)}\right]^{1/p} \bigg/ \left[1+\left(\frac{p-1}{2}\right)^{1/(2-p)}\right]$$

which tends to 2 as p tends to 1,

(ii) $$\varrho(p) \geqq 2\left(\frac{p-1}{p}\right)(3(p-1))^{-1/p} \mathrel{\hat{=}} \psi(p),$$

which tends to 2 as p tends to ∞.

(iii) *If $p > 2$, then*

$$\varrho(p) \leqq 1+[\psi(p)(2\psi(p)-1)]^{-1/(p-2)}$$

PROOF. (i) follows immediately from Theorem 2.2 by inserting

$$x = \left(\frac{p-1}{2}\right)^{1/(2-p)} \quad \text{and} \quad y = 1.$$

To prove (ii) we consider the function $(f_1, f_2; \lambda_1, \lambda_2) \in S(L_p)$ for which $f_1 = ap$ and $f_2 = a/2$. We have that

$$\lambda_1(f_1-a)^{p-1} \geqq \lambda_2(a-f_2)^{p-1} \quad \text{iff}$$

$$(1-(a/2)^p)(a(p-1))^{p-1} \geqq ((ap)^p-1)(a/2)^{p-1} \quad \text{iff}$$

$$a^p \leqq (2^p(p-1)^{(p-1)}+2)/((p-1)^{(p-1)}+2p^p).$$

By Theorem 2.1 (i)

$$\varrho(p)^p \geqq (2^p(p-1)^{(p-1)}+2)/((p-1)^{(p-1)}+2p^p) \geqq 2^p(p-1)^{(p-1)}/3p^p$$

which implies that

$$\varrho(p) \geqq 2\left(\frac{p-1}{p}\right)(3(p-1))^{-1/p}.$$

(iii) will follow from Theorem 2.1 (ii). Fix $p > 2$ and $1 < a < \varrho(p)$. Then suppose that $m = (f, g; \lambda_1, \lambda_2) \in S(L_p)$ satisfies (2.12) and $a\|m\|_{p-1}^{p-1} \leqq 1$. Put

$$\alpha = \int_0^1 |m(x)-a|^{p-2} m(x)dx.$$

One readily see that $2a-1 < f < 1/\alpha$ and $|m(x)-a| > a-1$. Therefore

$$1/a \geqq \|m\|_{p-1}^{p-1} = \int_0^1 m(x)^{p-1}dx = \frac{1}{\alpha}\int_0^1 |m(x)-a|^{p-2}dx \geqq (2a-1)(a-1)^{p-2}$$

so that $(a-1)^{(p-2)} \leqq 1/a(2a-1)$. Since $a < \varrho(p)$ was arbitrary we have

$$(\varrho(p)-1)^{(p-2)} \leqq 1/\varrho(p)(2\varrho(p)-1).$$

Inserting the estimate (ii) finishes the proof.

Finally, we prove the continuity of ϱ.

THEOREM 3.2. $\varrho : [1, \infty[\to [1, 2]$ *is continuous.*

PROOF. Continuity at $p = 1$ follows from Theorem 3.1 (i) and Proposition 0.1 (ii), while the continuity for $1 < p < \infty$ is immediate from Theorem 2.2.

REMARK. By Theorem 3.1 (ii), ϱ is discontinous at ∞ since $\varrho(\infty) = 1 \neq 2 = \lim_{p\to\infty} \varrho(p)$.

4. A characterization of Hilbert space.

THEOREM 4.1. *Let E be real, strictly convex and 1-complemented in E^{**}. Then every two-dimensional bounded convex subset is optimal, iff E is isometric to a Hilbert space.*

PROOF. The proof is a combination of Proposition 1.2 of [2] and the following theorem of Kakutani [6]:

THEOREM. *Let E be a real Banach space. If for each two-dimensional subspace G of E, each $x_0 \in E \setminus G$ and every finite sequence $(x_i)_{i=1}^n \subseteq G$, there exists*

$\bar{x}_0 \in G$ *such that*

$$\|\bar{x}_0 - x_i\| \leqq \|x_0 - x_i\|, \quad 1 \leqq i \leqq n.$$

Then E is a Hilbert space.

Assume that every two-dimensional bounded convex subset of E is optimal, and let G be a two-dimensional subspace. Suppose $x_1, x_2, \ldots, x_n \in G$ and that $x_0 \in E \setminus G$. Then there exists, by Proposition I.2 of [2],

$$\lambda \in \min\{x_i\} \subseteqq \min \operatorname{cvx}\{x_i\} = \operatorname{cvx}\{x_i\} \subseteqq G$$

such that

$$\|y - x_i\| \leqq \|x_0 - x_i\|, \quad 1 \leqq i \leqq n.$$

By Kakutani's theorem E is a Hilbert space.

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