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SHADOWS OF COLORED COMPLEXES*

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Abstract.

Let $\mathscr{F}$ be a family of k-subsets of a set V. The *shadow* of $\mathscr{F}$, $\Delta\mathscr{F}$ is defined by

$$\Delta\mathscr{F} = \{S \subset V : |S| = k - 1, S \subset T \in \mathscr{F}\}.$$

The well-known Kruskal-Katona Theorem determines the minimum cardinality of $\Delta\mathscr{F}$ as a function of the cardinality of $\mathscr{F}$. A family $\mathscr{F}$ is *r-colored* if there is a partition of V, $V = V_1 \cup V_2 \cup \ldots \cup V_r$ such that for every $S \in \mathscr{F}$ and every $1 \leqq i \leqq r$, $|V_i \cap S| \leqq 1$. The minimum size of the shadow of an r-colored family of k-sets $\mathscr{F}$, $|\mathscr{F}| = m$ is determined. These results generalize the Kruskal-Katona Theorem, and imply (combined with known results) a complete description of f-vectors of completely balanced Cohen-Macaulay complexes.

1. Introduction.

Let V be a set. Denote by $\binom{V}{k}$ the set of k-subsets of V. A family $\mathscr{F} \subset \binom{V}{k}$ is *r-colored* if there exists a partition $V = V_1 \cup \ldots \cup V_r$ of V ($V_1, \ldots, V_r$ are pairwise disjoint), such that for every $A \in \mathscr{F}$, $|A \cap V_i| \leqq 1$, $i = 1, 2, \ldots, r$. For a family $\mathscr{F} \subset \binom{V}{k}$, the *shadow* of $\mathscr{F}$, $\Delta\mathscr{F}$ is defined by

$$\Delta(\mathscr{F}) = \left\{R \in \binom{V}{k-1} : R \subset S \in \mathscr{F}\right\}.$$

More generally for $1 \leqq l \leqq k-1$ the l-shadow of $\mathscr{F}$, is

$$\Delta_l(\mathscr{F}) = \left\{R \in \binom{V}{l} : R \subset S \in \mathscr{F}\right\}.$$

The well-known Kruskal-Katona Theorem [5], [4] gives a sharp lower bound for the size of the shadow of a family $\mathscr{F} \subset \binom{V}{k}$, $|\mathscr{F}| = m$. In this paper we extend this result to r-colored familes.

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As in the case of the Kruskal-Katona Theorem the extremal families are certain lexicographically initial families, and the rather complicated expressions below become natural when the structure of those families is studied.

For integers $n, k, r, n \geqq k$, $r \geqq k > 0$ define $a = \lfloor n/r \rfloor$, $r_1 = n - ra$. Note that $0 \leqq r_1 < r$.

Let $X_1, \ldots, X_r$ be pairwise disjoint sets, $|X_i| = a+1$ for $1 \leqq i \leqq r_1$ and $|X_i| = a$ for $r_1 < i \leqq r$.

Define the complete r-colored k-graph $\mathscr{K}(n,k,r)$ by

$$\mathscr{K}(n,k,r) = \{F : |F| = k, \ |F \cap X_i| \leqq 1, \ i = 1, \ldots, r\}.$$

Define also $\binom{n}{k}_r = |\mathscr{K}(n,k,r)|$. Note that $\binom{n}{k}_r = \binom{n}{k}$ holds for $n \leqq r$. In general,

$$\binom{n}{k}_r = \sum_{j=0}^{k} \binom{r_1}{j}\binom{r-r_1}{k-j}(a+1)^j a^{k-j}. \tag{1.1}$$

LEMMA 1.1. *Let $r \geqq k$ be positive integers. Every positive integer m can be written uniquely in the form*

$$m = \binom{n_k}{k}_r + \binom{n_{k-1}}{k-1}_{r-1} + \ldots + \binom{n_{k-s}}{k-s}_{r-s}, \tag{1.2}$$

where $n_k > n_{k-1} > \ldots > n_{k-s} \geqq k-s > 0$.

Given this representation of m define

$$\partial^r_k(m) = \binom{n_k}{k-1}_r + \ldots + \binom{n_{k-s}}{k-s-1}_{r-s}. \tag{1.3}$$

For $1 \leqq l \leqq k-1$ define similarly

$$\partial^r_{k|l}(m) = \binom{n_k}{l}_r + \ldots + \binom{n_{k-s}}{l-s}_{r-s},$$

where $\binom{b}{0} = 1$, and $\binom{b}{c} = 0$ for $c < 0$.

THEOREM 1.2. *Let $\mathscr{F} \subset \binom{V}{k}$ be an r-colored family, $|\mathscr{F}| = m$. Then*

(i)

(1.4) $|\Delta \mathscr{F}| \geqq \partial^r_k(m)$.

(ii) $|\Delta_l \mathscr{F}| \geqq \partial^r_{k|l}(m)$.

Moreover, these bounds are sharp.

Note that part (ii) of Theorem 1.2 is obtained by repeated applications of part (i).

In Section 2 we study the lexicographic ordering for colored sets. An example of an r-colored family for which (1.4) holds with equality is provided by an initial family of sets with respect to this ordering. The proof of the theorem is given in Sections 3 and 4.

Some related results are considered in Section 5.

Let us mention that Theorem 1.2 combined with results of [7] and [1] gives a characterization of f-vectors of completely balanced Cohen-Macaulay complexes.

2. The lexicographic ordering for multi-colored sets.

For an integer $n \geqq 1$ define $[n] = \{1, \ldots, n\}$. Let $r \geqq k \geqq 1$ be fixed integers. Let N be the set of positive integers. For $1 \leqq i \leqq r$ set

$$\mathsf{N}_i = \{n \in \mathsf{N} : n \equiv i \pmod r\}.$$

For an integer $n \in \mathsf{N}$ define $\bar{n}$ to be its residue modulo r, that is $\bar{n} = i$ if and only if $n \in \mathsf{N}_i$ holds.

For a set $F \subset \mathsf{N}$ define its projection $\bar{F} = \{\bar{f}; f \in F\}$.

Let us define next the main object of our study, the set of all r-colored k-sets, $\mathcal{M}(k, r)$:

$$\mathcal{M}(k, r) = \binom{\mathsf{N}}{k}_r = \left\{F \in \binom{\mathsf{N}}{k} : |F \cap \mathsf{N}_i| \leqq 1 \text{ for all } i\right\}.$$

Equivalently, $F \in \binom{\mathsf{N}}{k}$ is in $\binom{\mathsf{N}}{k}_r$ if and only if $|\bar{F}| = k$.

The lexicographic order on $\binom{\mathsf{N}}{k}$ is defined by $F < G$ iff $\max\{i : i \in F - G\} <$ $< \max\{j : j \in G - F\}$.

We consider the restriction of this order to $\binom{\mathsf{N}}{k}_r$. It is clearly a linear order and e.g. setting

$$\binom{[n]}{k}_r = \binom{[n]}{k} \cap \binom{\mathsf{N}}{k}_r$$

one has

$$\left|\binom{[n]}{k}_r\right| = \binom{n}{k}_r$$

and these are the first $\binom{n}{k}_r$ sets in the restricted linear order.

In general, for arbitrary $m \geqq 1$ let $\mathcal{I}(m, k, r)$ denote the first m sets in $\mathcal{M}(k, r)$. When it causes no confusion we write simply $\mathcal{I}(m)$.

Proof of Lemma 1.1. For $k = 1$, $m = \binom{m}{1} = \binom{m}{1}_r$ gives the unique representation.

Apply induction on k and suppose that for $k' < k$ the lemma is verified.

Let us consider $\mathcal{I}(m) = \mathcal{I}(m,k,r)$. Let n be the largest integer such that $\mathcal{I}(m) \supset \binom{[n]}{k}_r$ holds. In case of equality $m = \binom{n}{k}_r$ gives the desired representation.

Now every member of $\mathcal{I}(m) - \binom{[n]}{k}_r$ contains $n+1$. Define

$$\mathcal{I}^* = \{F - \{n+1\} : n+1 \in F \in \mathcal{I}(m)\}.$$

Now $\mathcal{I}^*$ is a collection of $(r-1)$-colored $(k-1)$-subsets of $\mathbf{N} - \mathbf{N}_{\overline{n+1}}$.

By the induction assumption we obtain a unique representation:

$$|\mathcal{I}^*| = m - \binom{n}{k}_r = \binom{n_{k-1}}{k-1}_{r-1} + \dots + \binom{n_{k-s}}{k-s}_{r-s}.$$

Setting $n_k = n$ we obtain the desired representation, because

$$\binom{n}{k}_r + \binom{n}{k-1}_{r-1} = \binom{n+1}{k}_r$$

implies $n > n_{k-1}$.

The uniqueness follows by induction from

$$\binom{n}{k}_r = \binom{n-1}{k-1}_{r-1} + \dots + \binom{n-k+1}{1}_{r-k-1} + 1,$$

i.e., there could be no representation with $n_k < n$.

PROPOSITION 2.1. *$\Delta\mathcal{I}(m)$ is an initial set of $\mathcal{M}(k-1,r)$ and $|\Delta\mathcal{I}(m,k,r)| = \partial_k^r(m)$.*

The easy proof is left for the reader.

3. Shifting.

Suppose that we have a family $\mathcal{F} \subset \binom{\mathbf{N}}{k}_r$. Eventualy we want to show that replacing $\mathcal{F}$ by a lexicographically initial set of $\binom{\mathbf{N}}{k}_r$ of the same size, the shadow does not increase. As in many of the proofs of the Kruskal-Katona theorem the first (and easy) step is to replace $\mathcal{F}$ by an initial set of $\mathcal{M}(k,r)$ with respect to some weaker partial order. Define $<_p$ on $\mathcal{M}(k,r)$ as follows: For $S, T \in \mathcal{M}(k,r)$, $S = \{a_1, \dots, a_k\}$, $T = \{b_1, \dots, b_k\}$, $a_1 < \dots < a_k$ and $b_1 < \dots < b_k$ set $S <_p T$ if $a_i \leqq b_i$ for every $1 \leqq i \leqq k$. A family $\mathcal{F} \subset \mathcal{M}(k,r)$ is *shifted* if whenever $T \in \mathcal{F}$ and $S <_p T$, $S \in \mathcal{M}(k,r)$ then $S \in \mathcal{F}$.

PROPOSITION 3.1. *Let $\mathcal{F} \subset \mathcal{M}(k,r)$. There exists a shifted family $\mathcal{F}^*$ such that $|\mathcal{F}^*| = |\mathcal{F}|$ and $|\Delta\mathcal{F}^*| \leqq |\Delta\mathcal{F}|$.*

PROOF. The proof is standard cf. e.g., Katona [3].

4. Proof of the main theorem.

In the previous section we reduced Theorem 1.2 to the case of shifted colored families. In order to apply successfully an induction argument we need to extend the statement of the theorem in a somewhat technical way.

We need a few further definition.

For $G \subset \mathsf{N}$ let $s(G)$ denote the smallest element of G.

For $r - k + 1 \geqq j \geqq 0$ let us define the family

$$\mathscr{M}(j) = \mathscr{M}(k, r, j) = \left\{ G \in \binom{\mathsf{N}}{k}_r : |\bar{G} \cap [s(G)]| \leqq s(G) - j \right\},$$

in other words, there are at least j numbers i, $1 \leqq i \leqq s(G)$ such that $G \cap N_i = \emptyset$.

Note that

$$\mathscr{M}(0) = \binom{\mathsf{N}}{k}_r \quad \text{and} \quad \mathscr{M}(r-k+1) = \left\{ G \in \binom{\mathsf{N}}{k}_r : G \cap [r] = \phi \right\}.$$

Note also that for $G \in \mathscr{M}(j)$ always $G \cap [j] = \phi$ holds.

Call a family $\mathscr{F} \subset \mathscr{M}(j)$ *shifted* if $F \in \mathscr{F}$ and $G \leqq_p F$ imply $G \in \mathscr{F}$ for all $F, G \in \mathscr{M}(j)$.

Similarly, let $\mathscr{I}(m, k, r, j)$ denote the smallest m sets in $\mathscr{M}(j)$ in the lexicographic order.

In view of Proposition 3.1 the next result implies Theorem 1.2 (with $j = 0$).

THEOREM 4.1. *Let k, r, j be fixed integers, $r \geqq k > 0$, $r - k \geqq j \geqq 0$. Suppose that $\mathscr{F} \subset \mathscr{M}(k, r, j)$ is a shifted family, $|\mathscr{F}| = m$. Then*

$$|\Delta(\mathscr{F})| \geqq |\Delta(\mathscr{I}(m, k, r, j))| \tag{4.1}$$

holds.

PROOF. The case $m = 1$ is trivial. We apply induction on m. We will consider several families derived from $\mathscr{F}$ and $\mathscr{I}(m, k, r, j)$, and compare their sizes using the induction hypothesis. For convenience set $\mathscr{I} = \mathscr{I}(m, k, r, j)$.

We make the following definitions:

For $1 \leqq i \leqq k$,

$$\mathscr{F}^i = \{S \in \mathscr{F} : j+i \in S, \bar{S} \cap [j+i] = i\}.$$

For $S \in \mathscr{F}$ set

$$K(S) = \{j+i : S \in \mathscr{F}^i\}.$$

(Thus, $|K(S)| = |\{i : S \in \mathscr{F}^i\}|$.) For $0 \leqq l \leqq k$ define:

$$\mathscr{F}(l) = \{S \in \mathscr{F} : |K(S)| = l\}.$$

For $S \in \mathscr{F}$ define $\hat{S} = S \setminus K(S)$ and define

$$\hat{\mathscr{F}}(l) = \{\hat{S} : S \in \mathscr{F}(l)\}.$$

Define $\mathscr{I}^i$, $\mathscr{I}(l)$, and $\hat{\mathscr{I}}(l)$ in the same way.
The following lemma is crucial for the proof.

CLAIM 4.2.

(i) $|\Delta\mathscr{F}| = \sum_{l \geq 1} l|\mathscr{F}(l)|$.

(ii) $\Delta\hat{\mathscr{F}}(l) \subseteqq \hat{\mathscr{F}}(l+1)$.

PROOF. (i) For $S \in \mathscr{F}^i$ let $R^i(S) = S - \{i + j\}$. $R^i(S) \in \Delta\mathscr{F}$ and we will show that every set in $\Delta\mathscr{F}$ can be expressed uniquely as $R^i(S)$ for some $1 \leqq i \leqq k$ and some $S \in \mathscr{F}^i$. This gives

$$|\Delta\mathscr{F}| = \sum_{i=1}^{k} |\mathscr{F}^i| = \sum_{l \geqq 1} l|\mathscr{F}(l)|.$$

Note that if $S \in \mathscr{M}(k, r, j)$ and $R \subset S$, $|R| = k-1$ then $R \in \mathscr{M}(k-1, r, j)$. Suppose that $R \in \Delta\mathscr{F}$. Put $R_c = [r] \setminus \bar{R}$. It remains to show that there is a unique x, $x \in R_c$ such that $|\bar{R} \cap [x]| = x - j - 1$, and $S_x = R \cup \{x\} \in \mathscr{F}$. (Then $S_x \in \mathscr{F}^{x-j}$). The uniqueness is clear: the ath element x_a in R_c satisfies $|\bar{R} \cap [x_a]| = x_a - a$, therefore we must choose $x = x_{j+1}$. In order to show that $S_x \in \mathscr{F}$ we will show that S_x is the unique minimal set, with respect to the partial order $<_p$, which contains R and belongs to $\mathscr{M}(k, r, j)$. Indeed, let $T \supset R$, $|T| = k$. Let $y = T \setminus R$. If $\bar{y} < x$ then since $T \in \mathscr{M}(k, r, j)$, $y > r$, $T >_p S_x$. If $\bar{y} \geqq x$ then clearly $T \geqq_p S_x$.

(ii) Let $S \in \mathscr{F}(l)$, $K = K(S)$ and $a \in S \setminus K$. Put $R = S \setminus a$. We have to find b such that $T = R \cup \{b\} \in \mathscr{F}(l+1)$. Consider R_c and let x be the $(j+1)$th element in R_c. As in part (i), it is easy to see that $T = R \cup \{x\}$ is the unique minimal (with respect to $\leqq_p$) set in $\mathscr{M}(k, r, j)$ containing R and that $T \in \hat{\mathscr{F}}(l+1)$.

PROOF OF THEOREM 4.1. (*continued*). Next, we consider $\hat{\mathscr{I}}(l)$, $l \geqq 0$. For $S \in \mathscr{M}(k, r, j)$ denote by S^+ the successor of S in $\mathscr{M}(k, r, j)$ with respect to the order $<$. Define for an integer $i \geqq 0$, $S^{(0)} = S$ and $S^{(i)} = (S^{(i-1)})^+$. We need the following:

CLAIM 4.3. If $S \in \mathscr{I}(l)$, $l \geqq 1$, then $S^+ \in \mathscr{I}(l-1)$.

PROOF. Let $S \in \mathscr{I}(l)$, $a = \max K(S)$. Let us define

$$b = \min\{x \in \mathrm{N}, x > a, \bar{x} \notin \overline{S - \{a\}}\}.$$

Then $S^+ = S \setminus \{a\} \cup \{b\}$, $K(S^+) = K(S) \setminus \{a\}$, and thus $S^+ \in \mathscr{I}(l-1)$.

Now consider $\hat{\mathscr{I}}(l)$, $l \geqq 0$. By Claim 4.2, $\Delta\hat{\mathscr{I}}(l) \subseteqq \hat{\mathscr{I}}(l+1)$ holds. If $R = (S \setminus K(S)) \in \hat{\mathscr{I}}(l+1)$, then $R \subset (S^+ \setminus K(S^+))$. Let S be the last set in $\mathscr{I}$ and suppose that $S \in \mathscr{I}(l_0)$. Then $\Delta\hat{\mathscr{I}}(l-1) = \mathscr{I}(l)$ holds for all l except $l = l_0$. Define

$$\mathscr{I}^+ = \mathscr{I} \cup \{S^{(i)}: 1 \leqq i \leqq l_0\}.$$

Then $\mathscr{I}^+(l) = \mathscr{I}(l)$ for $l \geqq l_0$ and $|\mathscr{I}^+(l)| = |\mathscr{I}(l)| + 1$ when $0 \leqq l < l_0$. Clearly $\Delta\hat{\mathscr{I}}^+(l) = \hat{\mathscr{I}}^+(l+1)$ for every $l \geqq 0$.

We want to compare $\hat{\mathscr{I}}(l)$ and $\hat{\mathscr{F}}(l)$. Note that $\hat{\mathscr{F}}(l)$ may be regarded as a shifted family in $\mathscr{M}(k-l, r, j+1)$, so we may apply the induction hypothesis to $\hat{\mathscr{F}}(l)$.

We are ready for the heart of the proof of Theorem 4.1. By Claim 4.2 (i) we have to show that

$$\sum_{l \geq 1} l|\mathscr{F}(l)| \geqq \sum_{l \geq 1} l|\mathscr{I}(l)|. \tag{4.1}$$

CLAIM 4.4. $|\mathscr{F}(0)| \leqq |\mathscr{I}(0)|$.

PROOF. Assume on the contrary that $|\mathscr{F}(0)| > |\mathscr{I}(0)|$.

CASE I. $\mathscr{I} = \mathscr{I}^+$, then $\Delta(\hat{\mathscr{I}}(l)) = \hat{\mathscr{I}}(l+1)$ or every $l \geqq 0$. But Claim 4.2 (ii) asserts that $\Delta\hat{\mathscr{F}}(l) \subseteqq \hat{\mathscr{F}}(l+1)$ for every $l \geqq 0$. Therefore by the induction hypothesis of Theorem 4.1 $|\hat{\mathscr{F}}(l)| \geqq |\hat{\mathscr{I}}(l)$. So

$$\left|\sum_{l \geq 1} \mathscr{F}(l)\right| \geqq \left|\sum_{l \geq 1} \mathscr{I}(l)\right|$$

and

$$|\mathscr{F}(0)| = m - \sum_{l \geqq 1} |\mathscr{F}(l)| \leqq m - \sum_{l \geqq 1} |\mathscr{I}(l)| = |\mathscr{I}(0)|.$$

A contradiction.

CASE II. $\mathscr{I} \neq \mathscr{I}^+$. Then $|\mathscr{I}^+(0)| = |\mathscr{I}(0)| + 1$. From $|\mathscr{F}(0)| \geqq |\mathscr{I}^+(0)$ we get by the same argument as above that $|\hat{\mathscr{F}}(l)| \geqq |\hat{\mathscr{I}}^+(l)|$ for every l and

$$\sum_{l \geq 1} |\hat{\mathscr{F}}(l)| \geqq \sum_{l \geq 1} |\hat{\mathscr{I}}^+(l)| > \sum_{l \geq 1} |\hat{\mathscr{I}}(l)|.$$

A contradiction.

PROOF OF THEOREM 4.1. (*end*). Claim 4.4. says that

$$\sum_{l \geqq 1} |\hat{\mathscr{F}}(l)| \geqq \sum_{l \geqq 1} |\hat{\mathscr{I}}(l)|. \tag{4.2}$$

In order to prove (4.1) it is enough to show that for every $r \geqq 1$

$$\sum_{l \geqq r} |\hat{\mathscr{F}}(l)| \geqq \sum_{l \geqq r} |\hat{\mathscr{I}}(l)|. \tag{4.3}$$

It is enough to show that if $|\hat{\mathscr{F}}(\alpha)| > |\hat{\mathscr{I}}(\alpha)|$ then $|\hat{\mathscr{F}}(\beta)| \geqq |\hat{\mathscr{I}}(\beta)|$ for every $\beta > \alpha$. Indeed, if $|\hat{\mathscr{F}}(\alpha)| > |\hat{\mathscr{I}}(\alpha)|$ then $|\hat{\mathscr{F}}(\alpha)| \geqq |\mathscr{I}^+(\alpha)|$ and thus by Claim 4.2(ii) and the induction hypothesis we get (exactly as in the proof of Claim 4.4)

$$|\hat{\mathscr{F}}(\beta)| \geqq |\hat{\mathscr{I}}^+(\beta)| \geqq |\hat{\mathscr{I}}(\beta)|$$

for every $\beta > \alpha$. This proves (4.3) for every $r \geqq 1$, and with it completes the proof of Theorem 4.1.

Let us mention that Theorem 4.1 generalizes the Kruskal-Katona Theorem. Indeed, for $\mathscr{F} \subset \binom{[n]}{k}$, $\mathscr{F} \subset \binom{[n]}{k}_r$ holds for $r \geqq n$. Also, $\mathscr{I}(|\mathscr{F}|, k, r)$ consists simply of the lexicographically smallest $|\mathscr{F}|$ sets in $\binom{[n]}{k}$. Then Theorem 4.1 asserts $|\partial\mathscr{F}| \geqq |\partial\mathscr{I}(|\mathscr{F}|)|$, which is exactly the Kruskal-Katona Theorem.

Another important special case of Theorem 4.1 is when $k = r$. This could be called the Kruskal-Katona Theorem for k-partite k-graphs.

5. A numerical version.

The binomial coefficient $\binom{x}{k}$ can be defined by

$$\binom{x}{k} = \frac{x(x-1)\dots(x-k+1)}{k!}$$

for all real numbers x.

Lovász [6, Problem 13.31] proved that if $\mathscr{F} \subset \binom{\mathbf{N}}{k}$, $|\mathscr{F}| = \binom{x}{k}$, $x \geqq k$ then $|\partial_l\mathscr{F}| \geqq \binom{x}{l}$ holds for all $1 \leqq l \leqq k$.

The next result gives the corresponding statement for multicolored families.

THEOREM 5.1. *Suppose that* $\mathscr{F} \subset \binom{\mathbf{N}}{k}_r$ *and* $x \geqq 0$ *is defined by* $|\mathscr{F}| = \binom{r}{k}x^k$. *Then for all* $1 \leqq l \leqq k$ *one has*

$$|\partial_l\mathscr{F}| \geqq \binom{r}{l}x^l. \tag{5.1}$$

Note that for integer values of x, (5.1) is best possible.

PROOF. For $A \in \binom{[r]}{k}$ define $\mathscr{F}_A = \{F \in \mathscr{F} : \bar{F} = A\}$. For $B \in \binom{[r]}{l}$, $(\partial_l \mathscr{F})_B$ is defined similarly. The following should be clear

$$|\mathscr{F}| = \sum_A |\mathscr{F}_A|,$$

$$|\partial_l \mathscr{F}| = \sum_B |(\partial_l \mathscr{F})_B|,$$

$$(\partial_l \mathscr{F}_A)_B \subset (\partial_l \mathscr{F})_B.$$

Using these relations we infer

$$\binom{r-l}{k-l} |\partial_l \mathscr{F}| \geqq \sum_{A \in \binom{[r]}{k}} |\partial_l(\mathscr{F}_A)|. \tag{5.2}$$

Suppose now that (5.1) is proved for $r = k$. Define $x(A) \geqq 0$ by $|\mathscr{F}_A| = x(A)^k$. Using (5.1) for $\mathscr{F}_A$ and substituting into (5.2) gives

$$\binom{r-l}{k-l} |\partial_l \mathscr{F}| \geqq \binom{k}{l} \sum_{A \in \binom{[r]}{k}} x(A)^l.$$

Note that $y^{l/k}$ is a concave function and $x(A)^l = |\mathscr{F}(A)|^{l/k}$.

Applying the Jensen inequality to the RHS, we obtain

$$\binom{r-l}{k-l} |\partial_l \mathscr{F}| \geqq \binom{k}{l}\binom{r}{k} x^l,$$

which is equivalent to (5.1).

It remains to prove (5.1) for $r = k$.

In this case we may apply an inequality due to Shearer (cf. [2]) which states

$$|\mathscr{F}|^{\binom{k-1}{l-1}} \geqq \prod_{B \in \binom{[k]}{l}} |(\partial_l \mathscr{F})_B| \leqq \left(\sum |\partial_l \mathscr{F})_B| \Big/ \binom{k}{l} \right)^{\binom{k}{l}}.$$

Using $|\mathscr{F}| = x^k$ and taking the $(1/\binom{k}{l})$th power of the two extreme sides we obtain

$$x^1 \leqq |\partial_l \mathscr{F}| \Big/ \binom{k}{l},$$

as desired.

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