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EQUIVARIANT ALEXANDER-SPANIER COHOMOLOGY

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Introduction.

Let G be a finite group. In this paper we construct an equivariant cohomology theory defined on the category of all G-pairs. We call this theory "equivariant Alexander-Spanier cohomology", because our construction generalizes the construction of ordinary Alexander-Spanier cohomology (see [6, 6.4]) in approximately the same way as Illman's construction of equivariant singular cohomology (see [4]), valid for any topological group G, generalizes the construction of ordinary singular cohomology.

The contents of the paper are as follows: In section 1 we define the cohomology groups and in sections 2–4 we show that all the Eilenberg-Steenrod axioms for an equivariant cohomology theory, including the dimension axiom, are satisfied. In section 5 we prove a tautness property for this cohomology theory. The main result of section 6 is that for a paracompact G-space X the equivariant Alexander-Spanier cohomology groups can be interpreted as ordinary cohomology groups of the orbit space X/G with coefficients in a suitable sheaf. This fact is used in section 7 to show that equivariant Alexander-Spanier cohomology agrees with equivariant singular cohomology on G-locally contractible paracompact G-spaces. In the final section 8 we give some comments on equivariant Alexander-Spanier cohomology with compact supports.

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1. The construction.

As stated in the Introduction, G will always be a finite group.

Let X be a G-space. If $n \in N$, we denote by $V_n(X)$ the set of all G-maps ϕ: $G/H \times \{0, 1, \ldots, n\} \to X$ for various subgroups $H \leqq G$, where G acts trivially on $\{0, 1, \ldots, n\}$. We call H the *type* of the above $\phi \in V_n(X)$, denoted $H = t(\phi)$. We

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often write $\phi = (\phi_0, \phi_1, \ldots, \phi_n)$, where
$\phi = \phi_i|G/H \times \{i\}: G/H \to X, \quad 0 \leqq i \leqq$ n.
Let m be a contravariant coefficient system, i.e. a contravariant functor from the category of the G-spaces G/H ($H \leqq G$) and G-maps between them to the category of abelian groups (or R-modules, R a ring with unit).

Let $M = \oplus_{H \leqq G} m(G/H)$. The n'th *cochain group of X with coefficients m* is

$$C^n(X;m) = \{c\colon V_n(X) \to M \mid c(\phi) \in m(G/t(\phi))\ \forall\, \phi\}.$$

Equipped with the usual differential d: $C^n(X;m) \to C^{n+1}(X;m)$,

$$(d(c))(\phi_0, \ldots, \phi_{n+1}) = \sum_{i=0}^{n+1} (-1)^i c(\phi_0, \ldots, \hat{\phi}_i, \ldots, \phi_{n+1}),$$

$C^*(X;m)$ is clearly a cochain complex. A cochain $c \in C^n(X;m)$ is *equivariant*, if $c(\phi \circ \alpha) = m(\alpha)(c(\phi))$ whenever $\phi \in V_n(X)$ and α: $G/K \to G/t(\phi)$ is a G-map; here $\phi \circ \alpha = (\phi_0 \circ \alpha, \ldots, \phi_n \circ \alpha)$. We denote

$$C^n_G(X;m) = \{c \in C^n(X;m) \mid c \text{ is equivariant}\}.$$

Clearly $C^*_G(X;m)$ is a cochain subcomplex of $C^*(X;m)$.

We call a covering $\mathscr{U}$ of X a G-covering, if $gU \in \mathscr{U}$ for all $U \in \mathscr{U}$ and $g \in G$ (this is called an invariant covering in [1, Chapter III]). A cochain $c \in C^n(X)$ (we occasionally omit the coefficients m in the notation) is *locally zero* on X, if there is an open G-covering $\mathscr{U}$ of X such that $c(\phi) = 0$ whenever $\{\phi_0(eH), \ldots, \phi_n(eH)\} \subset U$ for some $U \in \mathscr{U}$ ($H = t(\phi)$).
We set

$$C^n_0(X;m) = \{c \in C^n(X;m) \mid c \text{ is locally zero on } X\},$$

$$C^n_{G,0}(X;m) = C^n_G(X;m) \cap C^n_0(X;m).$$

Evidently $dC^n_0(X) \subset C^{n+1}_0(X)$, and therefore the quotients

$$\bar{C}^n(X;m) = C^n(X;m)/C^n_0(X;m), \qquad \bar{C}^n_G(X;m) = C^n_G(X;m)/C^n_{G,0}(X;m)$$

form cochain complexes $\bar{C}^*(X;m)$ and $\bar{C}^*_G(X;m)$.

An equivariant function f: $X \to Y$ between G-spaces (f need not be continuous) induces a cochain map f^*: $C^*(Y) \to C^*(X)$ by the formula

$$(f^*(c))(\phi_0, \ldots, \phi_n) = c(f \circ \phi_0, \ldots, f \circ \phi_n); \qquad c \in C^n(Y),\ \phi \in V_n(X).$$

Obviously $f^*C^*_G(Y) \subset C^*_G(X)$. If f is a G-map (i.e. continuous), then $f^*C^n_0(Y) \subset C^n_0(X)$ (if $c \in C^n(Y)$ is locally zero with respect to the open G-covering $\mathscr{V}$ of Y, then $f^*(c) \in C^n(X)$ is locally zero with respect to the open G-covering $f^{-1}\mathscr{V} = \{f^{-1}(V) \mid V \in \mathscr{V}\}$ of X). Therefore the G-map f: $X \to Y$ induces cochain maps f^*: $\bar{C}^*(Y) \to \bar{C}^*(X)$ and f^*: $\bar{C}^*_G(Y) \to \bar{C}^*_G(X)$.

Let $A \subset X$ be a G-subspace and $i\colon A \to X$ the inclusion. The cochain map $i^*\colon \bar{C}^*_G(X) \to \bar{C}^*_G(A)$ is surjective: if $c \in C^n_G(A)$, then $c = i^*\tilde{c}$ where $\tilde{c} \in C^n_G(X)$ is defined by $\tilde{c}(\phi) = c(\phi)$ for $\phi \in V_n(A)$ and $\tilde{c}(\phi) = 0$ for $\phi \in V_n(X) \setminus V_n(A)$.

DEFINITION 1.1. $\bar{C}^*_G(X, A; m) = \ker\,[i^*\colon \bar{C}^*_G(X; m) \to \bar{C}^*_G(A; m)]$ is the *equivariant Alexander-Spanier cochain complex* of (X, A) with coefficients m. Its cohomology groups $\bar{H}^n_G(X, A; m) = H^n(\bar{C}^*_G(X, A; m))$ are the *equivariant Alexander-Spanier cohomology groups* of (X, A) with coefficients m.

THEOREM 1.2. *The functors $\bar{H}^n_G$ satisfy all the Eilenberg-Steenrod axioms for an equivariant cohomology theory, including the dimension axiom.*

Exactness is clear from the definition: the short exact sequence

$$0 \to \bar{C}^*_G(X, A) \to \bar{C}^*_G(X) \xrightarrow{i^*} \bar{C}^*_G(A) \to 0$$

of cochain complexes induces the long exact cohomology sequence

$$0 \longrightarrow \bar{H}^0_G(X, A; m) \longrightarrow \bar{H}^0_G(X; m) \longrightarrow \bar{H}^0_G(A; m) \longrightarrow \bar{H}^1_G(X, A; m) \longrightarrow \cdots$$
$$\cdots \longrightarrow \bar{H}^n_G(X, A; m) \longrightarrow \bar{H}^n_G(X; m) \longrightarrow \bar{H}^n_G(A; m) \longrightarrow \bar{H}^{n+1}_G(X, A; m) \longrightarrow \cdots$$

The remaining three axioms are proved in the next three sections.

2. The dimension axiom

PROPOSITION 2.1. *Let $H \leqq G$ be a subgroup. For $n \neq 0$, $\bar{H}^n_G(G/H; m) = 0$. For $n = 0$ there is an isomorphism $\bar{H}^0_G(G/H; m) \xrightarrow{\simeq} m(G/H)$, natural with respect to G-maps $G/H \to G/H'$.*

PROOF. Let $\pi_n = (\mathrm{id}_{G/H}, \ldots, \mathrm{id}_{G/H}) \in V_n(G/H)$ and define $D^n = \mathrm{Hom}_Z(Z\pi_n, m(G/H))$, $n \in \mathbf{N}$. The homomorphisms $d\colon D^n \to D^{n+1}$,

$$(d(c))(\pi_{n+1}) = \begin{cases} c(\pi_n), & n \text{ odd} \\ 0 \quad , & n \text{ even} \end{cases}; \qquad c \in D^n,$$

make D^* into a cochain complex. We can define a cochain map $u\colon C^*_G(G/H) \to D^*$ by $(u(c))(\pi_n) = c(\pi_n)$ for $c \in C^n_G(G/H)$. We show that u is surjective:

Given $c \in D^n$, define $\tilde{c} \in C^n(G/H)$ by $\tilde{c}(\phi_0, \ldots, \phi_n) = m(\phi_0)(c(\pi_n))$. If $\alpha\colon G/K \to G/t(\phi)$ is a G-map, then

$$\tilde{c}(\phi_0 \circ \alpha, \ldots, \phi_n \circ \alpha) = m(\phi_0 \circ \alpha)(c(\pi_n)) = m(\alpha)(m(\phi_0)(c(\pi_n))) = m(\alpha)(\tilde{c}(\phi_0, \ldots, \phi_n)),$$

so $\tilde{c} \in C^n_G(G/H)$. Clearly $u(\tilde{c}) = c$.

On the other hand, Lemma 2.2 below shows that $\ker u = C^*_{G,0}(G/H)$. Therefore u induces an isomorphism $\bar{C}^*_G(G/H) \xrightarrow{\simeq} D^*$ and

$$\bar{H}^n_G(G/H) \cong H^n(D^*) \cong \begin{cases} m(G/H), & n = 0 \\ 0 \quad , & n \neq 0. \end{cases}$$

LEMMA 2.2. *Let* $c \in C^n_G(G/H)$. *Then* $c \in C^n_{G,0}(G/H)$ *if and only if* $c(\pi_n) = 0$.

PROOF. The "only if"-part is clear. Conversely, assume that $c(\pi_n) = 0$. Then

$$c(\alpha, \ldots, \alpha) = c(\pi_n \circ \alpha) = m(\alpha)(c(\pi_n)) = 0$$

for any G-map $\alpha: G/K \to G/H$. This implies that c is locally zero with respect to the open G-covering $\{\{gH\} \mid gH \in G/H\}$ by singletons of G/H. Namely, if $\phi = (\phi_0, \ldots, \phi_n) \in V_n(G/H)$, $t(\phi) = K$ and $\phi_i(eK) \in \{gH\}$ $(i = 0, \ldots, n)$ for some $gH \in G/H$, then $\phi_0(eK) = \phi_1(eK) = \ldots = \phi_n(eK)$, so $\phi_0 = \phi_1 = \ldots = \phi_n$, and finally $c(\phi) = c(\phi_0, \ldots, \phi_0) = 0$.

REMARK 2.3. Contrary to the non-equivariant case, the passage from C^*_G to $\bar{C}^*_G$ is essential even for the dimension axiom, because $H^n(C^*_G(G/H))$ can be nonzero for $n \neq 0$, too.

For example, in the case $H = \{e\}$ we see that $C^*_G(G; m) \cong \operatorname{Hom}_{ZG}(F_*, m(G))$, where $0 \leftarrow Z \leftarrow F_0 \leftarrow F_1 \leftarrow \ldots$ is the standard resolution of Z by right ZG-modules, that is F_n is free abelian with basis $\{(g_0, \ldots, g_n) \mid g_i \in G\}$ and G acts on F_n by $(g_0, \ldots, g_n) \cdot g = (g_0 g, \ldots, g_n g)$; this follows easily from the fact that every G-map $G \to G$ is of the form $r_g: x \to xg$ for some $g \in G$.
Therefore

$$H^n(C^*_G(G; m)) \cong H^n(G; \mathrm{m}(G)).$$

3. The excision axiom.

Let X be a G-space, $A \subset X$ a G-subspace and $i: A \to X$ the inclusion. We first give a slightly more concrete description of the cochain complex $\bar{C}^*_G(X, A)$. Define

$$C^n_G(X, A; m) = \{c \in C^n_G(X; m) \mid i^*(c) \in C^n_0(A; m)\},$$

the G-cochains of X which are locally zero on A. Clearly $C^*_{G,0}(X) \subset C^*_G(X, A)$ and the image of the homomorphism

$$C^*_G(X, A) \hookrightarrow C^*_G(X) \to \bar{C}^*_G(\bar{X})$$

equals $\ker\,[i^*: \bar{C}^*_G(X) \to \bar{C}^*_G(A)] = \bar{C}^*_G(X, A)$, so

$$\bar{C}^*_G(X, A; m) \cong C^*_G(X, A; m)/C^*_{G,0}(X; m).$$

The excision axiom is a consequence of the following result:

PROPOSITION 3.1. *Let $B \subset A$ be a G-subset and W an open G-neighborhood of B in X such that $\bar{W} \subset \operatorname{int} A$. Then the inclusion $j\colon (X\backslash B, A\backslash B) \hookrightarrow (X, A)$ induces an isomorphism*

$$j^*\colon \bar{C}_G^*(X, A; m) \xrightarrow{\sim} \bar{C}_G^*(X\backslash B, A\backslash B; m).$$

PROOF. This is quite similar to the non-equivariant case (see [6, 6.4.4]). We prove surjectivity and leave injectivity to the reader.

Let $c \in C_G^n(X\backslash B, A\backslash B)$. Then c is locally zero on $A\backslash B$ with respect to an open G-covering $\mathscr{U}$ of $A\backslash B$. Define $\tilde{c} \in C_G^n(X)$ by

$$\tilde{c}(\phi) = \begin{cases} c(\phi) & \text{if } \phi \in V_n(X\backslash W) \\ 0 & \text{otherwise.} \end{cases}$$

Evidently $\tilde{c}$ locally zero on $A = (A\backslash B) \cup W$ with respect to the open G-covering $\{U \cup W \mid U \in \mathscr{U}\}$ of A, and thus $\tilde{c} \in C_G^n(X, A)$. By construction $j^*(\tilde{c}) - c$ is zero on $X\backslash W$ and locally zero on $A\backslash B$. Because the interiors of $X\backslash W$ and $A\backslash B$ cover $X\backslash B$, $j^*(c) - c$ is locally zero on $X\backslash B$, that is $j^*(\tilde{c}) = c$ in $\bar{C}_G^*(X\backslash B, A\backslash B)$.

4. The homotopy axiom.

In this section we adapt Spanier's proof of the homotopy axiom for ordinary Alexander-Spanier cohomology, given in [6, §6.5], to the equivariant case. First, following [6, 6.5.2], we represent the equivariant cohomology groups as a direct limit.

Given an open G-covering $\mathscr{U}$ of the G-space X, let $X(\mathscr{U})$ be the following simplicial complex: its set of vertices is $V_0(X)$ and the vertices $\phi_0, \phi_1, \ldots, \phi_n$ span a simplex, if $t(\phi_0) = t(\phi_1) = \ldots = t(\phi_n)$ and $\{\phi_0(eH), \ldots, \phi_n(eH)\} \subset U$ for some $U \in \mathscr{U}$, where $H = t(\phi_0)$. Further, let $C_*(\mathscr{U})$ be the ordered chain complex of $X(\mathscr{U})$, that is $C_n(\mathscr{U})$ is the free abelian group with basis

$$V_n(X, \mathscr{U}) = \{\phi \in V_n(X) \mid \{\phi_0(eH), \ldots, \phi_n(eH)\} \subset U \text{ for some } U \in \mathscr{U} (H = t(\phi))\}$$

and $\partial\colon C_n(\mathscr{U}) \to C_{n-1}(\mathscr{U})$ is given by

$$\partial(\phi_0, \ldots, \phi_n) = \sum_{i=0}^{n} (-1)^i (\phi_0, \ldots, \hat{\phi}_i, \ldots, \phi_n).$$

Let $A \subset X$ be a G-subspace. If $\mathscr{U}' \subset \mathscr{U}$ is a G-subset whose sets cover A, we call $(\mathscr{U}, \mathscr{U}')$ an open G-covering of the pair (X, A). Given such a $(\mathscr{U}, \mathscr{U}')$, let $A(\mathscr{U}')$ be the subcomplex of $X(\mathscr{U})$ consisting of those simplices $s = \{\phi_0, \ldots, \phi_n\}$ which satisfy $\{\phi_0(eH), \ldots, \phi_n(eH)\} \subset A \cap U'$ for some $U' \in \mathscr{U}'$. Also, let $C'_*(\mathscr{U}') \subset C_*(\mathscr{U})$ be the ordered chain complex of $A(\mathscr{U}')$.

We define a cochain complex $C_G^*(\mathscr{U}, \mathscr{U}'; m)$ as follows:

$$C_G^n(\mathscr{U}, \mathscr{U}'; m) = \{u: C_n(\mathscr{U}) \to M \mid u \text{ is a homomorphism};$$
$$u(\phi) \in m(G/t(\phi))\ \forall\ \phi \in V_n(X, \mathscr{U});\ u \mid C_n'(\mathscr{U}') = 0;$$
$$\text{if } \alpha: G/K \to G/t(\phi) \text{ is a } G\text{-map, then } u(\phi \circ \alpha) = m(\alpha)(u(\phi))\};$$

d: $C_G^n(\mathscr{U}, \mathscr{U}') \to C_G^{n+1}(\mathscr{U}, \mathscr{U}')$ is given by $(d(u))(\phi) = u(\partial\phi)$. If $(\mathscr{V}, \mathscr{V}')$ is a refinement of $(\mathscr{U}, \mathscr{U}')$, that is each $V \in \mathscr{V}$ is contained in some $U \in \mathscr{U}$ and each $V' \in \mathscr{V}'$ is contained in some $U' \in \mathscr{U}'$, then there is a canonical injection $(C_*(\mathscr{V}), C_*'(\mathscr{V}')) \hookrightarrow (C_*(\mathscr{U}), C_*'(\mathscr{U}'))$ and a canonical restriction homomorphism $C_G^*(\mathscr{U}, \mathscr{U}') \to C_G^*(\mathscr{V}, \mathscr{V}')$. We form $\varinjlim_{(\mathscr{U}, \mathscr{U}')}$ with respect to these restriction maps.

Let $c \in C_G^n(X, A)$. Then there is a G-set $\mathscr{U}_0'$ of open subsets of X covering A such that c is locally zero on A with respect to $\mathscr{U}_0'$. If we denote $\mathscr{U}_0 = \mathscr{U}_0' \cup \{X\}$, c determines an element $\lambda(c) \in C_G^n(\mathscr{U}_0, \mathscr{U}_0')$ by $(\lambda(c))(\phi) = c(\phi)$, $\phi \in V_n(X, \mathscr{U}_0)$. In this way we obtain a cochain map

$$\lambda: C_G^*(X, A; m) \to \varinjlim C_G^*(\mathscr{U}, \mathscr{U}'; m).$$

The following fact is evident from the definitions:

LEMMA 4.1. *λ is surjective and* $\ker \lambda = C_{G,0}^*(X; m)$.

COROLLARY 4.2. *λ induces an isomorphism*

$$\bar{H}_G^n(X, A; m) \xrightarrow{\ \simeq\ } \varinjlim_{(\mathscr{U}, \mathscr{U}')} H^n(C_G^*(\mathscr{U}, \mathscr{U}'; m)).$$

Let $I = [0, 1]$ be the closed unit interval with trivial G-action. To verify the homotopy axiom, it is enough to consider the G-maps i_0, i_1: $(X, A) \to (X \times I, A \times I)$, $i_0(x) = (x, 0)$, $i_1(x) = (x, 1)$, and prove the following result:

PROPOSITION 4.3. $i_0^* = i_1^*$: $\bar{H}_G^*(X \times I, A \times I; m) \to \bar{H}_G^*(X, A; m)$.

Let $(\mathscr{U}, \mathscr{U}')$ be an open G-covering of $(X \times I, A \times I)$. Given $x \in X$, we can using the compactness of I, find an open neighborhood V_x of x in X such that the following condition holds:

(4.4) there is an $n = n_x \in \mathrm{N}$ with the property that for each $k \in \{0, 1, \ldots, 2^n - 1\}$, $V_x \times [k/2^n, (k+1)/2^n] \subset U_k$ for some $U_k \in \mathscr{U}$ ($U_k \in \mathscr{U}'$ if $x \in A$).

We may assume that V_x is G_x-invariant (replace V_x with $\bigcap_{x \in G_x} g \cdot V_x$, if necessary). If y is in the orbit of x, $y = gx$ with $g \in G$, we let $V_y = g \cdot V_x$; this is independent of the choice of g. Performing this construction for all G-orbits of X, we obtain an open G-covering $\mathscr{V} = \{V_x \mid x \in X\}$ of X, and each $V_x \in \mathscr{V}$ satisfies 4.4. Let also $\mathscr{V}' = \{V_x \mid x \in A\}$. Then $(\mathscr{V}, \mathscr{V}')$ is an open G-covering of (X, A), and the maps i_0 and i_1 induce chain maps

$$i_{0*}, i_{1*}\colon (C_*(\mathscr{V}), C'_*(\mathscr{V}')) \to (C_*(\mathscr{U}), C'_*(\mathscr{U}')).$$

The compatibility condition in the next lemma causes the only additional difficulties compared to the non-equivariant case (cf. [6, 6.5.5]).

LEMMA 4.5. *There is a chain homotopy*

$$h\colon (C_*(\mathscr{V}), C'_*(\mathscr{V}')) \to (C_{*+1}(\mathscr{U}), C'_{*+1}(\mathscr{U}'))$$

from i_{0*} *to* i_{1*} *such that* $h(\phi \circ \alpha) = h(\phi) \circ \alpha$ *whenever* $\phi \in V_n(X, \mathscr{V})$, $t(\phi) = H$ *and* $\alpha\colon G/K \to G/H$ *is a G-map.*

PROOF. Given a simplex s of $X(\mathscr{V})$, let $n(s)$ be the smallest integer n such that for each $k \in \{0, 1, \ldots, 2^n - 1\}$ there is a $U_k \in \mathscr{U}$ containing $\{\phi(eH)\} \times [k/2^n, (k+1)/2^n]$ for all $\phi \in s$ $(H = t(\phi))$; if s is a simplex of $A(\mathscr{V}')$, we require that $U_k \in \mathscr{U}'$.

To construct the chain homotopy we use the method of acyclic models. Let $\mathscr{C}$ be the category whose objects are the subcomplexes of $X(\mathscr{V})$; a morphism $\tau\colon K \to K'$ in $\mathscr{C}$ is a simplicial embedding satifying $n(s) = n(\tau(s))$ for every simplex $s \in K$. If $K \in \operatorname{Ob}\mathscr{C}$, let $F(K) = C_*(K)$ be the ordered chain complex of K. Also let

$$\hat{K} = \bigcup_{s \in K} (s \times I)(\mathscr{U}(s, n(s)))$$

be the subcomplex of $(X \times I)(\mathscr{U})$ defined on p. 314 in [6]; a simplex of $(s \times I)(\mathscr{U}(s, n))$ is a subset

$$\{(\phi_0, t_0), \ldots, (\phi_q, t_q)\} \subset s \times I$$

such that $\{t_0, \ldots, t_q\} \subset [j/2^n, (j+1)/2^n]$ for some $j \in \{0, 1, \ldots, 2^n - 1\}$. Further let $F'(K) = C_*(\hat{K})$ be the ordered chain complex of $\hat{K}$. Then F and F' are functors from $\mathscr{C}$ to the category of augmented chain complexes over Z.

We define an equivalence relation $\sim$ on the set of simplices of $X(\mathscr{V})$ by

$$s \sim t \Leftrightarrow \text{there is a simplicial isomorphism } \tau\colon \bar{s} \to \bar{t} \text{ satisfying } n(s') = n(\tau(s')) \text{ for every } s \leqslant s;$$

here $\bar{s}$ is the simplicial complex whose simplices are the faces of s. Let S be a set consisting of one representative from each equivalence class, and define

$\mathcal{M} = \{\bar{s} \mid s \in S\} \subset \mathrm{Ob}\,\mathscr{C}$. Then the functor F is free with models $\mathcal{M}$. On the other hand, by [6, 6.5.3], $\tilde{C}_*(\hat{s})$ is acyclic for every $\bar{s} \in \mathcal{M}$ and so the functor F' is acyclic on the models $\mathcal{M}$. The maps i_0 and i_1 induce natural chain maps $i_{0*}, i_{1*}: F \to F'$ preserving argumentation and hence, by the acyclic model theorem (see [6, 4.3.3]), there is a natural chain homotopy h from i_{0*} to i_{1*}.

We contend that the chain homotopy

$$h = h(X(\mathscr{V})): C_*(\mathscr{V}) \to C_{*+1}((X(\mathscr{V}))^{\wedge}) \hookrightarrow C_{*+1}(\mathscr{U})$$

satisfies the required conditions. It is clear by naturality that $h(C'_*(\mathscr{V}')) \subset C'_{*+1}(\mathscr{U}')$. To prove the compatibility condition, assume that $\phi = (\phi_0, \ldots, \phi_n) \in V_n(X, \mathscr{V})$, $t(\phi) = H$ and α: $G/K \to G/H$ is a G-map. Then $s = \{\phi_0, \ldots, \phi_n\}$ and $s \circ \alpha = \{\phi_0 \circ \alpha, \ldots, \phi_n \circ \alpha\}$ are both q-simplices of $X(\mathscr{V})$, where $q = \mathrm{card}(s) - 1$, and we can define a simplicial isomorphism $\tilde{\alpha}$: $\bar{s} \to \overline{s \circ \alpha}$ by $\phi_i \mapsto \phi_i \circ \alpha$. If $\alpha(eK) = gH$, $g \in G$, then $(\phi_i \circ \alpha)(eK) = g \cdot \phi_i(eH)$ for $i \in \{0, \ldots, n\}$; because $\mathscr{U}$ is a G-covering, it follows that $n(s') = n(\tilde{\alpha}(s'))$ for every $s' \leqslant s$, that is $\tilde{\alpha} \in \mathrm{Mor}\,\mathscr{C}$. The formula $h(\phi \circ \alpha) = h(\phi) \circ \alpha$ now follows from the commutativity of the square

$$\begin{array}{ccc} F(\bar{s}) & \xrightarrow{h} & F'(\bar{s}) \\ \downarrow{\scriptstyle F(\tilde{\alpha})} & & \downarrow{\scriptstyle F'(\tilde{\alpha})} \\ F(\overline{s \circ \alpha}) & \xrightarrow{h} & F'(\overline{s \circ \alpha}) \end{array}$$

PROOF OF 4.3. Consider an element of

$$\bar{H}^n_G(X, A) \cong \varinjlim \mathrm{H}^n(C^*_G(\mathscr{U}, \mathscr{U}'))$$

represented by a cocycle $u \in C^n_G(\mathscr{U}, \mathscr{U}')$ where $(\mathscr{U}, \mathscr{U}')$ is an open G-covering of $(X \times I, A \times I)$. We construct the open G-covering $(\mathscr{V}, \mathscr{V}')$ of (X, A) as before Lemma 4.5 above and let

$$h: (C_*(\mathscr{V}), C'_*(\mathscr{V}')) \to (C_{*+1}(\mathscr{U}), C'_{*+1}(\mathscr{U}'))$$

be the chain homotopy given by 4.5. Let h^*u: $C_{n-1}(\mathscr{V}) \to M$ be defined by $(h^*u)(\phi) = u(h(\phi))$ for $\phi \in V_{n-1}(X, \mathscr{V})$. If α: $G/K \to G/t(\phi)$ is a G-map, then

$$(h^*u)(\phi \circ \alpha) = u(h(\phi) \circ \alpha) = m(\alpha)((h^*u)(\phi)),$$

and it follows that $h^*u \in C^{n-1}_G(\mathscr{V}, \mathscr{V}')$. Furthermore we have for $\phi \in V_n(X, \mathscr{V})$

$$\begin{aligned} d(h^*u)(\phi) &= (h^*u)(\partial\phi) = u(h\partial\phi) \\ &= u(i_{1*}\phi - i_{0*}\phi - \partial h\phi) \\ &= (i_1^*u)(\phi) - (i_0^*u)(\phi) - (du)(h\phi) \\ &= (i_1^*u)(\phi) - (i_0^*u)(\phi), \end{aligned}$$

that is $i_1^*u - i_0^*u = d(h^*u) \in C_G^n(\mathscr{V}, \mathscr{V}')$ is a coboundary. Thus $i_0^*u = i_1^*u$ in $\bar{H}_G^n(X, A) \cong \varinjlim H^n(C_G^*(\mathscr{V}, \mathscr{V}'))$.

5. Tautness.

Let X be a paracompact G-space and $A \subset X$ a closed G-subspace. If N is a G-neighborhood of A, we have the restriction homomorphism $\bar{H}_G^n(N; m) \to \bar{H}_G^n(A; m)$. These homomorphisms for various N determine a morphism

$$\varinjlim_N \bar{H}_G^n(N; m) \longrightarrow \bar{H}_G^n(A; m).$$

PROPOSITION 5.1. *This morphism is an isomorphism for all n.*

The proof can be carried out in exactly the same way as that of [6, 6.6.2], with aid of the following two lemmas:

LEMMA 5.2. *Every open G-covering $\mathscr{U}$ of a paracompact G-space X has an open star refinement, which is also a G-covering.*

PROOF. We can first find an open G-covering $\mathscr{U}'$, which is a locally finite refinement of $\mathscr{U}$ (cf. [1, p. 133]). If we then construct a star refinement $\mathscr{V}$ of $\mathscr{U}'$ in the usual way ([2, p. 167]), $\mathscr{V}$ is clearly a G-covering.

LEMMA 5.3. *Let X be a completely regular* (e.g. *paracompact*) *G-space and $A \subset X$ a G-subspace. Given an open G-covering $\mathscr{V}$ of X, there is an open G-neighborhood N of A and an equivariant function f: $N \to A$* (*not necessarily continuous*) *satisfying*
a) *$f(x) = x$ for $x \in A$, and*
b) if *$V \in \mathscr{V}$, then $f(V \cap N) \subset V^*$;*
here $V^* = \bigcup \{V' \in \mathscr{V} \mid V' \cap V \neq \emptyset\}$ is the star of $V \in \mathscr{V}$.

PROOF. If $V \in \mathscr{V}$ and $a \in A \cap V$, we can, by the Slice Theorem (see [2, II.5.4]), find an open neighborhood U_a of a such that $U_a \subset V$ and for all $x \in U_a$ there is a G-map $Gx \to Ga$ with $x \mapsto a$. We define

$$V' = \bigcup_{a \in A \cap V} U_a \subset V, \quad N' = \bigcup \{V' \mid A \cap V \neq \emptyset\}, \quad N = \bigcap_{g \in G} gN'.$$

Then N is an open G-neighborhood of A.

We now construct f: $N \to A$. Set $f(x) = x$ for $x \in A$. To define $f \mid N \setminus A$, let S be a set of representatives for the G-orbits of $N \setminus A$. Let $y \in S$ and choose $V \in \mathscr{V}$ such that $A \cap V \neq \emptyset$ and $y \in V'$ (notation as in the preceding paragraph). Then $y \in U_a \subset V'$ for some $a \in A \bigcap V$, and we may choose a G-map f_y: $Gy \to Ga$ with $f_y(y) = a$. We define $f \mid Gy = f_y$.

We must show that $f\colon N \to A$ satisfies a) and b). Condition a) is clear by definition. To prove b), assume that $W \in \mathscr{V}$ and $x \in W \cap N$; we claim that $f(x) \in W^*$. This obvious if $x \in A$. Let then $x \in N \setminus A$, $x = gy$ with $y \in S$, $g \in G$. Let V, V', and U_a be as above. Now $x = gy \in (gV) \cap W$, whence $(gV) \cap W \neq \emptyset$, and because $gV \in \mathscr{V}$, we have $gV \in W^*$. On the other hand $f(x) = g \cdot f(y) = g \cdot a \in gV$, so $f(x) \in W^*$.

REMARK 5.4. The equivariant versions of [6, 6.6.3, 6.6.5, and 6.6.6] are obviously true, too.

6. A connection with sheaf cohomology.

Let X be a paracompact G-space. Then the orbit space X/G is also paracompact, as can be seen directly from the definition. In this section we prove that $\bar{H}^*_G(X; m)$ equals the ordinary cohomology of X/G with coefficients in a suitable sheaf; the main result is Theorem 6.4. Two simple applications follow: in Corollary 6.6 we show that if X has finite covering dimension, then it has finite cohomological dimension with respect to $\bar{H}^*_G(\cdot\,; m)$; in Corollary 6.8 we show that if the coefficient system m is constant, then $\bar{H}^*_G(X; m)$ is the ordinary Alexander cohomology of X/G. In section 7 we use the sheaf theoretic interpretation to compare $\bar{H}^*_G(X; m)$ with equivariant singular cohomology.

Let $\pi\colon X \to X/G$ be the canonical projection. For each $n \in N$ we define a presheaf M^n on X/G by $M^n(U) = \bar{C}^n_G(\pi^{-1}U; m)$, $U \subset X/G$ open, with obvious restriction maps. Further, let $\bar{\mathscr{C}}^n_G$ be the sheaf associated to the presheaf M^n. The coboundary maps $d\colon \bar{C}^n_G(\pi^{-1}U) \to \bar{C}^{n+1}_G(\pi^{-1}U)$ define morphisms $d\colon M^n \to M^{n+1}$ and $d\colon \bar{\mathscr{C}}^n_G \to \bar{\mathscr{C}}^{n+1}_G$.

The following three lemmas are needed for theorem 6.4:

LEMMA 6.1. *The sheaves $\bar{\mathscr{C}}^n_G$ are fine.*

LEMMA 6.2. *The sequence $\bar{\mathscr{C}}^0_G \xrightarrow{d} \bar{\mathscr{C}}^1_G \xrightarrow{d} \bar{\mathscr{C}}^2_G \longrightarrow \ldots$ is exact.*

LEMMA 6.3. $\Gamma(X/G, \bar{\mathscr{C}}^n_G) = \bar{C}^n_G(X; m)$.

PROOF OF 6.1. This is entirely similar to [7, Proposition 3, p. 84]. It is enough to show that the presheaf M^n is fine. Let $\{U_\alpha \mid \alpha \in I\}$ be a locally finite open covering of X/G. For each $y \in X/G$ choose $\alpha_y \in I$ with $y \in U_{\alpha_y}$. For $\alpha \in I$ define w_α: $X/G \to \{0, 1\}$ by

$$w_\alpha(y) = \begin{cases} 1, & \alpha = \alpha_y \\ 0, & \alpha \neq \alpha_y. \end{cases}$$

If $U \subset X/G$ is open, define $l_\alpha\colon \bar{C}^n_G(\pi^{-1}U) \to \bar{C}^n_G(\pi^{-1}U)$ by

$$(l_\alpha(c))(\phi) = w_\alpha(\pi(\phi_0(eH)))\cdot c(\phi), \qquad \phi \in V_n(\pi^{-1}U),\ H = t(\phi).$$

The l_α's determine morphisms of presheaves $l_\alpha\colon M^n \to M^n$ with the required properties, that is $\operatorname{supp}(l_\alpha) \subset \bar{U}_\alpha$ and $\sum_{\alpha\in I} l_\alpha = \mathrm{id}$.

Proof of 6.2. We must show that for each $y \in X/G$, the sequence of stalks

$$(\bar{C}^0_G)_y \xrightarrow{d} (\bar{C}^1_G)_y \xrightarrow{d} (\bar{C}^2_G)_y \longrightarrow \cdots$$

is exact, that is $H^n((C^*_G)_y) = 0$ for $n > 0$. But, by applying the tautness result 5.1 to the closed G-subspace $\pi^{-1}(y) \subset X$, we can reduce this to showing that $\bar{H}^n_G(\pi^{-1}(y); m) = 0$ for $n > 0$, which is true by the dimension axiom.

Proof of 6.3. We must prove that the canonical homomorphism $M^n(X/G) \to \Gamma(X/G, \bar{C}^n_G)$ is an isomorphism. It follows easily from the definition of M^n that the presheaf M^n has no locally zero global sections except 0. Therefore $M^n(X/G) \to \Gamma(X/G, \bar{C}^n_G)$ is injective, and it remains to prove its surjectivity.

Let $s \in \Gamma(X/G, \bar{C}^n_G)$. By [7, Lemma 2 on p. 81], we find a locally finite open covering $\{U'_\alpha \mid \alpha \in I\}$ of X/G and sections $s'_\alpha \in M^n(U'_\alpha)$ such that

$$s'_\alpha \mapsto s \mid U'_\alpha \in \Gamma(U'_\alpha, \bar{C}^n_G) \qquad \text{for all } \alpha \in I$$

and

$$s'_\alpha \mid U'_\alpha \cap U'_\beta = s'_\beta \mid U'_\alpha \cap U'_\beta \in M^n(U'_\alpha \cap U'_\beta) \qquad \text{for all } \alpha, \beta \in I.$$

Let s'_α be represented by $c'_\alpha \in C^n_G(\pi^{-1}U'_\alpha)$; then $c'_\alpha - c'_\beta$ is locally zero on $\pi^{-1}(U'_\alpha \cap U'_\beta)$ with respect to an open G-covering $\mathscr{V}_{\alpha\beta}$ of $\pi^{+1}(U'_\alpha \bigcap U'_\beta)$.

Choose an open covering $\{U_\alpha \mid \alpha \in I\}$ of X/G satisfying $\bar{U}_\alpha \subset U'_\alpha$ for $\alpha \in I$. Let

$$s_\alpha = s'_\alpha \mid U_\alpha \in M^n(U_\alpha) \quad \text{and} \quad c_\alpha = c'_\alpha \mid \pi^{-1}U_\alpha \in C^n_G(\pi^{-1}U_\alpha).$$

Then $s_\alpha \mapsto s \mid U_\alpha \in \Gamma(U_\alpha, \bar{C}^n_G)$.

Let $x \in X$, $y = \pi(x) \in X/G$. We pick an open neighborhood W_y of y in X/G such that $\{\alpha \in I \mid U'_\alpha \cap W_y \neq \varnothing\}$ is finite, and $W_y \cap \bar{U}_\alpha \neq \varnothing$ only if $y \in \bar{U}_\alpha$. For all $(\alpha, \beta) \in I \times I$ with $y \in U'_\alpha \cap U'_\beta$ choose a $V_{\alpha\beta} \in \mathscr{V}_{\alpha\beta}$ such that $x \in V_{\alpha\beta}$. Define

$$V_y = \pi^{-1}(W_y) \cap \bigcap_{(\alpha,\beta)} V_{\alpha\beta},$$

an open neighborhood of x in X.

Let $\phi = (\phi_0, \ldots, \phi_n) \in V_n(X)$, $t(\phi) = H$, and denote $x_i = \phi_i(eH)$, $i = 0, 1, \ldots, n$. Assume that $\{x_0, \ldots, x_n\} \subset V_y$. Let $\alpha, \beta \in I$ and suppose that c_α and c_β are both "defined on ϕ", that is

$$\phi \in V_n(\pi^{-1}U_\alpha) \cap V_n(\pi^{-1}U_\beta).$$

Then $\{x_0, \ldots, x_n\} \subset V_y \cap \pi^{-1}(U_\alpha \cap U_\beta)$. In particular $W_y \cap U_\alpha \cap U_\beta \neq \emptyset$, and the choice of W_y implies that $y \in \bar{U}_\alpha \cap \bar{U}_\beta \subset U'_\alpha \cap U'_\beta$. Therefore $\{x_0, \ldots, x_n\} \subset V_{\alpha\beta} \in \mathscr{V}_{\alpha\beta}$, whence $c_\alpha(\phi) = c_\beta(\phi)$.

We denote $\mathscr{V} = \{gV_y \mid g \in G,\ y \in X/G\}$, an open G-covering of X. The preceding paragraph shows that the c_α's together determine a well-defined element $\tilde{c} \in C^n_G(\mathscr{V}; m)$. The class of $\tilde{c}$ is an element

$$\sigma \in M^n(X/G) = \bar{C}^n_G(X; m) \cong \varinjlim_{\mathscr{U}} C^n_G(\mathscr{U}; m),$$

and by construction $\sigma \mid U_\alpha = s_\alpha \in M^n(U_\alpha)$ for every $\alpha \in I$. It follows that $M^n(X/G) \ni \sigma \mapsto s \in \Gamma(X/G, \bar{C}^n_G)$.

We define $A = \ker[d\colon \bar{C}^0_G \to \bar{C}^1_G]$, a sheaf on X/G.

THEOREM 6.4. *$H^n(X/G; A) \cong \bar{H}^n_G(X; m)$ for all $n \in \mathbf{N}$.*

PROOF. By Lemmas 6.1 and 6.2, $\bar{C}^*_G$ is a fine resolution of A. Therefore

$$H^n(X/G; A) \cong H^n(\Gamma(X/G, \bar{C}^*_G)).$$

By Lemma 6.3, this equals $\bar{H}^n_G(X; m)$.

Next we give a more concrete description of the sheaf A. We call a 0-cochain $c \in C^0_G(X; m)$ *locally constant*, if there exists an open G-covering $\mathscr{V}$ of X with the property that $c(\phi) = c(\phi')$ for G-maps $\phi, \phi'\colon G/H \to X$ whenever $\phi(eH), \phi'(eH) \in V$ for some $V \in \mathscr{V}$.

PROPOSITION 6.5. a) *If $U \subset X/G$ is open, then*
$$\Gamma(U, A) = \{c \in C^0_G(\pi^{-1}U; m) \mid c \text{ is locally constant}\}.$$
b) *If $y \in X/G$, $x \in \pi^{-1}(y)$ and $H = G_x \leqq G$, then the stalk A_y is isomorphic to $m(G/H)$.*

PROOF. Define $A' = \ker[d\colon M^0 \to M^1]$, a presheaf on X/G. Let $U \subset X/G$ be open. Then

$$A'(U) = \ker[d\colon C^0_G(\pi^{-1}U) \to \bar{C}^1_G(\pi^{-1}U)],$$

and a cochain $c \in C^0_G(\pi^{-1}U)$ is in $A'(U)$ if and only if $d(c) \in C^1_G(\pi^{-1}U)$ is locally zero which, due to the formula $(d(c))(\phi, \phi') = c(\phi') - c(\phi)$, means that c is locally constant. Thus $A'(U) = \{c \in C^0_G(\pi^{-1}U) \mid c \text{ is locally constant}\}$.

Because sheafification is an exact functor, A is the sheaf associated to the presheaf A'. On the other hand, the above formula for $A'(U)$ shows that A' is already a sheaf, so $A = A'$.

b) If $U \subset X/G$ is an open neighborhood of y, let $\phi_x\colon G/H \to \pi^{-1}U$ be the G-map $gH \mapsto gx$; we can then define a homomorphism $\gamma_U\colon \Gamma(U, A) \to m(G/H)$ by

$\gamma_U(c) = c(\phi_x)$. The γ_U's together define a homomorphism

$$\gamma\colon A_y = \varinjlim_{U \ni y} \Gamma(U, A) \to m(G/H).$$

We claim that γ is an isomorphism.

Let V be a tube around the orbit $\pi^{-1}(y) = Gx$ and $r\colon V \to Gx$ a G-retraction. We may choose V so small that $V = \coprod_{gH \in G/H} gV_x$ (disjoint union), where V_x is an open H-neighborhood of x. Then $r(gV_x) = \{gx\}$ for every $g \in G$.

To prove the surjectivity of γ, let $a \in m(G/H)$. We define $c \in C_G^0(V)$ by $c(\phi) = m(r \circ \phi)(a) \in m(G/K)$ for a G-map $\phi\colon G/K \to V$. Then c is locally constant with respect to the open G-covering $\{gV_x \mid gH \in G/H\}$ of V, so $c \in \Gamma(\pi V, A)$, and clearly $\gamma_{\pi V}\colon c \mapsto a$. Thus $\gamma_{\pi V}$ is surjective, and consequently so is γ.

For injectivity, let $c \in \Gamma(U, A)$, where U is an open neighborhood of y, and assume that $\gamma_U(c) = c(\phi_x) = 0$. If $x' = gx \in Gx$, then $G_{x'} = gG_xg^{-1} = gHg^{-1}$, and the G-map $\phi_{x'}\colon G/gHg^{-1} \to \pi^{-1}U$, $u\cdot(gHg^{-1}) \mapsto ux'$, has the factorization

$$\phi_{x'}\colon G/gHg^{-1} \xrightarrow{\alpha} G/H \xrightarrow{\phi_x} Gx,$$

where $\alpha\colon u\cdot(gHg^{-1}) \mapsto ugH$. Thus $c(\phi_{x'}) = m(\alpha)(c(\phi_x)) = 0$.

By assumption, there is an open G-covering $\mathscr{W}$ of $\pi^{-1}U$ such that c is locally constant with respect to $\mathscr{W}$. We choose an open H-neighborhood $W_x \subset V_x$ of x, which is contained in some member of $\mathscr{W}$. Then πW_x is an open neighborhood of y, and we shall show that $c \mid \pi W_x = 0 \in \Gamma(\pi W_x, A)$.

Let $\phi\colon G/K \to \pi^{-1}\pi W_x$ be a G-map. Then $\phi(eK) \in gW_x$ for some $g \in G$. Because $r(gW_x) = \{gx\}$, the composite

$$\phi' = r \circ \phi\colon G/K \to \pi^{-1}\pi W_x \to Gx \hookrightarrow \pi^{-1}\pi W_x$$

satisfies

$$\phi'(uK) = ugx = \phi_{gx}(u\cdot(gHg^{-1})) \quad \text{for } u \in G.$$

Therefore $K \leqq gHg^{-1}$ and $\phi' = \phi_{gx} \circ \beta$, $\beta\colon G/K \to G/gHg^{-1}$ canonical surjection. We saw above that $c(\phi_{gx}) = 0$, and thus

$$c(\phi') = m(\beta)(c(\phi_{gx})) = 0.$$

Finally $\phi(eK), \phi'(eK) \in gW_x \subset W$ for some $W \in \mathscr{W}$, whence $c(\phi) = c(\phi') = 0$.

We now present the two simple applications of Theorem 6.4 referred to in the first paragraph of this section. Recall that X is a paracompact G-space.

COROLLARY 6.6. *If the covering dimension of X/G is finite, then $\bar{H}_G^n(X; m) = 0$ for $n > \dim(X/G)$.*

PROOF. This follows from the well-known fact that the sheaf cohomology $H^n(X/G; A)$ vanishes for $n > \dim(X/G)$, cf. [3, p. 236].

REMARK 6.7. The condition $\dim(X/G) < \infty$ holds if $\dim X < \infty$. Namely it is easy to see that $\dim X \leqq k$ implies $\dim(X/G) \leqq (k+1)|G| - 1$.

Let M be an abelian group, considered as the constant coefficient system $G/M \mapsto M$, each G-map $G/H \to G/K$ inducing the identity $M \to M$.

COROLLARY 6.8. *There is a natural isomorphism* $\bar{H}^n_G(X; M) \cong \bar{H}^n(X/G; M)$, *where the right hand side is the ordinary Alexander-Spanier cohomology of* X/G.

PROOF. Both sides of the asserted isomorphism can be calculated as sheaf cohomology groups of X/G, the left hand side with coefficients A as in Theorem 6.4, and the right hand side with constant coefficients M. We obtain a morphism of sheaves ω: $M \to A$ if we define for $U \subset X/G$ open

$$\begin{array}{ccc} \Gamma(U, M) & \xrightarrow{\quad\omega\quad} & \Gamma(U, A) \\ \| & & \| \\ \{f\colon U \to M \mid f \text{ locally constant}\} & & \{c \in C^0_G(\pi^{-1}U; M) \mid c \text{ locally constant}\} \end{array}$$

by the formula $(\omega(f))(\phi) = f(\pi\phi(eH))$, $\phi\colon G/H \to \pi^{-1}U$ a G-map. The stalks of both A and M equal M, and obviously ω induces the identity on stalks. Thus ω is an isomorphism.

7. Comparison with equivariant singular cohomology.

Let X be a G-space. In this section we show that, under suitable local conditions on X, the equivariant Alexander-Spanier cohomology groups $\bar{H}^n_G(X; m)$ are isomorphic to the equivariant singular cohomology groups $H^n_G(X; m)$.

Let $S^*_G(X; m)$ be the equivariant singular cochain complex of X with coefficients m, as defined in [4]; then $H^n_G(X; m) = H^n(S^*_G(X; m))$. We call a cochain $c \in S^n_G(X; m)$ *locally zero*, if there is an open G-covering $\mathscr{V}$ of X such that $c(\sigma) = 0$ for any equivariant singular simplex $\sigma\colon G/H \times \Delta^n \to X$ for which $\sigma(\{eH\} \times \Delta^n)$ is contained in some $V \in \mathscr{V}$. We denote by $S^*_{G,0}(X; m) \subset S^*_G(X; m)$ the cochain subcomplex of locally zero cochains. Let also $\bar{S}^*_G(X; m) = S^*_G(X; m)/S^*_{G,0}(X; m)$.

LEMMA 7.1. *The complex* $S^*_{G,0}(X; m)$ *is acyclic and hence the canonical surjection* $S^*_G(X; m) \to \bar{S}^*_G(X; m)$ *induces an isomorphism in cohomology*

$$H^*_G(X; m) \xrightarrow{\sim} H^*(\bar{S}^*_G(X; m)).$$

PROOF. Given an open G-covering $\mathscr{V}$ of X, let $S^*_G(X; m; \mathscr{V})$ be the cochain complex of cochains defined on singular simplices $\sigma\colon G/H \times \Delta^n \to X$ for which $\sigma(\{eH\} \times \Delta^n\}$ is contained in some $V \in \mathscr{V}$ (compare with [4, p. 34]; Illman uses only coverings by open G-subsets and requires that $\sigma(G/H \times \Delta^n) \subset V$ for some $V \in \mathscr{V}$). The proof of Proposition I.6.4 in [4] shows that the canonical morphism $S_G^*(X; m) \to S_G^*(X; m; \mathscr{V})$ is a homotopy equivalence. Therefore

$$K_{\mathscr{V}}^* = \ker [S_G^*(X; m) \to S_G^*(X; m; \mathscr{V})]$$

is acyclic. The assertion follows from this, for clearly $S_{G,0}^*(X; m) = \varinjlim_{\mathscr{V}} K_{\mathscr{V}}^*$.

We can define a natural cochain map $\lambda\colon C_G^*(X; m) \to S_G^*(X; m)$ as follows: given $c \in C_G^n(X; m)$, $\lambda(c) \in S_G^n(X; m)$ is defined by $(\lambda(c))(\sigma) = (\sigma_0, \ldots, \sigma_n)$, where σ: $G/H \times \Delta^n \to X$ is an equivariant singular simplex,

$$\sigma_i = \sigma \mid G/H \times \{v_i\}\colon G/H \to X$$

and $v_0, \ldots, v_n$ are the vertices of the standard simplex Δ^n. The morphism λ induces: $\lambda\colon \bar{C}_G^*(X; m) \to \bar{S}_G^*(X; m)$. Passing to cohomology and using Lemma 7.1 we obtain a natural transformation $\Lambda\colon \bar{H}_G^*(X; m) \to H_G^*(X; m)$.

THEOREM 7.2. *Λ: $\bar{H}_G^*(X; m) \to H_G^*(X; m)$ is an isomorphism provided that X is paracompact and every orbit $Gx \subset X$ is taut with respect to $H_G^*(\cdot\,; m)$.*

PROOF. Let S_G^n and $\bar{S}_G^n$ be the sheaves on X/G associated to the presheaves $U \mapsto S_G^n(\pi^{-1}U; m)$ and $U \mapsto \bar{S}_G^n(\pi^{-1}U; m)$, respectively. In section 6 we proved that the sequence $\bar{C}_G^0 \to \bar{C}_G^1 \to \bar{C}_G^2 \to \ldots$ is exact, the sheaves $\bar{C}_G^n$ are fine and $\bar{H}_G^*(X; m)$ can be computed from the complex $\bar{C}_G^*\colon \bar{H}_G^n(X; m) = H^n(\Gamma(X/G, \bar{C}_G^*))$.

On the other hand, under the present additional hypothesis it is known that $S_G^0 \to S_G^1 \to S_G^2 \to \ldots$ is an exact sequence of fine sheaves and $H_G^*(X, m)$ can be computed from the complex $S_G^*\colon H_G^n(X; m) = H^n(\Gamma(X/G, S_G^*))$, cf. [5, p. 441–442].

The above morphism $\bar{\lambda}$ defines a morphism of sheaf complexes $\bar{\lambda}\colon C_G^* \to \bar{S}_G^*$; also we have the canonical morphism $S_G^* \to \bar{S}_G^*$, which is in fact a quasi-isomorphism by Lemma 7.1. But now is clear that

$$\bar{C}_G^* \xrightarrow{\bar{\lambda}} \bar{S}_G^* \longleftarrow S_G^*$$

is a quasi-isomorphism. Namely, $\mathscr{H}^n(\bar{C}_G^*) = \mathscr{H}^n(S_G^*) = 0$ for $n > 0$, and if $y \in X/G$, then

$$[\mathscr{H}^0(\bar{C}_G^*)]_y \cong \bar{H}_G^0(\pi^{-1}(y); m)$$

by Proposition 5.1 and

$$[\mathscr{H}^0(S_G^*)]_y \cong H_G^0(\pi^{-1}(y); m)$$

by the tautness assumption in Theorem 7.2, whence $\bar{\lambda}$ induces an isomorphism between the stalks $[\mathscr{H}^0(\bar{C}_G^*)]_y$ and $[\mathscr{H}^0(S_G^*)]_y$. This proves the assertion.

REMARK 7.3. The tautness hypothesis in Theorem 7.2 is satisfied, if for example X is G-locally contractible, i.e. every orbit $Gx \subset X$ has arbitrarily small open G-neighborhoods V such that Gx is a G-deformation retract of V.

8. Remarks on cohomology with compact supports.

In this final section we indicate briefly, how the construction of ordinary Alexander-Spanier cohomology with compact supports given in [6, p. 320], can be generalized to the equivariant case.

Let us recall the following terminology: A subset Z of a topological space X is *bounded* if $\bar{Z}$ is compact, and *cobounded* if $X \backslash Z$ is bounded. A map $f: X \to Y$ is *proper* if $f^{-1}(Z) \subset X$ is bounded whenever $Z \subset Y$ is bounded.

Let now (X, A) be a G-pair. We define a cochain subcomplex $C_{G,c}^*(X, A; m)$ of $C_G^*(X, A; m)$ by

$$C_{G,c}^n(X, A; m) = \{c \in C_G^n(X, A; m) \mid c \text{ is locally zero on some cobounded } G\text{-subset of } X\}.$$

Clearly $C_{G,0}^*(X) \subset C_{G,c}^*(X, A)$, and we denote

$$\bar{C}_{G,c}^*(X, A; m) = C_{G,c}^*(X, A; m)/C_{G,0}^*(X; m).$$

The *equivariant Alexander-Spanier cohomology groups* of (X, A) *with compact supports* and coefficients m are

$$\bar{H}_{G,c}^n(X, A; m) = H^n(\bar{C}_{G,c}^*(X, A; m)) \tag{8.1}$$

A map of G-pairs $f: (X, A) \to (Y, B)$ such that $f: X \to Y$ is proper induces homomorphisms

$$f^*: \bar{H}_{G,c}^n(Y, B) \to \bar{H}_{G,c}^n(X, A).$$

There is a long exact cohomology sequence for the G-pair (X, A) only if A is closed in X. The other axioms for an equivariant cohomology theory are also satisfied with obvious modifications. For instance, the homotopy axioms holds for proper G-homotopies. A proof of this can be based on Lemma 4.5 in exactly the same way as the proof of 4.3, after the trivial observation that if $Z \subset X \times I$ is cobounded and $p: X \times I \to X$ is the projection, then $Z' = X \backslash p[(X \times I) \backslash Z]$ is cobounded in X and $Z' \times I \subset Z$.

The analog of [6, 6.6.11] evidently holds, too:

PROPOSITION 8.2. *If A is a compact (and Hausdorff) G-space and $B \subset A$ is a closed G-subspace, then for all $n \in \mathbf{N}$*

$$\bar{H}^n_{G,c}(A \setminus B; m) \cong \bar{H}^n_G(A, B; m).$$

There is the following connection between $\bar{H}^*_{G,c}(X; m)$ and sheaf cohomology of X/G with compact supports:

PROPOSITION 8.3. *If X is a locally compact G-space, then*

$$H^n_c(X/G; A) \cong \bar{H}^n_{G,c}(X; m)$$

for all $n \in \mathbf{N}$, where A is the sheaf on X/G described in Proposition 6.5.

PROOF. Since X/G is locally compact, we can use the fine resolution $\bar{C}^*_G$ to compute $H^n_c(X/G; A)$:

$$H^n_c(X/G; A) = H^n(\Gamma_c(X/G, \bar{C}^*_G)).$$

In Lemma 6.3 we proved that $\Gamma(X/G, \bar{C}^n_G) \cong \dot{\bar{C}}^n_G(X; m)$. Thus it only remains to show that in this isomorphism $\Gamma_c(X/G, \bar{C}^n_G)$ corresponds to $\bar{C}^n_{G,c}(X; m)$.

Let $c \in C^n_G(X; m)$ represent $s \in \Gamma(X/G, \bar{C}^n_G)$. If $s \in \Gamma_c(X/G, \bar{C}^n_G)$, then $X \setminus \pi^{-1}(\text{supp}(s))$ is cobounded in X and c is locally zero on $X \setminus \pi^{-1}(\text{supp}(s))$. Therefore $c \in C^n_{G,c}(X; m)$ in this case. Conversely, assume that $c \in C^n_{G,c}(X; m)$ is locally zero on the cobounded G-set $Z \subset X$. Then $\text{supp}(s)$ is contained in the compact set $\pi(\overline{X \setminus Z})$ and thus $s \in \Gamma_c(X/G, \bar{C}^n_G)$.

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