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ON π-REGULAR RINGS WITH NO INFINITE TRIVIAL SUBRING

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A ring R is called *π-regular* if for every x in R there exists a positive integer n (depending on x) and an element y of R such that $x^n = x^n y x^n$. A π-regular ring R for which the n in the above can be taken to be 1 for all x is called (*von Neumann*) *regular*. By a *trivial subring* of R we mean a subring S of R with $S^2 = 0$. Further a ring R is *orthogonally finite* if R has no infinite set of mutually orthogonal idempotents. In this paper, we shall generalize some results in [1] and [6]. We need the following lemma, which was proved in [2] for infinite alternative nil rings.

LEMMA 1. *Let R be an infinite nil ring. Then R has an infinite subring S with $S^2 = 0$.*

It is well known that the class of π-regular rings properly contains the class of strongly π-regular rings. Thus the following lemma slightly improves [1, Theorem 3].

LEMMA 2. *A ring R is an orthogonally finite π-regular ring with no infinite trivial subring if and only if $R = F \oplus S$, where F is a finite ring and S is a finite product of division rings.*

PROOF. It suffices to prove the only if part. Since R is π-regular, the Jacobson radical J of R is nil. Hence J is finite by Lemma 1. By [4, Theorem 2.1], R/J is Artinian semisimple. Using Lemma 1, we can easily see that $R/J = T \oplus D_1 \oplus \ldots \oplus D_n$ for some n, where T is a finite ring and D_i is an infinite division ring for each i. Let $e_1, e_2, \ldots, e_n$ be mutually orthogonal idempotents of R such that $e_i + J$ is the identity of D_i for each i. Clearly e_1Re_1 is a local ring with $e_1Re_1/e_1Je_1 \cong D_1$. Suppose now that $e_1Je_1 \neq 0$. Since J is a finite nil ring there exists a positive integer m such $(e_1Je_1)^m \neq 0$ and $(e_1Je_1)^{m+1} = 0$. Then the trivial subring $(e_1Re_1)^m$ is a non-zero vector space over the infinite division ring D_1, a contradiction. Hence e_1Re_1 is a division ring which is isomorphic to D_1. Since the trivial subring

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$e_1R(1-e_1) = \{e_1x - e_1xe_1 \mid x \in R\}$ is a left vector space over e_1Re_1, we obtain that $e_1R(1-e_1) = 0$. Similarly we have that $(1-e_1)Re_1 = 0$. These imply that e_1 is central. Similarly we can see that $e_2, \ldots, e_n$ are central. If we set $F = R(1 - e_1 - \ldots - e_n)$ and $S = Re_1 \oplus \ldots \oplus Re_n$, then we obtain the desired decomposition $R = F \oplus S$.

A ring R is said to be *strongly regular* if R is regular and each idempotent in R is central, or equivalently, if R is a regular ring with no non-zero nilpotent element.

LEMMA 3. *Let I be an ideal of a ring R. If I and R/I are strongly regular, then R is strongly regular.*

PROOF. By [5, Theorem 22, p. 112], R is regular. We can easily see that R has no non-zero nilpotent element. Hence R is strongly regular.

Using Lemma 3, we can easily see that any ring R has a unique largest strongly regular ideal M, and R/M has no non-zero strongly regular ideal.

We can now prove the main theorem in this paper.

THEOREM 1. *If R is a π-regular ring with no infinite trivial subring, then R has a strongly regular ideal M such that R/M is a finite ring.*

PROOF. Let M be the largest strongly regular ideal of R. We first show that every nilpotent element of $\bar{R} = R/M$ is a homomorphic image of a nilpotent element of R. Let $\bar{a} = a + M$ be an element of $\bar{R}$ with $\bar{a}^n = 0$. Then $a^n \in M$, and so there exists an idempotent e in M such that $a^n e = a^n$. Then we can easily see that $(a - ae)^n = 0$. Clearly, $\bar{a}$ is the homomorphic image of $a - ae$, which proves the claim.

Next we show that $\bar{R}$ has no infinite trivial subring. Suppose, to the contrary, that $\bar{R}$ has an infinite trivial subring S. Then, by the above, S is a homomorphic image of a subset T of R consisting of square-zero elements. Let a and b be two elements of T. Then $ab \in M$, and so $aba \in M$. Since $(aba)^2 = 0$, we see that $aba = 0$. Then $(ab)^2 = 0$, and hence $ab = 0$. Therefore T generates an infinite trivial subring of R, which is a contradiction.

Now we claim that $\bar{R}$ is orthogonally finite. Suppose, to the contrary, that $\bar{R}$ has infinitely many mutually orthogonal idempotents $e_1, e_2, \ldots$. Let N denote the set of all nilpotent elements of $\bar{R}$. Suppose that both the number of the i such that $e_i\bar{R} \cap N \neq 0$ and the number of the j such that $\bar{R}e_j \cap N \neq 0$ are finite. Then there exists k such that $\bar{R}e_k \cap N = 0 = e_k\bar{R} \cap N$, and so we obtain that $e_k\bar{R}(1-e_k) = 0 = (1-e_k)\bar{R}e_k$. Hence e_k is central and $\bar{R}e_k$ is a strongly regular ideal of $\bar{R}$. This contradicts the choice of M. Renumbering the e_i, we may therefore assume that $e_i\bar{R} \cap N \neq 0$ for all i. Since the trivial ring $e_1\bar{R}(1-e_1)$ is finite, the number of the i such that $e_1\bar{R}e_i \neq 0$ is finite. Similarly the number of the i such that $e_i\bar{R}e_1 \neq 0$ is finite. Hence there exists k_1 such that $e_1Re_k = 0 = e_kRe_1$

for all $k \geq k_1$. Applying the same reasoning to e_{k_1}, we get infinitely many idempotents $e_{k_0} = e_1, e_{k_1}, e_{k_2}, \ldots$ such that $e_{k_i}\bar{R}e_{k_j} = 0$ if $i \neq j$. Since $e_{k_i}\bar{R} \cap N \neq 0$, each $e_{k_i}\bar{R}$ contains a non-zero element a_i with $a_i^2 = 0$. Now it is easy to see that $a_0, a_1, \ldots$ generates an infinite trivial subring of $\bar{R}$, a contradiction. Therfore $\bar{R}$ is orthogonally finite. By Lemmas 2 and 3, we conclude that $\bar{R} = R/M$ is a finite ring.

REMARK. In Theorem 1, if we assume furthermore that R is orthogonally finite, then M is a finite direct sum of division rings, and hence M is a direct summand of R. Thus we can immediately deduce [1, Theorems 2 and 3] as well as Lemma 2 from Theorem 1.

We conclude this paper with the following corollary, which is a generalization of [6, Theorem 1].

COROLLARY 1. *If R is a periodic ring with no infinite trivial subring, then R has a commutative regular ideal M with R/M finite.*

PROOF. By Theorem 1, R has a strongly regular ideal M such that R/M is finite. Since M is periodic and since M has no non-zero nilpotent element, for every $x \in M$ there exists an integer $n(x) > 1$ such that $x^{n(x)} = x$. By [3, Theorem X.1.1] M is commutative.

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