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# THE CONTINUITY OF SUBTRACTION AND THE HAUSDORFF PROPERTY IN SPACES OF BOREL MEASURES

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**Summary.**

Let $X$ be a topological space, $M_\sigma(X)$ the space of all non-negative and finite Borel measures, endowed with the weak topology and $M_r(X)$, $M_\tau(X)$, $M_t(X)$ the subspaces consisting of all regular, $\tau$-smooth and tight measures. We show that for $\kappa \in \{\sigma, r, \tau, t\}$ the map

$$\Phi_\kappa\colon \{(\mu, v) \in M_\kappa(X)^2\colon \ \mu \geqq v\} \ni (\mu, v) \ \rightarrow \ \mu - v \in M_\kappa(X)$$

is continuous if and only if $M_\kappa(X)$ is a Hausdorff space. Furthermore we establish that $\Phi_\kappa(\Phi_r, \Phi_\tau, \Phi_t)$ is continuous if $X$ is perfectly normal (normal, regular, Hausdorff) and that weaker separation axioms are not sufficient.

**Notations.** Given a topological space $X$, denote by

(a) $\mathscr{G}(X)$ the family of all open sets in $X$,

(b) $\mathscr{F}(X)$ the family of all closed sets in $X$,

(c) $\mathscr{K}(X)$ the family of all compact sets in $X$,

(d) $\mathscr{B}(X)$ the family of all Borel sets in $X$.

The set of all non-negative and finite Borel measures is called $M_\sigma(X)$. Recall that a measure $\mu \in M_\sigma(M)$ is

(a) regular, if for each $B \in \mathscr{B}(X)$

$$\mu(B) = \sup\{\mu(F)\colon \ F \subset B,\, F \in \mathscr{F}(X)\},$$

(b) $\tau$-smooth, if for each subfamily $\mathscr{G} \subset \mathscr{G}(X)$ that is directed upwards by inclusion

$$\mu(\cup \mathscr{G}) = \sup\{\mu(G)\colon \ G \in \mathscr{G}\},$$

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(c) tight, if for each $B \in \mathscr{B}(X)$

$$\mu(B) = \sup\{\mu(K)\colon\ K \subset B,\ K \in \mathscr{K}(X) \cap \mathscr{B}(X)\}.$$

The subsets of all regular, $\tau$-smooth and tight measures in $M_\sigma(X)$ are denoted by $M_r(X)$, $M_\tau(X)$ and $M_t(X)$ respectively. We endow all these spaces of measures with the weak topology, i.e. the topology generated by the requirements (see [3])

$$\begin{aligned} &\mu \to \mu(X) \quad \text{is continuous,} \\ &\mu \to \mu(G) \quad \text{is lower semicontinuous for } G \in \mathscr{G}(X). \end{aligned}$$

**The results.** First of all we define the considered mapping. Abbreviate for $\kappa \in \{\sigma, r, \tau, t\}$

$$H_\kappa(X) = \{(\mu, \nu) \in M_\kappa(X)^2\colon\ \mu \geqq \nu\}$$

and define (suppressing the index $X$)

$$\Phi_\kappa\colon\ H_\kappa(X) \in (\mu, \nu) \to\ \mu - \nu \in M_\kappa(X).$$

If the weak topology is generated by the mappings $\mu \to \int f d\mu$ for bounded continuous real functions $f$ on $X$, the map $\Phi_\kappa$ is obviously continuous. But in general continuity of $\Phi_\kappa$ is a non-trivial problem.

THEOREM 1. *Let $X$ be a topological space. For each $\kappa \in \{\sigma, r, \tau, t\}$ the following are equivalent:*

(i) *$\Phi_\kappa$ is continuous,*

(ii) *$M_\kappa(X)$ is a Hausdorff space.*

PROOF. (i) $\Rightarrow$ (ii): Given a net $\rho_\alpha$, $\alpha \in A$, in $M_\kappa(X)$ and $\mu, \nu \in M_\kappa(X)$ such that $\rho_\alpha \to \mu$ and $\rho_\alpha \to \nu$, we have to show $\mu = \nu$. Define the constant nets $\mu_\alpha = \mu$, $\alpha \in A$, and $\nu_\alpha = \nu$, $\alpha \in A$. Then (i) and the continuity of addition in $M_\kappa(X)$ yield

$$\begin{aligned} \mu_\alpha &= (\mu_\alpha + \rho_\alpha) - \rho_\alpha \to \mu + \nu - \mu = \nu, \\ \nu_\alpha &= (\nu_\alpha + \rho_\alpha) - \rho_\alpha \to \nu + \mu - \nu = \mu. \end{aligned}$$

So $\mu$ and $\nu$ cannot differ on $\mathscr{G}(X)$ and are therefore equal.

(ii) $\Rightarrow$ (i): Let $(\mu_\alpha, \nu_\alpha)$, $\alpha \in A$, be a net in $H_\kappa(X)$ converging to $(\mu, \nu) \in H_\kappa(X)$. We will show that each subnet $\mu_\beta - \nu_\beta$, $\beta \in B$, of $\mu_\alpha - \nu_\alpha$, $\alpha \in A$, contains a subnet converging to $\mu - \nu$.

(1) The net $\mu_\beta - \nu_\beta$, $\beta \in B$, has an accumulation point in $D(\mu) = \{\rho \in M_\kappa(X)\colon \rho \leqq \mu\}$. Otherwise for each $\rho \in D(\mu)$ there would be an open set $\Gamma_\rho \subset M_\kappa(X)$ and $\beta_\rho \in B$ such that $\mu_\beta - \nu_\beta \notin \Gamma_\rho$ for all $\beta \geqq \beta_\rho$. Since $D(\mu)$ is quasicompact (see [2, (3.1)]) there are $\rho_1, \ldots, \rho_n \in D(\mu)$ and an index $\beta_0 \geqq \beta_{\rho_i}$, $1 \leqq i \leqq n$, such that

$$D(\mu) \subset \cup\{\Gamma_{\rho_i}\colon\ 1 \leqq i \leqq n\} = \Gamma \ \text{ and } \mu_\beta - \nu_\beta \notin \Gamma \text{ for all } \beta \geqq \beta_0.$$

Following the remark (2) in section 3 of [2], we get $D(\mu_\beta) \subset \Gamma$ eventually. Since $\mu_\beta - \nu_\beta \in D(\mu_\beta)$ the contradiction is derived.

(2) Denote by $\rho \in D(\mu)$ an accumulation point of $\mu_\beta - \nu_\beta$, $\beta \in B$. Then there is a subnet $\mu_\gamma - \nu_\gamma$, $\gamma \in C$, converging to $\rho$. So

$$\mu_\gamma = (\mu_\gamma - \nu_\gamma) + \nu_\gamma \to \rho + \nu$$

implies together with (ii) that $\rho + \nu = \mu$, that is, $\rho = \mu - \nu$.

The following theorem includes results of Topsøe, concerning the Hausdorff property of $M_\tau(X)$ and $M_t(X)$ (see [3, p. 49]).

THEOREM 2. *Let $X$ be a topological space and $\kappa \in \{\sigma, r, \tau, t\}$. Each of the conditions*

(a) *$\kappa = \sigma$ and $X$ perfectly normal,*
(b) *$\kappa = r$ and $X$ normal,*
(c) *$\kappa = \tau$ and $X$ regular,*
(d) *$\kappa = t$ and $X$ a Hausdorff space,*

*implies that $\Phi_\kappa$ is continuous and $M_\kappa(X)$ is a Hausdorff space.*

PROOF. We will show the continuity of $\Phi_\kappa$. So we have to establish that the map

$$H_\kappa(X) \in (\mu, \nu) \to \mu(G) - \nu(G)$$

is lower semicontinuous for each $G \in \mathcal{G}(X)$ (The continuity of this map in case of $G = X$ is evident). To this end it suffices to show

$$(+) \quad \mu(G) - \nu(G) = \sup\{\mu(G') - \nu(F') \colon G \supset G' \subset F',\ G' \in \mathcal{G}(X),\ F' \in \mathcal{F}(X)\}$$

for $\mu, \nu \in M_\kappa(X)$.

Since

$$(\mu - \nu)(G) \geqq (\mu - \nu)(G') \geqq \mu(G') - \nu(F')$$

holds for $G \supset G' \subset F'$, only the inequality "$\leqq$" of $(+)$ remains to be shown. Let a real $d$ be given such that $\mu(G) - \nu(G) > d$.

(a) In perfect spaces each open set is a union of countably many closed sets. So each Borel measure is regular and (a) is only a special case of (b).

(b) There are closed sets $F_1$, $F_2$ such that $F_1 \subset G$, $F_2 \subset X \setminus G$ and $\mu(F_1) - \nu(X \setminus F_2) > d$. Since $X$ is normal there exist disjoint open sets $G_i$ such that $F_i \subset G_i$ for $i = 1, 2$. Choosing now $G' = G \cap G_1$ and $F' = X \setminus G_2 \supset G_1 \supset G'$, we get

$$\mu(G') - \nu(F') \geqq \mu(F_1) - \nu(X \setminus F_2) > d.$$

(c) Since $X$ is regular, the family of all open sets whose closure is contained in $G$ is directed upwards by inclusion and converges to $G$. So there is an open set $G'$

whose closure $F'$ is a subset of $G$ such that $\mu(G') - \nu(G) > d$. Hence

$$\mu(G') - \nu(F') \geqq \mu(G') - \nu(G) > d.$$

(d) Since in a Hausdorff space compact sets can be separated as points, we get a proof of (*d*) by replacing in (b) closed sets by compact ones.

The following examples ensure that the separation properties in Theorem 2 are well chosen.

EXAMPLES. *The following conditions are not sufficient either for the continuity of* $\Phi_\kappa$ *or the Hausdorff property of* $M_\kappa(X)$:

(a′) $\kappa = \sigma$ *and* $X$ *normal,*

(b′) $\kappa = r$ *and* $X$ *completely regular,*

(c′) $\kappa = \tau$ *and* $X$ a *Hausdorff space,*

(d′) $\kappa = t$ *and* $X$ a $T_1$*-space.*

PROOF. (a′) Let $\omega_1$ be the first uncountable ordinal, $X = [0, \omega_1]$ endowed with the the order topology, $\nu_1 \in M_\sigma(X)$ the Dieudonné measure (see [1], p. 231, (10)) and $\nu_2$ the Dirac measure in $\omega_1$. Since $\nu_2(G) \leqq \nu_1(G)$ is true for $G \in \mathscr{G}(X)$, each neighbourhood of $\nu_2$ contains also $\nu_1$. So $M_\sigma(X)$ fails to be a $T_1$-space.

(b′) Let $\omega_2$ be the first ordinal of greater cardinality than $\omega_1$, $X_i = [0, \omega_i[$ endowed with the order topology and $\nu_i \in M_r(X_i)$ the Dieudonné measures for $i = 1, 2$, i.e. $\nu_i(B) = 1$ for each $B \in \mathscr{B}(X_i)$ containing a closed unbounded subset and $\nu_i(B) = 0$ else. Consider

$$X = [0, \omega_1] \times [0, \omega_2] \setminus \{(\omega_1, \omega_2)\}$$

and the measures $\rho_i \in M_r(X)$, which are defined as image measures of $\nu_i$ with respect to the mappings

$$p_1\colon X_1 \ni x \to (x, \omega_2) \in X, \qquad p_2\colon X_2 \ni x \to (\omega_1, x) \in X.$$

Assuming now $\rho_i \in \Gamma_i \in \mathscr{G}(M_r(X))$ for $i = 1, 2$, we will show $\Gamma_1 \cap \Gamma_2 \neq \emptyset$. Since the measures $\rho_i$ are 0-1-measures there are sets $G_i \in \mathscr{G}(X)$ and $\varepsilon > 0$ such that

$$\rho_i \in \{\rho \in M_r(X)\colon\ \rho(G_i) > 1 - \varepsilon \text{ and } \rho(X) < 1 + \varepsilon\} \subset \Gamma_i.$$

The existence of a Dirac measure in $\Gamma_1 \cap \Gamma_2$ is now ensured by establishing $G_1 \cap G_2 \neq \emptyset$.

Since $p_1^{-1}(X \setminus G_1)$ is a closed set of $\nu_1$-measure 0 it is bounded. This implies the existence of an ordinal $x_1 < \omega_1$ such that $]x_1, \omega_1[ \times \{\omega_2\} \subset G_1$. Analogously there is an ordinal $x_2 < \omega_2$ such that $\{\omega_1\} \times ]x_2, \omega_2[ \subset G_2$. Setting $U = ]x_1 \cdot \omega_1[ \times ]x_2, \omega_2[$ and denoting the cardinality of a set $A$ by $|A|$, we get

$$|(X \setminus G_1) \cap (\{x\} \times ]x_2, \omega_2[)| < \omega_2 \qquad \text{for each } x \in ]x_1, \omega_1[,$$

hence

$$|(X \setminus G_1) \cap U| \leqq \omega_1 \cdot \omega_1 = \omega_1,$$

and

$$|G_2 \cap (]x_1, \omega_1[ \times \{x\}| > 1 \quad \text{for each } x \in ]x_2, \omega_2[,$$

hence

$$|G_2 \cap U| \geqq \omega_2.$$

So $G_1 \cap G_2 \supset G_1 \cap G_2 \cap U \neq \emptyset$.

(c′) Let $\lambda$ denote Lebesgue measure on $[0, 1]$ and $C \subset [0, 1]$ be a set of inner Lebesgue measure 0 and outer Lebesgue measure 1. Endow $X = [0, 1]$ with the topology generated by the Euclidean open sets and $C$. Open sets of $X$ are therefore of type $G_1 \cup (G_2 \cap C)$ with Euclidean open sets $G_i$. Since any Borel set in $X$ is of type $(B_1 \cap (X \setminus C)) \cup (B_2 \cap C)$ with Euclidean Borel sets $B_i$, by

$$\rho_i((B_1 \cap (X \setminus C)) \cup (B_2 \cap C)) = \lambda(B_i),$$

we get well defined measures $\rho_i \in M_\sigma(X)$ (see [1, p. 71(2)]). These measures are $\tau$-smooth since $X$ has a countable basis.
Given Euclidean open sets $G_i$, it follows

$$\rho_1(G_1 \cup (G_2 \cap C)) = \lambda(G_1) \leqq \lambda(G_1 \cup G_2) = \rho_2(G_1 \cup (G_2 \cap C)).$$

So each neighbourhood of $\rho_1$ contains $\rho_2$.

(d′) Take any $T_1$-space which is not a Hausdorff space. Since one point sets are Borel measurable in $X$, the map $X \ni x \to \varepsilon_x \in M_t(X)$ is an embedding. So $M_t(X)$ cannot be a Hausdorff space.

## REFERENCES

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