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COMPARISON OF STATES AND DARBOUX-TYPE PROPERTIES IN VON NEUMANN ALGEBRAS

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0. Introduction and preliminaries.

In connection with his study of quantum comparative probability [3], W. Ochs considered the following problem. Let φ be a normal state on $\boldsymbol{B}(H)$ and let $\leqq_\varphi$ be the relation defined for pairs of projections from $B(H)$ by $p \leqq_\varphi q \overset{\text{df}}{=} \varphi(p) \leqq \varphi(q)$. Does $\leqq_\varphi = \leqq_\psi$ imply $\varphi = \psi$? We give the complete solution to the problem for a not necessarily normal state on an arbitrary von Neumann algebra in Section 3. The comparison of states on finite-dimensional von Neumann algebras is described in Section 4.

In [1], H. Choda, M. Enomoto and M. Fujii proved an interesting result: if φ and ψ are states on a non-atomic von Neumann algebra M, with φ normal, and if, for every projection p of M, $\varphi(p) = \frac{1}{2}$ implies $\psi(p) = \frac{1}{2}$, then $\varphi = \psi$. One easily notices the strong connection between the theorem and the problem of Ochs (see Section 3). In order, however, that the theorem be applicable to the problem in a nontrivial way, it should be appropriately generalized. We do not require the state φ to be normal, and the algebra M (although possibly atomic) should not contain a direct summand of type I_n, $n < \infty$. The above-mentioned theorem was proved in [1] by using a "dyadic" method and a simple Darboux-type property of a normal state φ: if $\varphi(p) = \alpha > 0$ for some projection p, then φ takes all values less than α (but $\geqq 0$) at some subprojections of p. So to generalize the theorem, one should generalize the property. Our Darboux-type properties (being of interest in themselves) are described in section 1, and the generalized theorem in section 2.

In the sequel, M denotes a von Neumann algebra, Z its center, $\operatorname{Proj} M$ the lattice of all orthogonal projections of M and φ, ψ (not necessarily normal) positive linear functionals on M. Moreover, for $r \in \operatorname{Proj} M$ we put

$$\mathscr{L}_r = \{q \in \operatorname{Proj} M\,;\, q \leqq r\},$$
$$\mathscr{P}_r = \{q \in \mathscr{L}_r\,;\, q \sim r - q \sim r\}$$

(with $\mathscr{P} = \mathscr{P}_1$).

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The following theorem is the generalization of the "dyadic" method of [1], suited to our purposes.

THEOREM 0.1. *Fix* $r \in \operatorname{Proj} M$. *Let* $Q \subset \mathscr{L}_r$, $Q \neq \phi$, *satisfy the following conditions*:

(i) *if* $q \in Q$, *then* $r-q \in Q$;
(ii) *if* $0 < \gamma < \varphi(q)$ *for some* $q \in Q$, *then there is a* $q_0 \in Q$, $q_0 \leqq q$ *such that* $q-q_0 \in Q$ *and* $\varphi(q_0) = \gamma$.

Assume that $0 < \alpha < \varphi(r)$ *and* $0 \leqq \beta$, *and suppose that* $\varphi(p) = \alpha$ *implies* $\psi(p) = \beta$ *for* $p \in Q$.

Then $\psi(q) = (\beta/\alpha)\varphi(q)$ *for* $q \in Q$.

PROOF. Note that $\varphi(q_1) = \alpha/n$ implies $\psi(q_1) = \beta/n$ for any projection $q_1 \in Q$ and a positive integer n satisfying $\alpha(n+1)/n \leqq \varphi(r)$. Indeed, by (i), $r-q_1 \in Q$, and since $\varphi(r-q_1) \geqq \alpha$, there exist by (ii) mutually orthogonal projections $q_i \in Q$, $q_i \leqq r-q_1$ $(i = 2, 3, \dots, n+1)$ such that $\varphi(q_i) = \alpha/n$ for each i. Denote $p = \sum_{i=1}^{n+1} q_i$. By supposition,

$$\varphi(p-q_1) = \varphi(p-q_i) = \alpha$$

implies

$$\psi(p-q_1) = \psi(p-q_i) = \beta \quad \text{for } i = 2, 3, \dots, n+1.$$

Thus $\psi(q_1) = \psi(q_i)$ and $\psi(q_i) = \psi(p-q_1)/n = \beta/n$.

Take now an arbitrary $q \in Q$. If $0 < \varphi(q) < \varphi(r)$, then there are positive integers k, n (with n arbitrarily great) such that $\alpha k/n \leqq \varphi(q) \leqq \alpha(k+1)/n$ and that $\alpha(k+1)/n \leqq \varphi(r)$. Hence, mutually orthogonal projections $q_i \in Q$ $(i = 1, 2, \dots, k+1)$ can be chosen satisfying $\sum_{i=1}^{k} q_i \leqq q \leqq \sum_{i=1}^{k+1} q_i$ and $\varphi(q_i) = \alpha/n$ for each i. Therefore, $\beta k/n \leqq \psi(q) \leqq \beta(k+1)/n$ and the conclusion follows. If $\varphi(q) = \varphi(r)$, find $q_1, q_2 \in Q$ such that $0 < \varphi(q_i) < \varphi(r)$ and that $q_1 + q_2 = q$, and apply the above result to each of the q_i $(i = 1, 2)$ to obtain $\psi(q) = (\beta/\alpha)\varphi(q)$. Similarly, $\psi(r) = (\beta/\alpha)\varphi(r)$. If $\varphi(q) = 0$, then $\varphi(r-q) = \varphi(r)$ and what we have got so far shows that $\psi(q) = 0$. All the cases having been considered, the proof of the theorem is finished.

1. Darboux-type properties.

We shall need the following simple result (cf. [5; Lemma 1]).

PROPOSITION 1.1. *Let* $p, q \in \operatorname{Proj} M$, $p \sim q$, $pq = 0$ *and* $\varphi(p) \leqq \gamma \leqq \varphi(q)$. *Then there is a projection r such that*

$$\varphi(r) = \gamma, r \leqq p+q \quad \text{and} \quad r \sim p \sim p+q-r.$$

PROOF. Let $u \in M$ be such that $u^*u = p$, $uu^* = q$. Define a norm-continuous function $\omega: [0, 1] \to \operatorname{Proj} M$ by

$$\omega(\lambda) = (1-\lambda^2)p + \lambda^2 q + \lambda(1-\lambda^2)^{\frac{1}{2}}(u+u^*).$$

Then $\omega(0) = p$, $\omega(1) = q$, and the sought – for projection r may be chosen from among the values of the function ω.

A stronger result will be proved below (Theorem 1.4). We shall use the following Wold-type decomposition [4; Theorem 1.1].

THEOREM 1.2. *If $e+r \sim e+s$ for mutually orthogonal $e,r,s \in \operatorname{Proj} M$, then there are mutually orthogonal projections $r_1, r_2, s_1, s_2, f, g_n, h_n$ $(n \geqq 1)$ such that $r = r_1 + r_2$, $s = s_1 + s_2$,*

$$e = f + \sum_{n \geqq 1}{}^{\cdot} (g_n + h_n)$$

and $r_1 \sim s_1$, $r_2 \sim g_n$, $s_2 \sim h_n$ for $n \geqq 1$.

LEMMA 1.3. *Any equivalent projections $p,q \in M$ can be decomposed* (*in* Proj M) *as follows:*

$$p = r + f + \sum_{n \geqq 1} (g_n + h_n),$$

$$q = \tilde{s} + \tilde{f} + \sum_{n \geqq 1} (\tilde{g}_n + \tilde{h}_n)$$

so that, for any $K \subset \mathbf{N}$ with $\# K = \# \mathbf{N} \setminus K$, the projections

$$p_K = r + f + \sum_{n \in K} (g_n + h_n)$$

and

$$q_K = \tilde{s} + \tilde{f} + \sum_{n \in K} (\tilde{g}_n + \tilde{h}_n)$$

are unitarily equivalent.

PROOF. Choose $t \in \operatorname{Proj} M$ so that $pt = tp$ and $q = vtv$ for some unitary $v \in M$ (see, for example, [4; 3.9]). Put $e = pt$, $r = p - e$, $s = t - e$ and apply Theorem 1.2. Let further

$$p_K = f + r + \sum_{n \in K} (g_n + h_n), \quad t_K = p_K - r + s.$$

Then

$$p_K = f + r_1 + \left(r_2 + \sum_{n\in K} g_n\right) + \sum_{n\in K} h_n$$

$$\sim f + s_1 + \sum_{n\in K} g_n + \left(s_2 + \sum_{n\in K} h_n\right) = t_K,$$

$$1 - p_K = 1 - (e + r + s) + s_1 + \sum_{n\notin K} g_n + \left(s_2 + \sum_{n\notin K} h_n\right)$$

$$\sim 1 - (e + r + s) + r_1 + \left(r_2 + \sum_{n\notin K} g_n\right) + \sum_{n\notin K} h_n = 1 - t_K$$

Put now $\tilde{s} = vsv^*$, $\tilde{f} = vfv^*$, $\tilde{g}_n = vg_nv^*$, $\tilde{h}_n = vh_nv^*$ and conclude that p_K and $q_K = vt_Kv^*$ are unitarily equivalent.

THEOREM 1.4. *Let $p \sim q$ for some $p, q \in \operatorname{Proj} M$, and let $\varepsilon > 0$. Then there exists a continuous (in norm) function $\omega: [0,1] \to \operatorname{Proj} M$ such that*

$$1^\circ \quad \omega(0) \leqq p,\ \omega(1) \leqq q;$$

$$2^\circ \quad \varphi(\omega(0)) > \varphi(p) - \varepsilon,\ \varphi(\omega(1)) > \varphi(q) - \varepsilon.$$

PROOF. Consider the decomposition from lemma 1.3 and take a sequence of disjoint subsets $N_i \subset \mathbf{N}$ with $\# N_i = \# \mathbf{N} \setminus N_i$. For one of them, the inequalities

$$\varphi\left(\sum_{n\in N_i} (g_n + h_n)\right) < \varepsilon, \quad \varphi\left(\sum_{n\in N_i} (\tilde{g}_n + \tilde{h}_n)\right) < \varepsilon$$

hold. The projections

$$\omega(0) = r + f + \sum_{n\notin N_i} (g_n + h_n),$$

$$\omega(1) = \tilde{s} + \tilde{f} + \sum_{n\notin N_i} (\tilde{g}_n + \tilde{h}_n)$$

satisfy 1° and 2°, and are unitarily equivalent. Thus the required function ω exists (see, for example, [2; Theorem 1], or use the connectedness of the unitary group of M).

Two more Darboux-type properties will be used in the sequel.

PROPOSITION 1.5. *Let $p \in \operatorname{Proj} M$ be properly infinite and let $\varphi(p) > \gamma > 0$. Then $\varphi(r) = \gamma$ for some $r \in \operatorname{Proj} M$ such that $r \leqq p$ and $r \sim p - r \sim p$.*

PROOF. There is a sequence $\{p_n\}$ of mutually orthogonal projections from M

such that $p_n \sim p$ and $p = \sum p_n$ (see [7; Proposition 4.12]). If $q = p_n$ with sufficiently large n, then $\varphi(q) < \gamma$, $\varphi(p-q) > \gamma$ and, obviously, $q \sim p-q \sim p$. By Proposition 1.1, there is a projection r in M such that $\varphi(r) = \gamma$ and $r \sim p-r \sim p$.

PROPOSITION 1.6. *Let $p \in \operatorname{Proj} M$ be finite and continuous and let $\varphi(p) \geqq \gamma \geqq 0$. Then $\varphi(r) = \gamma$ for some $r \in \operatorname{Proj} M$ with $r \leqq p$.*

PROOF. We may assume that M is of type II_1, $p = 1$ and $\varphi(p) = 1$. Note also that it suffices to prove the proposition for $0 < \gamma \leqq 1/2$. There are two possibilities:

1°. $\varphi(q) = 1/2$ for each $q \in \operatorname{Proj} M$ such that $q \sim 1-q$.

Let T denote the canonical center-valued trace on M, μ a positive linear functional on Z, and let $\tau = \mu \circ T$. Moreover, put

$$Q = \{p \in \operatorname{Proj} M : T(p) = \beta 1 \text{ for some } \beta,\ 0 \leqq \beta \leqq 1\}.$$

By assumption, $T(q) = (1/2)1$ implies $\varphi(q) = 1/2$ for $q \in \operatorname{Proj} M$. In view of the Darboux-type property of T (see [7; Proposition 7.17]), we may apply Theorem 0.1 with τ and φ in place of φ and ψ to conclude that $T(q) = \beta 1$ implies $\varphi(q) = \beta$ for each $\beta \in [0,1]$ and $q \in \operatorname{Proj} M$. Hence, $\varphi(r) = \gamma$ for a (clearly existing) projection r such that $T(r) = \gamma 1$.

2°. $\varphi(q) = \delta < 1/2 < 1 - \delta = \varphi(1-q)$ for some $q \in \operatorname{Proj} M$

satisfying $q \sim 1-q$. There are positive integers k,n $k \leqq 2^n$, such that $\beta = 2^n\gamma/k \in [\delta, 1-\delta]$. By Proposition 1.1, $\varphi(s) = \beta$ for some $s \in \operatorname{Proj} M$. By repeated use of Proposition 1.1, we get a sequence of mutually orthogonal projections $r_1, \ldots, r_{2^n}$ from M such that $\varphi(r_m) = \beta/2^n$ for each m, $1 \leqq m \leqq 2^n$, and that $r_1 + \ldots + r_{2^n} = s$. Put $r = r_1 + \ldots + r_k$ to get $\varphi(r) = k\beta/2^n = \gamma$.

2. A sufficient condition for the equality of states.

LEMMA 2.1. *Let M be properly infinite (respectively of type II_1), $0 < \alpha < \varphi(1)$ and $0 \leqq \beta$. If $\varphi(p) = \alpha$ implies $\psi(p) = \beta$ for $p \in \mathscr{P}$ (respectively $p \in \operatorname{Proj} M$), then $\psi(a) = (\beta/\alpha)Q(q)$ for each $q \in \mathscr{P}$ (respectively $q \in \operatorname{Proj} M$). (For the definition of $\mathscr{P}$ see Introduction.)*

PROOF. Follows at once from Proposition 1.5 (respectively Proposition 1.6) and Theorem 0.1 with $Q = \mathscr{P}$ (respectively $Q = \operatorname{Proj} M$).

LEMMA 2.2. *Let M be properly infinite. If $\varphi = \psi$ on $\mathscr{P}$, then $\varphi = \psi$ (on $\operatorname{Proj} M$).*

PROOF. Will be carried out in two steps.

STEP 1. $\varphi = \psi$ on $\mathscr{P}_z$ for any $z \in Z$. Fix $p \in \mathscr{P}_z$ and $\varepsilon > 0$. By Proposition 1.5, we may find a projection $q \in \mathscr{P}_{1-z}$ satisfying $\varphi(q) < \varepsilon$, $\psi(q) < \varepsilon$. Note that $p+q \in \mathscr{P}$. By assumption, $\varphi(p+q) = \psi(p+q)$ and, consequently, $|\psi(p)-\varphi(p)| < 2\varepsilon$.

STEP 2. $\varphi = \psi$ on $\operatorname{Proj} M$. Fix $p \in \operatorname{Proj} M$. By the comparability theorem (see [6; Theorem V.1.8]), there are $x, y \in Z$ such that $x+y = 1$, $px \lesssim (1-p)x$, $(1-p)y \lesssim py$. Choose $q_1, \ldots, q_4 \in \operatorname{Proj} M$ so that $q_1+q_2 = (1-p)x$, $q_1 \sim q_2 \sim (1-p)x$, $q_3+q_4 = py$, $q_3 \sim q_4 \sim py$. Note that $q_1, q_2 \in \mathscr{P}_x$, $q_3, q_4 \in \mathscr{P}_y$, and $px = x-q_1-q_2$. By Step 1,

$$\psi(p) = \psi(px)+\psi(py)$$

$$= \psi(x-q_1)-\psi(q_2)+\psi(q_3)+\psi(q_4) = \varphi(p),$$

which ends the proof.

THEOREM 2.3. *Let M be a von Neuman algebra without a direct summand of type I_n $(n < \infty)$, $0 < \alpha < \varphi(1)$ and $0 \leqq \beta$. If $\varphi(p) = \alpha$ implies $\psi(p) = \beta$ for $p \in \operatorname{Proj} M$, then $\psi = (\beta/\alpha)\varphi$. In particular, if φ and ψ are states, then $\varphi = \psi$ (and $\alpha = \beta$).*

PROOF. Let z be the maximal projection in the center Z of M, such that M_z is of type II_1. Then $M(1-z)$ is properly infinite. Let us note that, by Propositions 1.5 and 1.6, $Q = \mathscr{L}_z+\mathscr{P}_{1-z}$ satisfies the assumptions of Theorem 0.1. Thus $\psi = (\beta/\alpha)\varphi$ on $\mathscr{L}_z+\mathscr{P}_{1-z}$ and the equality must hold on $\mathscr{P}_{1-z}$, on $\mathscr{L}_z$ and, by Lemma 2.2, on $\mathscr{L}_{1-z}$. The proof is finished.

We have proved, in fact, the following

PROPOSITION 2.4. *Let $0 < \alpha < \varphi(1)$, $0 \leqq \beta$, and, for a projection $z \in Z$, let*

(i) *$0 < \gamma < \varphi(p)$, $p \in \mathscr{L}_z$, imply $\varphi(q) = \gamma$ for some $q \in \mathscr{L}_p$;*
(ii) *$M(1-z)$ be properly infinite.*

If $\varphi(p) = \alpha$ implies $\psi(p) = \beta$ for $p \in \mathscr{L}_z\mathscr{P}_{1-z}$, then $\psi = (\beta/\alpha)\varphi$.

It follows easily from Theorem 3.3 that if $M \neq \mathrm{C}1$ and M has a nonzero direct summand of type I_n (for some $n < \infty$), then there are two distinct (and equivalent) states φ, ψ on M such that $\varphi(p) = 1/2$ implies $\psi(p) = 1/2$ for $p \in \operatorname{Proj} M$. However, we have the following

COROLLARY 2.5. *Let M be a von Neumann algebra with no factor of type I_n, $n < \infty$, as a direct summand, $0 < \alpha < \varphi(1)$ and $0 \leqq \beta$. Suppose that φ is* *normal (at least on the finite discrete part of M), and that $\varphi(p) = \alpha$ implies $\psi(p) = \beta$ for $p \in \operatorname{Proj} M$. Then $\psi = (\beta/\alpha)\varphi$.*

PROOF. Let z be the smallest projection in Z such that $M(1-z)$ is properly infinite. Then $Mz = M_1 \oplus M_2$, where M_1 is of type II_1, M_2 is (finite, discrete and) non-atomic and φ is normal on M_2. By Proposition 1.6 and [1; Theorem 1] (see Introduction), the conditions $0 < \gamma < \varphi(p)$, $p \in \operatorname{Proj} Mz$, imply $\varphi(q) = \gamma$ for some $q \in \mathscr{L}_p$. Thus, Proposition 2.4 can be used to end the proof.

3. Equivalent and exclusive states.

Each state φ on a von Neumann algebra M gives rise to the following relation of comparative probability (cf. [3]) on the lattice of projections of M:

$$p \leqq_{\varphi} q \quad \text{iff} \quad \varphi(p) \leqq \varphi(q), \quad p,q \in \operatorname{Proj} M.$$

A state ψ is said to be equivalent to a state φ if $\leqq_{\varphi} = \leqq_{\psi}$, i.e. if, for $p,q \in \operatorname{Proj} M$,

$$\varphi(p) \leqq \varphi(q) \quad \text{is equivalent to } \psi(p) \leqq \psi(q).$$

We shall also say that ψ is similar to φ if, for $p,q \in \operatorname{Proj} M$,

$$\varphi(p) < \varphi(q) \quad \text{implies} \quad \psi(p) \leqq \psi(q).$$

We denote by $E(\varphi)$ (respectively $S(\varphi)$) the set of all states equivalent (respectively similar) to φ. If $E(\varphi) = \{\varphi\}$ (respectively $S(\varphi) = \{\varphi\}$), then the state φ is called exclusive (respectively strongly exclusive). Note that the equivalence is, in fact, an equivalence relation, and that $\psi \in S(\varphi)$ iff $\varphi \in S(\psi)$ i.e., the relation of similarity is symmetric.

The notions of quantum comparative probability, equivalence and exclusiveness of states were introduced by Ochs [3]. He showed that (with the equivalence relation restricted to the set of normal states) each normal state on a factor of type I_∞ is exclusive. He also stated a necessary and sufficient condition for the exclusiveness of a state on a factor of type $\mathrm{I}_n, n < \infty$, and proved that each nonfaithful state on such a factor is exclusive.

In this section we shall describe those von Neumann algebras which admit only exclusive states. The subsequent section contains a thorough description of the sets $E(\varphi)$ and $S(\varphi)$ for factors of type I_n, $n < \infty$.

We shall start with two simple lemmas.

LEMMA 3.1. *Let K be a commutative von Neumann algebra. There is a state φ on K such that $\varphi(\operatorname{Proj} K) = \{0, 1\}$.*

PROOF. We may assume that $K = L^\infty(\Omega, \mathscr{F}, \nu)$ where $(\Omega, \mathscr{F}, \nu)$ is a finite measure space. Let U be an ultrafilter in Ω containing the complements of all ν-negligible subsets of Ω. For $A \in \mathscr{F}$, put $\mu(A) = 0$ when $A \notin U$ and $\mu(A) = 1$ when $A \in U$. Since μ is a finitely additive measure on $\mathscr{F}$, absolutely continuous with respect to ν, it yields a state φ with the desired property.

LEMMA 3.2. *Let $M(1-z)$ be of type II_1 and let Mz be properly infinite for some $z \in Z$. If ψ is similar to φ, then $\varphi(p) = 1/2$ implies $\psi(p) = 1/2$ for any $p \in \mathscr{L}_{1-z} + \mathscr{P}_z$.*

PROOF. Let $\varphi(p) = 1/2$, $p \in \mathscr{L}_{1-z} + \mathscr{P}_z$, and $\varepsilon > 0$. By Propositions 1.5 and 1.6, we can always find mutually orthogonal projections $r_n \in \mathscr{L}_{1-z} + \mathscr{P}_z$, $r_n \leqq p$, satisfying $\varphi(r_n) > 0$ for every n. For sufficiently large n_0, $\psi(r_{n_0}) < \varepsilon$. Hence, $\varphi(p) < \varphi(1-p+r_{n_0})$ implies

$$\psi(p) \leqq \psi(1-p+r_{n_0}) < \psi(1-p)+\varepsilon.$$

Thus, $\psi(p) \leqq \psi(1-p)$ and, replacing p by $1-p$, $\psi(p) \geqq \psi(1-p)$, which gives the assertion of the lemma.

THEOREM 3.3. *For a von Neumann algebra M, the following conditions are equivalent*:

(i) *$M = \mathbf{C}1$ or M has no direct summand of type I_n, $n < \infty$*;
(ii) *each state on M is strongly exclusive*;
(iii) *each state on M is exclusive.*

PROOF. (i) $\Rightarrow$ (ii). We may assume that M has no direct summand of type I_n, $n < \infty$. By virtue of Lemma 3.2 and Proposition 1.6, we can use Proposition 2.4 with $\alpha = \beta = 1/2$ to obtain (ii).

(ii) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (i). Assume the contrary. There are two cases:

1°. M has a nonzero commutative direct summand K and $M \neq \mathbf{C}1$. We may assume that $M = K \oplus N$ for some nonzero von Neumann subalgebra N. Let φ be state on K such that $\varphi(\operatorname{Proj} K) = \{0, 1\}$, which exists by Lemma 3.1, and let ψ be any state on N. Put

$$\varphi_1 = (2/3)\varphi \oplus (1/3)\psi \quad \text{and} \quad \varphi_2 = (3/4)\varphi \oplus (1/4)\psi.$$

Then the inequality

$$\varphi_i(p \oplus r) \leqq \varphi_i(q + s)$$

is equivalent to the alternative $\varphi(p) < \varphi(q)$ or $\varphi(p) = \varphi(q)$, $\psi(p) \leqq \psi(q)$. Hence, φ_1 and φ_2 are distinct and equivalent.

2°. M has a nonzero direct summand M_n of type I_n, $1 < n < \infty$. Write

M_n as $F_n \bar{\otimes} K$ where F_n is a factor of type I_n and K is commutative. By Theorem 4.5, there are two distinct and equivalent states ψ_1, ψ_2 on F_n. Let φ be a state on K such that $\varphi(\operatorname{Proj} K) = \{0, 1\}$ (Lemma 3.1). Define a Fubini mapping h on $F_n \otimes K$ by $h(a \otimes b) = a\varphi(b)$ (cf. [6; Section 9.8]). Since $F_n \otimes K$ is norm-dense in M_n, we may extend h to the whole algebra M_n. It is easy to see that h is a homomorphism of M_n onto F_n. Hence, $\psi_1 \circ h$ and $\psi_2 \circ h$ are distinct and equivalent states on M_n which can be extended to distinct and equivalent states on M in an obvious way. Thus, the proof of the theorem is finished.

4. Equivalence and similarity of states in factors of type I_n, $n < \infty$.

Throughout this section, M is a factor of type I_n, $n < \infty$, τ is the normalized trace on M, $\zeta = \{e \in \operatorname{Proj} M;\ \tau(e) = 1/n\}$ and $\zeta_p = \zeta \cap \mathscr{L}_p$ for any $p \in \operatorname{Proj} M$. The following lemma generalizes Lemma 2 from [3].

LEMMA 4.1. *Let $\varphi = \tau(v\,\cdot)$ and $\psi = \tau(w\,\cdot)$ be two arbitrary hermitian functionals on M with density operators*

$$v = \sum_{i=1}^{k} \alpha_i p_i, \quad w = \sum_{i=1}^{m} \beta_i q_i,$$

where $p_1, \ldots, p_k$ (respectively $q_1, \ldots, q_m$) are nonzero mutually orthogonal projections from M,

$$\sum_{i=1}^{k} p_i = \sum_{i=1}^{m} q_i = 1$$

and $\alpha_1 < \ldots < \alpha_k$, $\beta_1 < \ldots < \beta_m$. If φ is not a multiple of τ and

$$\varphi(e) < \varphi(f) \quad \textit{implies} \quad \psi(e) < \psi(f) \quad \textit{for} \quad e, f \in \zeta,$$

then $k = m$, $p_i = q_i$ for $i = 1, \ldots, k$ and $\beta_i = \gamma\alpha_i + \delta$ for some $\gamma, \delta \in \mathrm{R}$, $i = 1, \ldots, k$.

PROOF. Take $j < k$ and assume that $m \geqq j$ and $p_i = q_i$ for $i < j$. Then

(1) $p_j + \ldots + p_k = q_j + \ldots + q_m$;
(2) $e \in \zeta_{p_j}$, $f \in \zeta_{p_j + \ldots + p_k} \setminus \zeta_{p_j}$ implies $\varphi(e) < \varphi(f)$.

Take $e, f \in \zeta_{p_j}$ and $\varepsilon > 0$. We can choose $g \in \zeta_{p_j + \ldots + p_k} \setminus \zeta_{p_j}$ so that $\|f - g\| < \varepsilon$. By (2), $\varphi(e) < \varphi(g)$, which implies $\psi(e) < \psi(g) < \psi(f) + \varepsilon\|\psi\|$. Hence

(3) $e, f \in \zeta_{p_j}$ implies $\psi(e) = \psi(f)$.

By (1) and (2), $f \in \zeta_{p_j+\ldots+p_k} \setminus \zeta_{p_j}$ implies $f \in \zeta_{q_j+\ldots+q_m} \setminus \zeta_{q_j}$, so that $\zeta_{q_j} \subset \zeta_{p_j}$. Since $\zeta_{q_j} \neq \emptyset$ and $\zeta_{p_j} \subset \zeta_{q_j+\ldots+q_m}$, therefore (3) implies $\zeta_{q_j} = \zeta_{p_j}$. Thus, $p_j = q_j$ and $m \geqq j+1$.

We have proved so far that $m \geqq k$ and that $p_i = q_i$ for $i < k$. Since φ is not a multiple of τ, we must have $k \geqq 2$, which implies $p_1 = q_1$. Replacing φ by $-\varphi$ and ψ by $-\psi$, we get $p_k = q_m$, which gives $k = m$ and $p_i = q_i$ for $i = 1, \ldots, k$.

Choose now, for each $i = 1, \ldots, k$, a projection $e_i \in \zeta_{p_i}$. Let

$$f_\lambda = \lambda e_1 + (1-\lambda)e_k + \lambda^{1/2}(1-\lambda)^{1/2}(u+u^*)$$

where $u^*u = e_1$ and $uu^* = e_k$ $(0 \leqq \lambda \leqq 1)$. For $i = 2, \ldots, k-1$, we choose λ_1 so that $\beta_i = \lambda_i\beta_1 + (1-\lambda_i)\beta_k$. Then $\psi(e_i) = \psi(f_{\lambda_i})$, which implies $\varphi(e_i) = \varphi(f_{\lambda_i})$. The last equality yields $\alpha_i = \lambda_i\alpha_1 + (1-\lambda_i)\alpha_k$, and the existence of γ and δ such that $\beta_i = \gamma\alpha_i + \delta$ follows. This ends the proof of the lemma.

Now, let φ be a state on M. We shall examine the following sets of states:

$$X_\varphi = \{\psi\,;\, \varphi(p) < \varphi(q),\ \tau(p) = \tau(q) \text{ imply } \psi(p) < \psi(q) \text{ for } p,q \in \operatorname{Proj} M\};$$

$$Y_\varphi = \{\psi\,;\, \varphi(p) < \varphi(q),\ \tau(p) \neq \tau(q) \text{ imply } \psi(p) < \psi(q) \text{ for } p,q \in \operatorname{Proj} M\};$$

$$Z_\varphi = X_\varphi \cap Y_\varphi.$$

Observe that

(4) $E(\varphi) = \{\psi \in Z_\varphi\,;\, \varphi \in Z_\psi\} \subset Z_\varphi$;
(5) $\psi \in S(\varphi)$ iff $(1-\varepsilon)\psi + \varepsilon\varphi \in Z_\varphi$ for each $0 < \varepsilon \leqq 1$.

The following characteristic δ_φ of φ will play an important role in the sequel:

$$\delta_\varphi = \gamma_1 + \ldots + \gamma_s - \gamma_{n-s+2} - \ldots - \gamma_n \quad \text{with } n = 2s \text{ or } n = 2s-1$$

$$(\delta_\varphi = \gamma_1 \quad \text{for } n = 1, 2),$$

where $\varphi = \operatorname{tr}(v\,\cdot)$ and

(6) $$v = \sum_{i=1}^{n} \gamma_i e_i$$

with $\gamma_1 \leqq \ldots \leqq \gamma_n$ and mutually orthogonal $e_i \in \zeta$ $(\sum_{i=1}^n e_i = 1)$. Obviously, $\delta_\varphi \leqq \gamma_1 \leqq 1/n$, and $\delta_\varphi = 1/n$ iff $\varphi = \tau$. Moreover, if $\varphi \neq \tau$, $\gamma > 0$ and $\psi = \gamma\varphi + (1-\gamma)\tau$, then

(7) ($\psi \geqq 0$ and then $\delta_\psi > 0$) iff $\gamma < (1-n\delta_\varphi)^{-1}$.

LEMMA 4.2. *X_τ is the set of all states on M, and*

$$X_\varphi = \{\psi = \gamma\varphi + (1-\gamma)\tau\,;\gamma > 0, \psi \geqq 0\} \quad \textit{for } \varphi \neq \tau.$$

PROOF. The assertion is, of course, valid for $\varphi = \tau$. If $\varphi \neq \tau$, the inclusion "$\supset$" is obvious and the inclusion "$\subset$" is an immediate consequence of Lemma 4.1.

LEMMA 4.3. *If $\delta_\varphi > 0$, then $Y_\varphi = \{\psi\,;\delta_\psi > 0\}$.*

PROOF. STEP 1. Let $\varphi = \mathrm{tr}(v\,\cdot)$ with v given by (6). Then the condition $\delta_\varphi > 0$ is equivalent to:

$$\tau(p) < \tau(q) \quad \text{implies} \quad \varphi(p) < \varphi(q) \quad \text{for } p,q \in \operatorname{Proj} M.$$

In fact, it is not difficult to check that the following conditions are equivalent:

$$\gamma_1 + \ldots + \gamma_s > \gamma_{n-s+2} + \ldots + \gamma_n, \text{ where } n = 2s \quad \text{or} \quad n = 2s-1;$$

$$\gamma_1 + \ldots + \gamma_j > \gamma_{n-j+2} + \ldots + \gamma_n, \text{ for each } j = 1, \ldots, n;$$

$$\varphi(e_1 + \ldots + e_j) = \gamma_1 + \ldots + \gamma_j > \gamma_{n-1+1} + \ldots + \gamma_n = \varphi(e_{n-i+1} + \ldots + e_n)$$
$$\text{for each } 0 \leqq i < j \leqq n;$$

$$\varphi(q) > \varphi(p) \text{ for each } p,q \in \operatorname{Proj} M,\ \tau(p) = i/n < j/n = \tau(q).$$

STEP 2. Assume that $\delta_\varphi > 0$. By Step 1, the condition $\varphi(p) < \varphi(q)$, $\tau(p) \neq \tau(q)$ is equivalent to $\tau(p) < \tau(q)$. Using Step 1 once more (with φ replaced by ψ), we get $\psi \in Y_\varphi$, iff $\delta_\psi > 0$.

LEMMA 4.4. *If $\delta_\varphi < 0$, then $Z_\varphi = \{\varphi\}$.*

PROOF. Let $\delta_\varphi < 0$. By Lemma 4.2, it is enough to prove that $\psi = \gamma\varphi + (1-\gamma)\tau \in Y_\varphi$ implies $\gamma = 1$. There are two cases to be considered:

1°. $\gamma_s > 0$ ($n = 2s$ or $n = 2s-1$, γ_i as in (6)). Put

$$p = e_1 + \ldots + e_{s-1}, \quad q = e_{n-s+2} + \ldots + e_n.$$

Then $pq = 0$, $p \sim q$ and $\varphi(p) < \varphi(p+e_s) < \varphi(q)$. For any sufficiently small $\varepsilon > 0$, there are, by Proposition 1.1, projections $r_1, r_2 \sim p$ satisfying

$$\varphi(r_1) = \varphi(p+e_s) - \varepsilon, \quad \varphi(r_2) = \varphi(p+e_s) + \varepsilon.$$

Thus

$$\tau(r_1) = \tau(r_2) = \tau(p + e_s) - 1/n, \quad \psi(r_1) < \psi(p + e_s) < \psi(r_2)$$

and, consequently, $-\varepsilon\gamma/n < 1-\gamma < \varepsilon\gamma/n$. Hence $\gamma = 1$.

2°. $\gamma_s = 0$. Then $\psi \geqq 0$ implies $\gamma \leqq 1$ and, for p,q and r_2 as in 1°, we have $\psi(p+e_s) < \psi(r_2)$ and $1-\gamma < \varepsilon\gamma/n$. Hence $\gamma = 1$.
The proof is finished.

We sum up our results in the following

THEOREM 4.5. *For a state φ on M:*

(i) $\delta_\varphi < 0$ *implies* $E(\varphi) = S(\varphi) = \{\varphi\}$;
(ii) $\delta_\varphi = 0$ *implies* $E(\varphi) = \{\varphi\}$, $S(\varphi) = \{\gamma\varphi+(1-\gamma)\tau\,;\, 0 \leqq \gamma \leqq 1\}$;
(iii) $0 < \delta_\varphi < 1/n$ *implies* $E(\varphi) = \{\psi = \psi\varphi + (1-\gamma)\tau; 0 < \gamma < (1-n\delta_\varphi)^{-1}\}$,
$S(\varphi) = \{\psi = \gamma\varphi + (1-\gamma)\tau; 0 \leqq \gamma \leqq (1-n\delta_\varphi)^{-1}\}$;

(iv) $\delta_\varphi = 1/n$ *is equivalent to* $\varphi = \tau$, *and then*, $E(\tau) = \{\tau\}$, $S(\tau) = \{\psi\,;\,\delta_\psi \geqq 0\}$.

PROOF. (i). Follows from (4), (5), and Lemma 4.4.
(iii). Let $0 < \delta_\varphi < 1/n$. By Lemmas 4.2 and 4.3,

$$Z_\varphi = \{\psi = \gamma\varphi+(1-\gamma)\tau\,;\,\gamma > 0, \psi \geqq 0, \delta_\psi > 0\}.$$

Thus $0 < \delta_\psi < 1/n$ and $\varphi = \gamma^{-1}\psi + (1-\gamma^{-1})\tau \in Z_\psi$ for $\psi = \gamma\varphi+(1-\gamma)\tau \in Z_\varphi$. By (4) and (7),

$$E(\varphi) = Z_\varphi = \{\psi = \gamma\varphi+(1-\gamma)\tau\,; 0 < \gamma < (1-n\delta_\varphi)^{-1}\}.$$

Now, the form of $S(\varphi)$ is easily obtained from (5).
(iv). By Lemmas 4.2 and 4.3, $Z_\tau = \{\varphi\,;\,\delta_\varphi > 0\}$, and $\tau \notin Z_\varphi$ if $\varphi \neq \tau$. Hence, $E(\tau) = \{\tau\}$ by (4), and $S(\tau) = \{\varphi\}\,;\,\delta_\varphi \geqq 0\}$ by (5).
(ii). Suppose $\delta_\psi = 0$. By (i), (iv), and (iii) with (7), $\psi \notin E(\varphi)$ if $\delta_\varphi \neq 0$. Hence

$$E(\psi) = \{\varphi\,;\,\psi \in E(\varphi)\} \subset \{\varphi\,;\,\delta_\varphi = 0\}.$$

Since $E(\psi) \subset X_\psi$, the equality $E(\psi) = \{\psi\}$ follows from Lemma 4.2. Similarly, by (i),

$$S(\psi) = \{\varphi\,;\,\psi \in S(\varphi)\} \subset \{\varphi\,;\,\delta_\varphi \geqq 0\}. \tag{8}$$

By (5) and Lemma 4.2,

$$S(\psi) \subset \bar{X}_\psi = \{\varphi = \lambda\psi+(1-\lambda)\tau\,;\,\lambda \geqq 0, \varphi \geqq 0\}. \tag{9}$$

If $\varphi = \lambda\psi+(1-\lambda)\tau \geqq 0$ and $0 < \delta_\varphi < 1/n$, then, by (iii),

$$\psi \in S(\varphi) = \{\gamma\varphi+(1-\gamma)\tau\,; 0 \leqq \gamma \leqq (1-n\delta_\varphi)^{-1} = \lambda^{-1}\}. \tag{10}$$

By (9), (10) and (iv),

$$S(\psi) \subset \{\varphi = \lambda\psi + (1-\lambda)\tau;\, \varphi \geqq 0, 0 < \delta_\varphi < 1/n\} \cup \{\tau, \psi\}$$
$$\subset \{\varphi;\, \psi \in S(\varphi)\} = S(\psi):$$

Now, if $\varphi = \lambda\psi + (1-\lambda)\tau \in S(\psi)$, then, by (9), $\lambda \geqq 0$. Also, $\delta_\varphi = (1-\lambda)/n \geqq 0$ by (8). Hence, $0 \leqq \lambda \leqq 1$, and the proof of the theorem is finished.

REFERENCES

1. H. Choda, M. Enomoto, and M. Fujii, *Non-commutative Liapounoff's theorem*, Math. Japon. 28 (1983), 651–653.
2. S. Maeda, *On arcs in the space of projections of a* C*-*algebra*, Math. Japon. 21 (1976), 371–374.
3. W. Ochs, *Gleason measures and quantum comparative probability* in *Quantum Probability and Applications* II (Proc. Heidelberg, 1984), eds. L. Accardi and W. von Waldenfels. (Lecture Notes in Math. 1136), pp. 388–396. Springer-Verlag, Berlin - New York, 1985.
4. A. Paszkiewicz, *Wold-type decomposition for equivalent and commuting projections*, to appear in Bull. Polon Acad. Sci. (1989).
5. T. Sakaue, M. O'uchi, and S. Maeda, *Connected components of projections of a C*-algebra*, Math. Japon. 29 (1984), 427–431.
6. Ş. Strătilă, *Modular Theory in Operator Algebras*, Abacus Press, Tunbridge Wells, 1981.
7. Ş. Strătilă and L. Zsidó, *Lectures on von Neumann Algebras*, Abacus Press, Tunbridge Wells, 1979.

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