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DOMINATED AND UNIFORMLY DOMINATED FAMILIES OF LOEB-MEASURES

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Abstract.

It is shown in this paper that each dominated family of Loeb-measures, derived from an internal family of probability contents, is uniformly dominated. As a corollary we obtain some surprising "nonstandard equivalences" for uniform domination in the standard world. An essential tool is an extension of a well-known theorem of Halmos and Savage which is proven by nonstandard methods in a rather direct way.

1. Notations.

Let P and Q be probability contents (p-contents) on an algebra $\mathscr{C}$. Then

Q weakly dominates P iff $C \in \mathscr{C}$ and $Q(C) = 0$ imply $P(C) = 0$;
Q dominates P iff for each $\varepsilon > 0$ there exists $\delta > 0$ such that $C \in \mathscr{C}$ and $Q(C) < \delta$ imply $P(C) < \varepsilon$.

If P, Q are p-measures on a σ-algebra then both concepts coincide.

Let $\mathscr{P}$ be a family of p-contents on $\mathscr{C}$. Then $\mathscr{P}$ is (weakly) dominated iff there exists a p-content Q, which (weakly) dominates $\mathscr{P}$; that is Q (weakly) dominates each $P \in \mathscr{P}$.

$\mathscr{P}$ is uniformly dominated iff there exists a p-content Q, which uniformly dominates $\mathscr{P}$; i.e. for each $\varepsilon > 0$ there exists $\delta > 0$ such that $C \in \mathscr{C}$ and $Q(C) < \delta$ imply $P(C) < \varepsilon$ for all $P \in \mathscr{P}$.

We assume in this paper that we have a structure containing the real numbers R, and a polysaturated nonstandard model of this structure.

Let $\mathscr{B}$ be an internal algebra, and $Q : \mathscr{B} \to {}^*[0, 1]$ be an internal p-content. Then $Q_L(B) := {}^\circ(Q(B))$, $B \in \mathscr{B}$, defines a p-measure on $\mathscr{B}$ and the system $L(Q)$ of all sets C with

$$\sup\{Q_L(B) : C \supset B \in \mathscr{B}\} = \inf\{Q_L(B) : C \subset B \in \mathscr{B}\}$$

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is the σ-algebra of all Carathéodory-measurable sets with respect to $Q_L|\mathscr{B}$. The common value of the above expressions defines the unique extension of $Q_L|\mathscr{B}$ to a measure on the complete σ-algebra $L(Q)$. This extension is also denoted by Q_L; Q_L is the Loeb-measure associated with Q. The described construction was given in [3], [4]. Put

$$L_u(\mathscr{B}) = \bigcap \{L(Q): Q \text{ internal } p\text{-content on } \mathscr{B}\}.$$

If $\mathscr{G}$ is a family of internal p-contents on $\mathscr{B}$, let $\mathscr{G}_L|\mathscr{B} = \{Q_L|\mathscr{B}: Q \in \mathscr{G}\}$.

2. The results.

The following Theorem of Halmos and Savage (see [1]) is an important tool in mathematical statistics and especially in the theory of sufficiency. We give a short and transparent proof using nonstandard techniques.

If P is a p-content on an algebra $\mathscr{A}$, put $N(P) := \bigcup\{{}^*N : N \in \mathscr{A}, P(N) = 0\}$. As our model is polysaturated, Theorem 1 of [2] implies that $N(P) \in L_u({}^*\mathscr{A})$ and ${}^*P_L(N(P)) = 0$.

The following Lemma will be used several times in the proofs of our results.

1. Lemma. *Let Q be an internal p-content on an internal algebra $\mathscr{B}$.*

(1) *If $Q_L|\mathscr{B}$ is weakly dominated by a p-content $v|\mathscr{B}$, then it is dominated by $v|\mathscr{B}$.*

(2) *If $P|\mathscr{B}$ is an internal p-content such that P_L dominates Q_L on $\mathscr{B}$, then it dominates Q_L on $L_u(\mathscr{B})$.*

(3) *Let P and Q be p-contents on an algebra $\mathscr{A}$. Then $P|\mathscr{A}$ dominates $Q|\mathscr{A}$ iff ${}^*P_L|L_u(\mathscr{A})$ dominates ${}^*Q_L|L_u(\mathscr{A})$.*

Proof. (1) As $\mathscr{B}$ is an internal algebra, $v|\mathscr{B}$ is a p-measure and can be extended to a unique p-measure on $\sigma(\mathscr{B})$. It suffices to show that $v|\sigma(\mathscr{B})$ weakly dominates $Q_L|\sigma(\mathscr{B})$. Let $C \in \sigma(\mathscr{B})$ with $v(C) = 0$. Assume indirectly that $Q_L(C) > 0$. Then there exists $B \in \mathscr{B}$ with $B \subset C$ and $Q_L(B) > 0$, contradicting $v(B) = 0$.

(2) Let $C \in L_u(\mathscr{B})$ with $P_L(C) = 0$. If $Q_L(C) > 0$, we obtain a contradiction as in (1).

(3) By transfer it can be seen that $P|\mathscr{A}$ dominates $Q|\mathscr{A}$ iff ${}^*P_L|{}^*\mathscr{A}$ dominates ${}^*Q_L|{}^*\mathscr{A}$. Now by (2), applied to $P|B = {}^*P_L|{}^*\mathscr{A}$ and $Q|B = {}^*Q_L|{}^*\mathscr{A}$, we obtain (3).

2. Theorem. *Let $\mathscr{P}$ be a dominated family of p-measures on a σ-algebra $\mathscr{A}$. Then there exists $P_n \in \mathscr{P}$, $n \in \mathbb{N}$, such that $\sum_{n \in \mathbb{N}} 2^{-n} P_n$ dominates $\mathscr{P}$.*

PROOF. Let $\mathscr{P}$ be dominated by a p-content μ. Let $\{P_n : n \in \mathbf{N}\} \subset \mathscr{P}$ be such that

$$(1) \qquad {}^*\mu_L\Big(\bigcap_{n\in\mathbf{N}} N(P_n)\Big) = \inf\Big\{{}^*\mu_L\Big(\bigcap_{P\in\mathscr{P}_0} N(P)\Big) : \mathscr{P}_0 \subset \mathscr{P} \text{ countable}\Big\}.$$

By (1) we have for each $Q \in \mathscr{P}$ that

$$N := \bigcap_{n\in\mathbf{N}} N(P_n) \subset N(Q) \qquad {}^*\mu_L|L_u({}^*\mathscr{A})\text{-a.e.}$$

As ${}^*Q_L|L_u({}^*\mathscr{A})$ is dominated by ${}^*\mu_L|L_u({}^*\mathscr{A})$ according to Lemma 1, we obtain $N \subset N(Q)$ ${}^*Q_L$-a.e. Put

$$P_0 := \sum_{n\in\mathbf{N}} \frac{1}{2^n} P_n$$

and let $A \in \mathscr{A}$ with $P_0(A) = 0$. Then ${}^*A \subset N(P_0) \subset \bigcap_{n\in\mathbf{N}} N(P_n) = N$. Hence $Q(A) = {}^*Q_L({}^*A) = 0$ for all $Q \in \mathscr{P}$.

Using once more nonstandard techniques, we obtain the following generalization of the Theorem of Halmos-Savage.

3. COROLLARY. *Let $\mathscr{P}$ be a dominated family of p-contents on an algebra $\mathscr{A}$. Then there exist $P_n \in \mathscr{P}$, $n \in \mathbf{N}$, such that* $\sum_{n\in\mathbf{N}} 2^{-n} P_n$ dominates $\mathscr{P}$.

PROOF. Let $\mathscr{P}$ be dominated by a p-content μ. By Lemma 1 we have that $\{{}^*P_L : P \in \mathscr{P}\}$ is dominated by ${}^*\mu_L$ on $\sigma({}^*\mathscr{A})$, the σ-algebra generated by ${}^*\mathscr{A}$. According to Theorem 2 there exist $P_n \in \mathscr{P}$, $n \in \mathbf{N}$, such that $\sum_{n\in\mathbf{N}} 2^{-n}({}^*P_n)_L$ dominates $\{{}^*P_L : P \in \mathscr{P}\}$ on $\sigma({}^*\mathscr{A})$. Hence $\sum_{n\in\mathbf{N}} 2^{-n} P_n$ dominates $\mathscr{P}$.

Now we prove a result for certain families of Loeb-measures which is obviously false for general families of measures.

4. THEOREM. *Let $\mathscr{B}$ be an internal algebra and let $\mathscr{G}$ be an internal family of p-contents on $\mathscr{B}$. If $\mathscr{G}_L|\mathscr{B}$ is weakly dominated, then there exists an internal p-content $\nu|\mathscr{B}$ such that $\nu_L|\mathscr{B}$ uniformly dominates $\mathscr{G}_L|\mathscr{B}$.*

PROOF. According to Lemma 1, $\mathscr{G}_L|\mathscr{B}$ is dominated. Hence there exist $Q_n \in \mathscr{G}$, $n \in \mathbf{N}$, such that $\sum_{n\in\mathbf{N}} 2^{-n}(Q_n)_L$ dominates $\mathscr{G}_L|\mathscr{B}$ (use Corollary 3). Since $\mathscr{G}$ is an internal set, and since our model is polysaturated, there exists an internal extension $(Q_H)_{H\in{}^*\mathbf{N}} \subset \mathscr{G}$ of $(Q_n)_{n\in\mathbf{N}}$. Put

$$\nu = \sum_{H\in{}^*\mathbf{N}} \frac{1}{2^H} Q_H.$$

Then $\nu|\mathscr{B}$ is an internal p-content. Furthermore, $\nu_L|\mathscr{B}$ dominates $\mathscr{G}_L|\mathscr{B}$.

If $B \in \mathscr{B}$ and $\nu_L(B) = 0$, then $Q_n(B) \approx 0$ for all $n \in \mathsf{N}$ and hence $Q_L(B) = 0$ for all $Q \in \mathscr{G}$; therefore $\mathscr{G}_L$ is dominated by ν_L according to Lemma 1.

Let $\varepsilon \in \mathsf{R}_+$ be fixed and put for each $\delta \in \mathsf{R}_+$

$$\mathscr{G}_\delta := \{Q \in \mathscr{G} | (\forall B \in \mathscr{B})(\nu(B) < \delta \Rightarrow Q(B) < \varepsilon)\}.$$

As $\nu_L|\mathscr{B}$ dominates $\mathscr{G}_L|\mathscr{B}$, we obtain $\mathscr{G} = \bigcup_{\delta \in \mathsf{R}_+} \mathscr{G}_\delta$. As $\mathscr{G}$ and $\mathscr{G}_\delta$, $\delta \in \mathsf{R}_+$, are internal sets and since our model is polysaturated, there exists $\delta \in \mathsf{R}_+$ such that $\mathscr{G} = \mathscr{G}_\delta$. Consequently $\mathscr{G}_L|\mathscr{B}$ is uniformly dominated by $\nu_L|\mathscr{B}$.

5. THEOREM. *Let $\mathscr{A}$ be an algebra, and let $\mathscr{G}$ be an internal family of p-contents on $^*\mathscr{A}$. If for each $Q \in \mathscr{G}$ there exists a p-content $P|\mathscr{A}$ such that $^*P_L|^*\mathscr{A}$ dominates $Q_L|^*\mathscr{A}$, then there exists a p-content $\mu|\mathscr{A}$ such that $^*\mu_L|^*\mathscr{A}$ uniformly dominates $\mathscr{G}_L|^*\mathscr{A}$.*

PROOF. Let $\varepsilon, \delta \in \mathsf{R}_+$, $P|\mathscr{A}$ be a p-content and put

$$\mathscr{G}_{P,\delta,\varepsilon} := \{Q \in \mathscr{G} | (\forall A \in {}^*\mathscr{A})({}^*P(A) < \delta \Rightarrow Q(A) < \varepsilon)\}.$$

By assumption we obtain for each $\varepsilon \in \mathsf{R}_+$ that $\mathscr{G} = \bigcup\{\mathscr{G}_{P,\delta,\varepsilon} : \delta \in \mathsf{R}_+,\ P|\mathscr{A}\ p\text{-content}\}$.

Since $\mathscr{G}$ and $\mathscr{G}_{P,\delta,\varepsilon}$ are internal and since our model is polysaturated there exists p-contents $P_1^\varepsilon, \ldots, P_{n(\varepsilon)}^\varepsilon$ on $\mathscr{A}$ and $\delta(\varepsilon) \in \mathsf{R}_+$ such that for all $A \in {}^*\mathscr{A}$ and all $Q \in \mathscr{G}$:

$$^*P_i^\varepsilon(A) < \delta(\varepsilon) \quad \text{for } i = 1, \ldots, n(\varepsilon) \ \Rightarrow\ Q(A) < \varepsilon.$$

Let P_n, $n \in \mathsf{N}$, be a denumeration of $\{P_\nu^\varepsilon : \nu \leqq n(\varepsilon),\ \varepsilon = 1/m,\ m \in \mathsf{N}\}$ and put $\mu = \sum_{n \in \mathsf{N}} 2^{-n} P_n$. Then $^*\mu_L|^*\mathscr{A}$ uniformly dominates $\mathscr{G}_L|^*\mathscr{A}$.

6. COROLLARY. *Let $\mathscr{P}$ be a family of p-contents on an algebra $\mathscr{A}$. Then the following four conditions are equivalent:*

(i) *$\mathscr{P}|\mathscr{A}$ is uniformly dominated;*
(ii) *$^*\mathscr{P}_L|^*\mathscr{A}$ is weakly dominated;*
(iii) *$^*\mathscr{P}_L|^*\mathscr{A}$ is uniformly dominated;*
(iv) *for each $Q \in {}^*\mathscr{P}$ there exists a p-content $P|\mathscr{A}$ such that $^*P_L|^*\mathscr{A}$ dominates $Q_L|^*\mathscr{A}$.*

PROOF. (i) $\Rightarrow$ (iv) by transfer; (iv) $\Rightarrow$ (iii) by Theorem 5; (iii) $\Rightarrow$ (ii) is trivial. (ii) $\Rightarrow$ (i): If (ii) holds, then by Theorem 4 there exists an internal p-content $\nu|^*\mathscr{A}$ such that $\nu_L|^*\mathscr{A}$ uniformly dominates $^*\mathscr{P}_L|^*\mathscr{A}$. Hence $\mathscr{P}|\mathscr{A}$ is uniformly dominated by the p-content $P|\mathscr{A}$, given by $P(A) := \nu_L({}^*A)$ for $A \in \mathscr{A}$.

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