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CONTINUITY PROPERTIES OF RIESZ POTENTIALS AND BOUNDARY LIMITS OF BEPPO LEVI FUNCTIONS

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Abstract.

This paper deals with various properties of α-potentials of functions f satisfying the condition that $\int_{R^n} \Phi(|f(y)|)dy < \infty$, where Φ is a positive nondecreasing function on R^1 such that for any $\varepsilon > 0$, $Ar^{n/\alpha} < \Phi(r) < Br^{n/\alpha+\varepsilon}$ whenever $r > 1$ with positive constants A and B. Of corse, there are many known results in case $\Phi(r) = r^p$, which belong to the nonlinear potential theory. According as $\alpha p \leqq n$ or $\alpha p > n$, the results will take on a different aspect. Our results given below will be similar to those in the case $\alpha p > n$.

The results obtained for Riesz potentials will be valid for Beppo Levi functions, by the aid of integral representations. We shall also be concerned with the existence of boundary limits of Beppo Levi functions in a half space of R^n.

1. Introduction.

For a nonnegative locally integrable function f on R^n, we write $R_\alpha f(x) = \int R_\alpha(x-y)f(y)dy$, where $R_\alpha(x) = |x|^{\alpha-n}$, $0 < \alpha < n$. We then note that $R_\alpha f \not\equiv \infty$ if and only if $\int_{R^n - B(x,1)} R_\alpha(x-y)f(y)\,dy < \infty$ for some x, where $B(x,r)$ denotes the open ball with center at x and radius r; this is equivalent to

$$(1) \qquad \int (1+|y|)^{\alpha-n} f(y)\,dy < \infty.$$

In this note we are concerned with the following properties:

(1) Continuity and differentiability of $R_\alpha f$.
(2) Behavior at infinity of $R_\alpha f$.
(3) The existence of boundary limits of Beppo Levi functions.

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In case $f \in L^p(R^n)$, many authors discussed these properties for $R_\alpha f$, and obtained a great number of results (see e.g. Aikawa [1], Kurokawa-Mizuta [2], Meyers [4], Mizuta [5], [7] and Ohtsuka [9]). If α is a positive integer, then $R_\alpha f$ will be seen to be a Beppo Levi function. Conversely, Beppo Levi functions will have integral representations with kernel functions introuced by modifications of Riesz kernels (cf. [5], [7]). Thus, the results obtained below for Riesz potentials will also be valid for Beppo Levi functions.

In what follows we study several problems concerning such properties for α-potentials of nonnegative functions f satisfying (1) together with the following condition:

$$(2) \qquad \int f(y)^p \omega(f(y))\, dy < \infty,$$

where $p = n/\alpha > 1$ and ω is a positive nodecreasing function on the interval $(0, \infty)$ such that

$$(\omega_1) \qquad \int_1^\infty \omega(t)^{-1/(p-1)} t^{-1}\, dt < \infty$$

and

(ω_2) $\qquad \omega(2r) < A\omega(r)$ for any $r > 0$ with a positive constant A.

As typical examples of ω, we give

$$\omega(r) = [\log(2 + r)]^\delta, [\log(2 + r)]^{p-1} [\log(2 + \log(2 + r))]^\delta, \ldots,$$

where $\delta > p - 1 > 0$.

The author has already obtained some results in the papers [6] and [8], and the present paper is an extension of these papers.

2. Continuity and differentiability.

First of all we note that if $\alpha p > n$ and f is a nonnegative measurable function in $L^p(R^n)$ such that $R_\alpha f \not\equiv \infty$, then $R_\alpha f$ is continuous on R^n; this fact follows readily from Sobolev's imbedding theorem. In case $\alpha p \leqq n$, $R_\alpha f$ may not be continuous anywhere but it is quasi continuous in a certain sense (cf. Meyers [4]).

Our first result will asure the continuity of $R_\alpha f$ if f satisfies condition (2) with $p = n/\alpha$; the case $\alpha p > n$ is reduced to the present case if we replace ω by $r^\varepsilon \omega(r)$ with $\varepsilon > 0$.

THEOREM 1. *If f is a nonnegative measurable function on R^n satisfying* (1) *and* (2) *with $p = n/\alpha > 1$, then $R_\alpha f$ is continuous on R^n.*

For a proof of this theorem we need the following Hölder type inequality.

LEMMA 1. *There exists a positive constant M such that*

$$\int_{\{y;g(y)\geqq a\}} R_\alpha(x-y)g(y)\,dy$$
$$\leqq M\left(\int_{\{y;g(y)\geqq a\}} g(y)^p\,\omega(g(y))\,dy\right)^{1/p}\left(\int_a^\infty \omega(t)^{-1/(p-1)}t^{-1}\,dt\right)^{1/p'}$$

for any $a>0$ and any nonnegative measurable function g on R^n, where $\alpha p=n$ and $1/p+1/p'=1$.

PROOF. Define $G_j=\{y\in R^n; 2^{j-1}a\leqq g(y)<2^j a\}$ for each positive integer j, and take $r_j\geqq 0$ such that $|G_j|=|B(0,r_j)|$, where $|E|$ denotes the Lebesgue measure of a set $E\subset R^n$. Then we note that

$$\int_{\{y;g(y)\geqq a\}} R_\alpha(x-y)g(y)\,dy=\sum_{j=1}^\infty\int_{G_j}|x-y|^{\alpha-n}g(y)\,dy$$
$$\leqq\sum_{j=1}^\infty 2^j a\int_{G_j}|x-y|^{\alpha-n}\,dy\leqq\sum_{j=1}^\infty 2^j a\int_{B(x,r_j)}|x-y|^{\alpha-n}\,dy$$
$$=M_1\sum_{j=1}^\infty 2^j a|G_j|^{\alpha/n}$$
$$\leqq M_2\left(\sum_{j=1}^\infty (2^{j-1}a)^p\,\omega(2^{j-1}a)|G_j|\right)^{1/p}\left(\sum_{j=1}^\infty \omega(2^j a)^{-1/(p-1)}\right)^{1/p'}$$
$$\leqq M_3\left(\int g(y)^p\,\omega(g(y))\,dy\right)^{1/p}\left(\int_a^\infty \omega(t)^{-1/(p-1)}t^{-1}\,dt\right)^{1/p'},$$

where M_1, M_2 and M_3 are positive constants independent of g, x and a. Thus Lemma 1 is proved.

PROOF OF THEOREM 1. We have only to prove that $R_\alpha f$ is continuous at the origin 0. For $x\in R^n$ and $r>0$, we write

$$R_\alpha f(x)=\int_{B(0,r)} R_\alpha(x-y)f(y)\,dy+\int_{R^n-B(0,r)} R_\alpha(x-y)f(y)\,dy$$
$$=u_r'(x)+u_r''(x).$$

Applying Lemma 1 with $a=1$, we have

$$u_r'(x)\leqq M\left\{r^\alpha+\left(\int_{B(0,r)} f(y)^p\,\omega(f(y))\,dy\right)^{1/p}\right\},$$

where M is a positive constant independent of r. Hence $u'_r(0)$ is finite, so that $R_\alpha f(0)$ is finite. Moreover, letting $r = 2|x|$, we see that $\lim_{x\to 0} u'_{2|x|}(x) = 0$.

If $y \in R^n - B(0,2|x|)$, then $|x - y| \geqq (1/2)|y|$. Since $R_\alpha f(0) < \infty$ as seen above, it follows from Lebesgue's dominated convergence theorem that $\lim_{x\to 0} u''_{2|x|}(x) = R_\alpha f(0)$. Thus Theorem 1 is established.

We next consider the differentiability properties of $R_\alpha f$. A function u is said to be totally m times differentiable at x_0 if there exists a polynomial P of degree at most m such that $\lim_{x\to x_0} |x - x_0|^{-m}[u(x) - P(x)] = 0$. To evaluate the size of the exceptional sets, we use the Bessel capacities; $B_{\beta,p}$ is used to denote the Bessel capacity of index (β, p) (see Meyers [3] for the definition and properties of Bessel capacities). If $\beta = 0$, then $B_{\beta,p}$ is understood as the n-dimensional Lebesgue measure.

As a generalization of the result in [6], we give the following result.

THEOREM 2. *Let f be as in theorem 1. If $\alpha p = n$ and m is a positive integer such that $m \leqq \alpha$, then there exists a subset E of R^n such that $B_{\alpha-m,p}(E) = 0$ and $R_\alpha f$ is totally m times differentiable at any point of $R^n - E$.*

The proof is similar to that of [6; Theorem 1], where we were concerned with the special case: $\omega(r) = [\log(2 + r)]^\delta$ with $\delta > p - 1$. Thus we give a sketch of a proof for readers' convenience. For this purpose we prepare several lemmas.

LEMMA 2. *For a nonnegative integer m, we set*

$$K_m(x,y) = R_\alpha(x-y) - \sum_{|\lambda|\leqq m} (\lambda!)^{-1} x^\lambda [(\partial/\partial x)^\lambda R_\alpha](-y).$$

Then there exists a positive constant M such that

$$|K_m(x,y)| \leqq M|x|^{m+1}|y|^{\alpha-n-m-1} \quad \textit{whenever } y \in R^n - B(0,2|x|).$$

This follows readily from the mean value theorem.

LEMMA 3. *Let f be a nonnegative measurable function on R^n satisfying (2). For a positive number β, we define*

$$E = \left\{x \in R^n; \limsup_{r\downarrow 0} r^{\alpha-\beta-n} \int_{B(x,r)} |f(y) - f(x)|\,dy > 0\right\}.$$

Then $H_{\beta p}(E) = 0$. Moreover, if $\beta < \alpha$ and ω is assumed in addition to be continuous and satisfy

$$(\omega_3) \qquad \omega(r^2) \leqq A\omega(r) \quad \text{for } r \in (1,\infty)$$

with a positive constant A, then $H_h(E) = 0$, where $h(r) = r^{\beta p}\,\omega^*(r^{-1})$ with $\omega^*(r) = \left(\int_r^\infty \omega(t)^{-1/(p-1)}\,t^{-1}\,dt\right)^{1-p}$ and H_h denotes the Hausdorff measure with the measure function h.

REMARK. From condition (ω_3) it follows that $\lim_{r\to\infty} r^{-\delta}\omega(r) = 0$ and $h(r) = r^\delta\,\omega^*(r^{-1})$ is nondecreasing on some interval $(0, C_\delta)$, $C_\delta > 0$, for $\delta > 0$.

PROOF OF LEMMA 3. We shall give a proof only in the case $\beta < \alpha$, because the remaining case can be proved similarly. By Lemma 1 we have

$$r^{\alpha-\beta-n}\int_{B(x,r)} f(y)\,dy \leqq r^{-\beta}\int_{B(x,r)} |x-y|^{\alpha-n} f(y)\,dy$$
$$\leqq M_1 r^{-\beta}\left\{ar^\alpha + \omega^*(a)^{-1/p}\left(\int_{B(x,r)} f(y)^p\,\omega(f(y))\,dy\right)^{1/p}\right\}$$

for any $a > 0$, where M_1 is a positive constant independent of a, x and r. Hence, if we take $a = r^{-\delta}$, $0 < \delta < \alpha - \beta$, then by condition (ω_3) there exists $M_2 > 0$ such that

$$\limsup_{r\downarrow 0} r^{\alpha-\beta-n}\int_{B(x,r)} f(y)\,dy$$
$$\leqq M_2 \limsup_{r\downarrow 0}\left(h(r)^{-1}\int_{B(x,r)} f(y)^p\,\omega(f(y))\,dy\right)^{1/p}.$$

Since $f(y)^p\,\omega(f(y)) \in L^1(R^n)$, with the aid of the fact in [3; p. 165], we obtain the desired result.

PROOF OF THEOREM 2. For $x_0 \in R^n$ and a multi-index λ with $|\lambda| \leqq m$, define

$$A_\lambda = \lim_{r\to 0}\int_{R^n - B(x_0,r)} [(\partial/\partial x)^\lambda R_\alpha](x_0 - y) f(y)\,dy.$$

If $|\lambda| = m = \alpha$, then the limit exists and is finite for almost every x_0 (see [10; Theorem 4 in §11]), and if $|\lambda| < \alpha$, then the limit exists and is finite for x_0 such that $\int |x_0 - y|^{\alpha-|\lambda|-n} f(y)\,dy < \infty$. Letting $g(y) = f(y)$ if $f(y) > 1$ and $g(y) = 0$ otherwise, we see that $g \in L^p(R^n)$ and $\int |x_0 - y|^{\beta-n} f(y)\,dy = \infty$ if and only if $\int |x_0 - y|^{\beta-n} g(y)\,dy = \infty$, where $\beta > 0$. Hence we can find a set $E_1 \subset R^n$ such that $B_{\alpha-m,p}(E_1) = 0$ and if $x_0 \in R^n - E_1$, then A_λ exists and is finite for any multi-index λ with $|\lambda| \leqq m$. In what follows we assume that $x_0 \in R^n - E_1$.

We next note that $\int_{B(0,1)} R_\alpha(x-y)\,dy$ is infinitely differentiable in $B(0,1)$. We thus let $B_\lambda = 0$ if $|\lambda| < m$ and

$$B_\lambda = (\partial/\partial x)^\lambda \int_{B(0,1)} R_\alpha(x-y)\,dy\Big|_{x=0} \quad \text{if } |\lambda| = m,$$

and consider the numbers $C_\lambda = A_\lambda + f(x_0)B_\lambda$. We now define

$$P(x) = \sum_{|\lambda| \leq m} (\lambda!)^{-1} C_\lambda (x - x_0)^\lambda.$$

Letting $K_l(x,y) = R_\alpha(x-y) - \sum_{|\lambda| \leq l} (\lambda!)^{-1}(x-x_0)^\lambda [(\partial/\partial x)^\lambda R_\alpha](x_0 - y)$, we write

$$\begin{aligned}
&|x-x_0|^{-m}\{R_\alpha f(x) - P(x)\} \\
= \;& |x-x_0|^{-m} \int_{R^n - B(x_0,1)} K_m(x,y) f(y)\,dy \\
&+ |x-x_0|^{-m} \int_{B(x_0,1) - B(x_0, 2|x-x_0|)} K_m(x,y)[f(y) - f(x_0)]\,dy \\
&- |x-x_0|^{-m} \sum_{|\lambda| \leq m} (\lambda!)^{-1}(x-x_0)^\lambda \\
&\times \lim_{r \downarrow 0} \int_{B(x_0, 2|x-x_0|) - B(x_0, r)} (\partial/\partial x)^\lambda R_\alpha(x_0 - y)[f(y) - f(x_0)]\,dy \\
&+ f(x_0)|x-x_0|^{-m}\left(\int_{B(x_0,1)} K_{m-1}(x,y)\,dy - \sum_{|\lambda| = m} (\lambda!)^{-1} B_\lambda (x-x_0)^\lambda\right) \\
&+ |x-x_0|^{-m} \int_{B(x_0, 2|x-x_0|)} R_\alpha(x-y)[f(y) - f(x_0)]\,dy \\
= \;& I_1 + I_2 + I_3 + I_4 + I_5,
\end{aligned}$$

since $\int_{B(0,r)-B(0,s)} (\partial/\partial x)^\lambda R_m(x)\,dx = 0$ for any $r, s > 0$ and any λ with $|\lambda| = m = \alpha$.

By condition (1) and Lemma 2, we can apply Lebesgue's dominated convergence theorem to prove that I_1 tends to zero as $x \to x_0$. In view of Lemma 3, we can find a set $E_2 \subset R^n$ such that $H_{mp}(E_2) = 0$ and $\lim_{r \downarrow 0} r^{\alpha - m - n} \int_{B(x_0, r)} |f(y) - f(x_0)|\,dy = 0$ for any $x_0 \in R^n - E_2$. Hence it follows that I_2 and I_3 tend to zero as $x \to x_0$ for $x_0 \in R^n - E_2$. Further, the definition of B_λ implies that I_4 tends to zero as $x \to x_0$. Finally, setting $g(y) = |f(y) - f(x_0)|$ for simplicity, we establish with the aid of Lemma 1

$$|I_5| \leqq M_1 a|x - x_0|^{\alpha - m} + |x - x_0|^{-m} \int_{\{y \in B(x_0, 2|x - x_0|); g(y) > a\}} R_\alpha(x - y) g(y)\,dy$$

$$\leqq M_1 a|x - x_0|^{\alpha - m} + M_2|x - x_0|^{-m}\left(\int_{B(x, 2|x - x_0|)} g(y)^p \omega(g(y))\,dy\right)^{1/p} \times \left(\int_a^\infty \omega(t)^{-1/(p-1)} t^{-1}\,dt\right)^{1/p'},$$

where M_1 and M_2 are positive constants independent of x and a. Define

$$E_3 = \{x \in R^n; \limsup_{r \downarrow 0} r^{-mp} \int_{B(x,r)} |f(y)^p \omega(f(y)) - f(x)^p \omega(f(x))|\,dy > 0\}.$$

Then we have $H_{mp}(E_3) = 0$. Moreover, if $x_0 \in R^n - E_3$, then, since $(t + s)^p \omega(t + s) - t^p \omega(t) \geqq s^p \omega(s)$ for $t, s \geqq 0$, we see that $\limsup_{x \to x_0} |I_5| \leqq M_1 a$, which implies that $\lim_{x \to x_0} I_5 = 0$. Thus $R_\alpha f$ is totally m times differentiable at $x_0 \in R^n - E_1 \cup E_2 \cup E_3$ and, in view of [3; Theorem 21], $B_{\alpha - m, p}(E_1 \cup E_2 \cup E_3) = 0$. Now the proof of Theorem 2 is completed.

We next consider the existence of weak sense derivatives. For $z \in R^n$ and a function u on R^n, we set $\Delta_z u(x) = u(x + z) - u(x)$, and define $\Delta_z^m = \Delta_z(\Delta_z^{m-1})$ inductively with $\Delta_z^1 = \Delta_z$. Note here that $\Delta_z^m u$ is of the form $\sum_{k=0}^{m} a_k u(x + kz)$, where $a_k = (-1)^{m-k}\binom{m}{k}$.

THEOREM 3. *Let $p = n/\alpha > 1$ and ω be a nondecreasing function on $(0, \infty)$ satisfying conditions (ω_1), (ω_2) and (ω_3). If f is a nonnegative measurable function on R^n satisfying (1) and (2), then, for a positive integer m such that $m \leqq \alpha$ and a positive number β,*

$$\lim_{x \to 0} |x|^{\beta - m} \Delta_x^m R_\alpha f(x_0) = 0 \tag{3}$$

holds when $x_0 \in F_1 \cup F_2$, where

$$F_1 = \left\{x \in R^n; \lim_{r \downarrow 0} r^{\alpha - n - m + \beta} \int_{B(x,r)} f(y)\,dy = 0\right\}$$

and

$$F_2 = \left\{x \in R^n;\ \lim_{r \downarrow 0} h_\beta(r)^{-1} \int_{B(x,r)} f(y)^p \omega(f(y))\,dy = 0\right\}$$

with $h_\beta(r) = r^{(m-\beta)p} \omega^*(r^{-1})$.

REMARK. In view of Lemma 3, $H_{h_\beta}(R^n - F_1 \cup F_2) = 0$.

PROOF OF THEOREM 3. For $x_0 \in R^n$, we write

$$R_\alpha f(x) = \int_{R^n - B(x_0,1)} |x-y|^{\alpha-n} f(y)\,dy + \int_{B(x_0,1)} |x-y|^{\alpha-n} [f(y) - f(x_0)]\,dy$$

$$+ f(x_0) \int_{B(x_0,1)} |x-y|^{\alpha-n}\,dy = u_1(x) + u_2(x) + f(x_0)\,u_3(x).$$

Then it is easy to see that u_1 and u_3 are infinitely differentiable on $B(x_0, 1)$. Thus it suffices to show that u_2 satisfies (3) for $x_0 \in F_1 \cup F_2$. For simplicity, we assume that $x_0 = 0$, $f(0) = 0$ and f vanishes outside $B(0, 1)$; in this case, $u_2 = R_\alpha f$. Write

$$\Delta_x^m R_\alpha f(0) = \int_{R^n - B(0,(m+2)|x|)} (\Delta_x^m R_\alpha)(-y) f(y)\,dy$$

$$+ \int_{B(0,(m+2)|x|)} (\Delta_x^m R_\alpha)(-y) f(y)\,dy = U'(x) + U''(x).$$

If $y \in R^n - B(0,(m+2)|x|)$, then we see by the mean value theorem that $|\Delta_x^m R_\alpha(-y)| \leqq M_1 |x|^m |y|^{\alpha-n-m}$ with a positive constant M_1. Hence

$$|x|^{\beta-m} |U'(x)| \leqq M_1 |x|^\beta \int_{R^n - B(0,(m+2)|x|)} |y|^{\alpha-m-n} f(y)\,dy$$

$$\leqq M_1 |x|^\beta \int_{R^n - B(0,\varepsilon)} |y|^{\alpha-m-n} f(y)\,dy$$

$$+ M_1 |x|^\beta \int_{(m+2)|x|}^{\varepsilon} \left(\int_{B(0,r)} f(y)\,dy\right) d(-r^{\alpha-m-n})$$

$$\leqq M_1 |x|^\beta \int_{R^n - B(0,\varepsilon)} |y|^{\alpha-m-n} f(y)\,dy + M_2 A(\varepsilon),$$

where $A(\varepsilon) = \sup_{0<r\leqq\varepsilon} r^{\alpha-n-m+\beta} \int_{B(0,r)} f(y)\,dy$ and M_2 is a positive constant independent of ε and x. Thus it follows that $|x|^{\beta-m} U'(x)$ tends to zero as $x \to 0$. On the

other hand, Lemma 1 gives

$$|x|^{\beta-m}|U''(x)| \leqq M_3|x|^{\beta-m}\sum_{k=0}^{m}\int_{B(0,(m+2)|x|)}|kx-y|^{\alpha-n}f(y)\,dy$$

$$\leqq M_4|x|^{\beta-m}\left(\omega^*(a)^{-1}\int_{B(0,(m+2)|x|)}f(y)^p\,\omega(f(y))\,dy\right)^{1/p}+M_4\,a|x|^{\beta-m+\alpha}$$

for any $a > 0$, where M_3 and M_4 are positive constants independent of a and x. Hence, taking $a = |x|^{-\beta/2}$, we see that $|x|^{\beta-m}U''(x)$ tends to zero as $x \to 0$. Thus Theorem 3 is proved.

THEOREM 4. *Let f be as in Theorem 3. For a nonnegative integer m such that $m \leqq \alpha$, we set*

$$F_1 = \left\{x \in R^n;\ A_\lambda = \lim_{r\downarrow 0}\int_{B(x,r)}((\partial/\partial x)^\lambda R_\alpha)(x-y)f(y)\,dy\right.$$
$$\left.\text{exists and is finite for any } \lambda \text{ with } |\lambda| = m\right\},$$

$$F_2 = \left\{x \in R^n;\ \lim_{r\downarrow 0} r^{\alpha-m-n}\int_{B(x,r)}|f(y)-f(x)|\,dy = 0\right\}$$

and

$$F_3 = \left\{x \in R^n;\ \lim_{r\downarrow 0} r^{-mp}\,\omega^*(r^{-1})^{-1}\int_{B(x,r)}f(y)^p\,\omega(f(y))\,dy = 0\right\}.$$

If $x_0 \in F_1 \cup F_2 \cup F_3$, then

$$\lim_{x\to 0}|x|^{-m}[\Delta_x^m R_\alpha f(x_0) - P_{x_0}(x)] = 0,$$

where $P_{x_0}(x) = \sum_{|\lambda|=m}(C_\lambda/\lambda!)\,x^\lambda$ and $C_\lambda = A_\lambda + B_\lambda f(x_0)$ with B_λ defined in the proof of Theorem 2.

This fact can be proved in the same way as Theorems 2 and 3. Here we note that $H_h(R^n - F_2 \cup F_3) = 0$, where $h(r) = r^{mp}\omega^*(r^{-1})$, whereas $H_{mp}(R^n - F_1) = 0$. Hence we only know that $H_{mp}(R^n - F_1 \cup F_2 \cup F_3) = 0$.

3. Behavior at infinity of $R_\alpha f$.

Let f be a nonnegative function in $L^p(R^n)$ satisfying (1). Then we note that

(i) if $\alpha p > n$, then $|x|^{(n-\alpha p)/p} R_\alpha f(x) \to 0$ as $|x| \to \infty$;

(ii) if $\alpha p \leqq n$, then $|x|^{(n-\alpha p)/p} R_\alpha f(x) \to 0$ as $|x| \to \infty$, $x \in R^n - E$, where E is (α, p)-thin at infinity;

(see [2] for a proof of these facts).

For a function f as in Theorem 1, we investigate the limit at infinity of $R_\alpha f$. As before, we consider the function $\omega^*(r) = \left(\int_r^\infty \omega(t)^{-1/(p-1)} t^{-1} \, dt\right)^{1-p}$, where ω is a nondecreasing positive function on $(0, \infty)$ satisfying (ω_1), (ω_2) and the following condition:

(ω_4) There exists $A > 0$ such that $\omega(r) \leqq A\omega(r^2)$ for any $r \in (0,1)$.

Then it is easy to see that ω^* also satisfies condition (ω_4) and $\lim_{r \downarrow 0} r^{-\delta} \omega(r) = \lim_{r \downarrow 0} r^{-\delta} \omega^*(r) = \infty$ for $\delta > 0$.

Now we prove the following result.

THEOREM 5. *Let f be a nonnegative measurable function on R^n satisfying* (1) *and* (2) *with $p = n/\alpha > 1$. Then $\omega^*(|x|^{-1})^{1/p} R_\alpha f(x)$ tends to zero as $|x| \to \infty$.*

REMARK. Since ω is nondecreasing, $\omega^*(r) \leqq \omega(1) [\log r^{-1}]^{1-p}$ for $r \in (0, 1/2)$. If $\lim_{r \downarrow 0} \omega(r) > 0$, then there exists a positive constant $M > 0$ such that $[\log(1/r)]^{1-p} \leqq M\omega^*(r)$ for $r \in (0, 1)$. Hence, in this case, $(\log|x|)^{-1/p'} R_\alpha f(x) \to 0$ as $|x| \to \infty$.

PROOF OF THEOREM 5. For $x \in R^n - \{0\}$, we write

$$R_\alpha f(x) = \int_{B(x,|x|/2)} R_\alpha(x-y) f(y) \, dy + \int_{R^n - B(x,|x|/2)} R_\alpha(x-y) f(y) \, dy$$
$$= u_1(x) + u_2(x).$$

First we note that $|y| \leqq |x| + |x-y| \leqq 3|x-y|$ if $y \in R^n - B(x, |x|/2)$, so that we can find a positive constant M_1 such that $u_2(x) \leqq M_1 \int (|x| + |y|)^{\alpha - n} f(y) \, dy$. Since (1) holds, we have by Lebesgue's dominated convergence theorem

$$\lim_{|x| \to \infty} u_2(x) = 0.$$

We are next concerned with the estimate of u_1. By Lemma 1 there exists a positive constant M_2 such that

$$\int_{\{y;\, f(y)>a\}} R_\alpha(x-y)f(y)\,dy$$
$$\leqq M_2\left(\int_{\{y;\, f(y)>a\}} f(y)^p\,\omega(f(y))\,dy\right)^{1/p}\left(\int_a^\infty \omega(t)^{-1/(p-1)}t^{-1}\,dt\right)^{1/p'}$$

for any $a>0$ and any $x\in R^n-\{0\}$. Hence, taking $a=|x|^{-\delta}$, $\delta>\alpha$, and using condition (ω_4), we obtain

$$u_1(x)\leqq M_3\left\{|x|^{\alpha-\delta}+\left(\omega^*(|x|^{-1})^{-1}\int_{B(x,|x|/2)} f(y)^p\,\omega(f(y))\,dy\right)^{1/p}\right\},$$

so that

$$\lim_{|x|\to\infty}\omega^*(|x|^{-1})^{1/p}u_1(x)=0.$$

Thus the required equality follows.

PROPOSITION 1. *Let $\alpha p=n$ and ω be a nondecreasing function on the interval $(0,\infty)$ satisfying (ω_1), (ω_2) and (ω_4). Then for any positive nondecreasing function $a(r)$ on R^1 such that $\lim_{r\to\infty}a(r)=\infty$, there exists a nonnegative measurable function f satisfying* (1) *and* (2) *such that*

$$\limsup_{|x|\to\infty} a(|x|)\,\omega^*(|x|^{-1})^{1/p}R_\alpha f(x)=\infty.$$

PROOF. Let $\{k_j\}$ be a sequence of positive integers such that $2k_j<k_{j+1}$ and $\sum_{j=1}^\infty a(2^{2k_j})^{-1}<\infty$. Setting $e_j=(2^{2k_j},0,\dots,0)\in R^n$, we define

$$f(y)=a(|e_j|)^{-1/p}\,\omega^\sim(|e_j|^{-1})^{-1/p}|e_j-y|^{-\alpha}\left[\omega(|e_j-y|^{-1})\right]^{-1/(p-1)}$$

if $1<|e_j-y|<2^{k_j-1}$ and $f(y)=0$ elsewhere, where $\omega^\sim(r)=\int_r^\infty \omega(t)^{-1/(p-1)}t^{-1}\,dt$. Then, since $f(y)\leqq M_1|e_j-y|^{-\alpha/2}$ on account of (ω_4) with a positive constant M_1, we have

$$\int f(y)^p\,\omega(f(y))\,dy$$
$$\leqq M_2\sum_{j=1}^\infty a(|e_j|)^{-1}\,\omega^\sim(|e_j|^{-1})^{-1}\int_{B(0,2^{k_j-1})}|y|^{-n}\,\omega(|y|^{-1})^{-p/(p-1)+1}\,dy$$
$$\leqq M_3\sum_{j=1}^\infty a(|e_j|)^{-1}<\infty,$$

$$\int (1 + |y|)^{\alpha-n} f(y)\,dy \leqq M_4 \sum_{j=1}^{\infty} a(|e_j|)^{-1/p} \omega^{\sim}(|e_j|^{-1})^{-1/p} |e_j|^{\alpha-n} \times \int_{B(0,2^{k_j-1})} |y|^{-\alpha} \omega(|y|^{-1})^{-1/(p-1)}\,dy$$

$$\leqq M_5 \sum_{j=1}^{\infty} 2^{k_j(\alpha-n)} \omega^{\sim}(2^{-k_j})^{1/p'} < \infty$$

and

$$\int |e_j - y|^{\alpha-n} f(y)\,dy$$

$$\geqq a(|e_j|)^{-1/p} \omega^{\sim}(|e_j|^{-1})^{-1/p} \int_{B(0,2^{k_j-1})-B(0,1)} |y|^{-n} \omega(|y|^{-1})^{-1/(p-1)}\,dy$$

$$\geqq M_6\, a(|e_j|)^{-1/p} \omega^{*}(|e_j|^{-1})^{-1/p},$$

where $M_2, \ldots, M_6$ are positive constants independent of j. Thus f has all the conditions in the proposition.

4. Logarithmic potentials.

For a nonnegative locally integrable function f on R^n, we define $Lf(x) = \int \log(1/|x-y|) f(y)\,dy$. Then it is noted that $Lf \not\equiv -\infty$ if and only if

(4) $$\int \log(2 + |y|) f(y)\,dy < \infty.$$

Here we collect several results concerning logarithmic potentials.

THEOREM 1′. *Let f be a nonnegative measurable function on R^n satisfying* (4) *and*

(5) $$\int f(y) \log(2 + f(y))\,dy < \infty.$$

Then Lf is continuous on R^n.

THEOREM 2′. *If f is as in Theorem 1′, then Lf is totally n times differentiable almost everywhere on R^n.*

THEOREM 3′. *If f is as in Theorem 1′, then*

$$Lf(x) + \left(\int f(y)\,dy\right) \log|x| \to 0 \qquad \text{as } |x| \to \infty.$$

These theorems can be proved in a way similar to proofs of Theorems 1, 2 and 5, respectively, if we note, instead of Lemma 1, the following lemma.

LEMMA 1′. *There exists a positive constant M such that*

$$\int_{\{y;f(y)>1\}} [\log(1/|x-y|)]f(y)\,dy \leqq M\,F\log(1/F)$$

for any nonnegative measurable function f on R^n, where

$$F = \int_{\{y;f(y)>1\}} f(y)\log(2+f(y))\,dy < e^{-1}.$$

5. Boundary limits of *p*-precise functions.

In this section we study the boundary limits of functions in the Beppo Levi space $BL_1(L^p_{\mathrm{loc}}(R^n_+))$, where R^n_+ is the half space $\{(x', x_n) \in R^{n-1} \times R^1; x_n > 0\}$. Such functions with certain quasi continuity are called locally p-precise (see Ohtsuka [9]). We know several existence theorems of boundary limits for locally p-precise functions u on R^n_+ such that $\int_{R^n_+} |\mathrm{grad}\, u(x)|^p x_n^\alpha\,dx < \infty$. In these discussions, one of the main tools is a canonical or potential type integral representation of u (see [5] and [7]). This section is concerned with the existence of nontangential limits of u satisfying a stronger condition when we restrict ourselves to the case $p = n$. We say that a function u on R^n_+ has a nontangential limit at $\xi \in \partial R^n_+$ if $u(x)$ tends to a number as $x \to \xi$ along the cone $\Gamma(\xi, a)$ for any $a > 0$, where $\Gamma(\xi, a) = \{x = (x', x_n) \in R^{n-1} \times R^1;\ |(x', 0) - \xi| < ax_n\}$, $a > 0$. Before giving our result, we remark that if $p > n$ and u satisfies the above inequality, then u has nontangential limits at boundary points except those in a set E such that $B_{1-\alpha/p,p}(E) = 0$. However, if $p \leqq n$, then u may fail to have a nontangential limit at any boundary point; in this case, we are concerned with the existence of fine nontangential limits (cf. [5], [7]).

Letting ω be a positive nondecreasing function on R^1 satisfying (ω_1) with $p = n$ and (ω_2), we state our result concerning the existence of nontangential limits of locally n-precise functions on R^n_+.

THEOREM 6. *Let u be a locally n-precise function on R^n_+ such that* $\int_{R^n_+} |\mathrm{grad}\, u(x)|^n \omega(|\mathrm{grad}\, u(x)|)\, x_n^\alpha\,dx < \infty$. *If $0 < \alpha < n-1$ and u is continuous on R^n_+, then there exists a set $E \subset \partial R^n_+$ such that $B_{1-\alpha/n,n}(E) = 0$ and u has a nontangential limit at any boundary point in $\partial R^n_+ - E$.*

PROOF. Let u be as in Theorem 6. In view of the proof of Theorem 1 in [7], u can be extended to a function $\bar{u}$ which is locally p-precise on R^n for p such that $1 < p < n/(\alpha + 1)$ and satisfies $\int_{R^n} |\text{grad}\, \bar{u}(x)|^n \omega(|\text{grad}\, \bar{u}(x)|) |x_n|^\alpha \, dx < \infty$. Further we note the following integral representation of u:

$$u(x) = c \sum_{j=1}^{n} \int k_j(x, y)(\partial/\partial y_j) \bar{u}(y) \, dy + A \tag{6}$$

for almost every $x \in R^n$, where c is a constant depending only on the dimension n, A is a number which may depend on u and

$$k_j(x, y) = \begin{cases} [(\partial/\partial x_j) R_2](x - y) & \text{if } |y| < 1, \\ [(\partial/\partial x_j) R_2](x - y) - [(\partial/\partial x_j) R_2](-y) & \text{if } |y| \geqq 1. \end{cases}$$

For $N > 0$ and $x_0 \in R^n_+$ such that $x_{0,n} > 2N$, write

$$u_j(x) = \int_{B(x_0, N)} k_j(x, y)(\partial/\partial y_j) \bar{u}(y) \, dy$$

and

$$v_j(x) = \int_{R^n - B(x_0, N)} k_j(x, y)(\partial/\partial y_j) \bar{u}(y) \, dy.$$

Since $|k_j(x, y)| \leqq M_1 |x| \, |y|^{-n}$ whenever $|y| \geqq 1$ and $|y| \geqq 2|x|$ with a positive constant M_1, we see easily that v_j is continuous on $B(x_0, N)$. On the other hand, since $|k_j(x, y)| \leqq M_2 |x - y|^{1-n}$ whenever $y \in B(x, 2|x|)$ with a positive constant M_2, as in the proof of Theorem 1, we can show that u_j is continuous on R^n. Thus the right hand side of (6) is continuous on R^n_+, and hence equation (6) holds for any $x \in R^n_+$. For $x = (x', x_n) \in R^n_+$, we write

$$u_j'(x) = \int_{B(x, x_n/2)} k_j(x, y)(\partial/\partial y_j) \bar{u}(y) \, dy,$$

$$u_j''(x) = \int_{R^n - B(x, x_n/2)} k_j(x, y)(\partial/\partial y_j) \bar{u}(y) \, dy$$

and

$$U_j(x) = u_j'(x) + u_j''(x).$$

For simplicity, set $f_j(y) = |(\partial/\partial y_j) \bar{u}(y)|$. Since $|k_j(x, y)| \leqq M_2 |x - y|^{1-n}$ whenever $y \in B(x, x_n/2)$, it follows from Lemma 1 that

$$\begin{aligned} |u_j'(x)| &\leqq M_3 x_n + M_3 \left(\int_{B(x, x_n/2)} f_j(y)^n \omega(f_j(y)) \, dy \right)^{1/n} \omega^*(1)^{-1/n} \\ &\leqq M_3 x_n + M_4 \left(x_n^{-\alpha} \int_{B(x, x_n/2)} f_j(y)^n \omega(f_j(y)) \, y_n^\alpha \, dy \right)^{1/n} \end{aligned}$$

with positive constants M_3 and M_4 independent of x. Define

$$E_j = \left\{x \in \partial R^n_+;\ \limsup_{r \downarrow 0} r^{-\alpha} \int_{B(x,r)} f_j(y)^n\, \omega(f_j(y))\, |y_n|^\alpha\, dy > 0\right\}.$$

If $\xi \in \partial R^n_+ - E_j$, then we easily see that $u_j'(x)$ has nontangential limit zero at ξ.

We next consider the set

$$F_j = \left\{\xi \in \partial R^n_+;\ \int_{B(\xi,1)} |\xi - y|^{1-n} f_j(y)\, dy = \infty\right\}.$$

By Lebesgue's dominated convergence theorem we can show that u_j'' has a nontangential limit l at $\xi \in \partial R^n_+ - F_j$, where $l = \int_{R^n} k_j(\xi, y)(\partial/\partial y_j)\, u(y)\, dy$, which is finite. Thus U_j has a nontangential limit at any boundary point in $\partial R^n_+ - E_j \cup F_j$.

Since $H_\alpha(E_j) = 0$, we find by [3; Theorem 21] that $B_{1-\alpha/n, n}(E_j) = 0$. Moreover $B_{1-\alpha/n,n}(F_j) = 0$ because of $f_j \in L^n(R^n)$. Thus the proof of Theorem 6 is completed.

REMARK 1. We do not know whether Theorem 6 is best possible as to the size of the exceptional sets, or not.

REMARK 2. In Theorem 6, if $\alpha = 0$, then u has a finite limit at any boundary point. To show this fact, for fixed $\xi \in \partial R^n_+$, for r with $0 < r < 1$ and for $x \in R^n_+ \cap B(\xi, r/2)$, we write $u_j''(x) = u_{j,r}''(x) + v_{j,r}''(x)$, where

$$u_{j,r}''(x) = \int_{B(\xi,r) - B(x,x_n/2)} k_j(x, y)(\partial/\partial y_j)\, \bar{u}(y)\, dy$$

and

$$v_{j,r}''(x) = \int_{R^n - B(\xi,r)} k_j(x, y)(\partial/\partial y_j)\, \bar{u}(y)\, dy.$$

Then $|u_{j,r}''(x)| \leqq M_1 \int_{B(\xi,r)} |x - y|^{1-n} f_j(y)\, dy$, and by Lemma 1 we have

$$|u_{j,r}''(x)| \leqq M_2 \left\{ r + \left(\int_{B(\xi,r)} f_j(y)^n\, \omega(f_j(y))\, dy \right)^{1/n} \right\},$$

where M_1 and M_2 are positive constants independent of x and r. Hence it follows that $\int_{B(\xi,1)} |\xi - y|^{1-n} f_j(y)\, dy < \infty$ and $u_{j,r}''(x)$ tends to zero as $x \to \xi$. Moreover, with the aid of the inequality, we can apply Lebesgue's dominated convergence theorem to prove that $v_{j,r}''$ with $r = 2|\xi - x|$ has a finite limit as x tends to ξ. This implies that u_j'' has a finite limit at any boundary point. Thus the proof of Theorem 6 assures the existence of boundary limits of u at all boundary points.

Now we consider the following condition on ω, which is weaker than condition (ω_3).

(ω_5) $$\omega(r)^{1/(n-1)} \int_r^\infty \omega(t)^{-1/(n-1)} t^{-1}\, dt \to \infty \quad \text{as } r \to \infty.$$

We remark here that $\varphi(r) = r^\delta \omega^*(r^{-1})$, $\delta > 0$, is nondecreasing on some interval $(0, A_\varphi)$, $A_\varphi > 0$, and hence the Hausdorff measure H_φ with the measure function φ is defined.

THEOREM 7. *Let u be as in Theorem 6 and $0 < \alpha < n-1$. If in addition $\lim_{x_n \downarrow 0} u(x', x_n) = 0$ for almost every $x' \in R^{n-1}$, then there exists a set $E \subset \partial R^n_+$ such that $H_h(E) = 0$ and u has nontangential limit zero at any point of $\partial R^n_+ - E$, where $h(r) = r^\alpha \omega^*(r^{-1})$.*

PROOF. If we set $u = 0$ outside R^n_+, then, as in the above proof, we see that u is locally p-precise on R^n for p with $1 < p < n/(\alpha+1)$ and it has the following integral representation (for this fact, see also the proof of Theorem 2 in [7]):

$$u(x) = c \sum_{j=1}^{n} \int [k_j(x, y) - k_j(\bar{x}, y)]\, (\partial/\partial y_j)\, u(y)\, dy$$

for every $x \in R^n_+$, where $\bar{x} = (\bar{x}, -x_n)$ for $x = (x', x_n)$. For $x = (x', x_n) \in R^n_+$, we write

$$u'_j(x) = \int_{B(x, x_n/2)} [k_j(x, y) - k_j(\bar{x}, y)] (\partial/\partial y_j)\, u(y)\, dy,$$

$$u''_j(x) = \int_{R^n - B(x, x_n/2)} [k_j(x, y) - k_j(\bar{x}, y)] (\partial/\partial y_j)\, u(y)\, dy$$

and

$$u_j(x) = u'_j(x) + u''_j(x).$$

For simplicity, set $f_j(y) = |(\partial/\partial y_j)\, u(y)|$. Then it follows from Lemma 1 that

$$|u'_j(x)| \leqq M_1 a x_n + M_1 \left(\int_{B(x, x_n/2)} f_j(y)^n \omega(f_j(y))\, dy \right)^{1/n} \omega^*(a)^{-1/n}$$

with positive constants M_1 and M_2 independent of $x \in R^n_+$ and $a > 0$. Hence, letting $a = \varepsilon x_n^{-1}$, $\varepsilon > 0$, we have

$$|u'_j(x)| \leqq M_1 \varepsilon + M_2 \left(h(x_n)^{-1} \int_{B(x, x_n/2)} f_j(y)^n \omega(f_j(y))\, y_n^\alpha\, dy \right)^{1/n}$$

with a positive constant M_2 independent of x, where $h(r) = r^\alpha \omega^*(r^{-1})$.

Define

$$E_j = \left\{ x \in \partial R^n_+ ;\ \limsup_{r \downarrow 0} h(r)^{-1} \int_{B(x,r)} f_j(y)^n \omega(f_j(y)) |y_n|^\alpha \, dy > 0 \right\}.$$

If $\xi \in \partial R^n_+ - E_j$, then it follows that $\limsup_{x \to \xi, x \in \Gamma(\xi, b)} |u_j'(x)| \leqq M_1 \varepsilon$ for any $b > 0$, which implies that $u_j'(x)$ has nontangential limit zero at ξ; we must note here that M_1 is independent of ε. Next we are concerned with the estimate of u_j''. For this purpose, we see first that if $x \in \Gamma(\xi, a)$, where $\xi \in \partial R^n_+$ and $a > 0$, then

$$\begin{aligned} |u_j''(x)| &\leqq M_3 x_n \int_{R^n - B(x, x_n/2)} |x - y|^{1-n} |\bar{x} - y|^{-2} y_n f_j(y) \, dy \\ &\leqq M_4 x_n \int (|\xi - y| + x_n)^{-n-1} y_n f_j(y) \, dy \\ &\leqq M_4 x_n \left\{ \int_0^\varepsilon \left(\int_{B(\xi, r)} y_n f_j(y) \, dy \right) d(-(r + x_n)^{-n-1}) \right. \\ &\qquad \left. + (\varepsilon + x_n)^{-n-1} \int_{B(\xi, \varepsilon)} y_n f_j(y) \, dy \right\} \\ &\quad + M_4 x_n \int_{R^n - B(\xi, \varepsilon)} (|\xi - y| + x_n)^{-n-1} y_n f_j(y) \, dy \end{aligned}$$

for $\varepsilon > 0$. Hence if follows that

$$\limsup_{x \to \xi, x \in \Gamma(\xi, a)} |u_j''(x)| \leqq M_5 \sup_{0 < r \leqq \varepsilon} r^{-n} \int_{B(\xi, r)} y_n f_j(y) \, dy,$$

where M_3, M_4 and M_5 are positive constants independent of $x \in \Gamma(\xi, a)$ and ε. Set

$$F_j = \{ \xi \in \partial R^n_+ ;\ \limsup_{r \downarrow 0} r^{-n} \int_{B(\xi, r)} y_n f_j(y) \, dy > 0 \}.$$

If $\xi \in \partial R^n_+ - F_j$, then we see that u_j'' has nontangential limit zero at ξ. Thus, what remains is to show that $H_h(F_j) = 0$. Let $\varepsilon > 0$. Then we have by Hölder's inequality

$$\begin{aligned} & r^{-n} \int_{\{y \in B(\xi, r);\, f_j(y) > \varepsilon |y_n|^{-1}\}} y_n f_j(y) \, dy \\ &\leqq r^{-n} \left(\int_{B(\xi, r)} y_n^\alpha f_j(y)^n \omega(f_j(y)) \, dy \right)^{1/n} \\ &\qquad \times \left(\int_{B(\xi, r)} y_n^{n'(1-\alpha/n)} \omega(\varepsilon |y_n|^{-1})^{1/(1-n)} \, dy \right)^{1/n'} \\ &\leqq M_6 \left(r^{-\alpha} \omega^*(r^{-1})^{-1} \int_{B(\xi, r)} y_n^\alpha f_j(y)^n \omega(f_j(y)) \, dy \right)^{1/n} \end{aligned}$$

with a positive constant M_6 which may depend on ε. On the other hand,

$$r^{-n}\int_{\{y\in B(\xi,r);\,f_j(y)\leqq\varepsilon|y_n|^{-1}\}} y_n f_j(y)\,dy \leqq M_7\varepsilon,$$

where M_7 is a positive constant independent of r and ε. Therefore, if $\xi\notin G_j$, where

$$G_j \equiv \left\{x\in R^n;\ \limsup_{r\downarrow 0} h(r)^{-1}\int_{B(x,r)} |y_n|^{\alpha} f_j(y)^n\,\omega(f_j(y))\,dy > 0\right\},$$

then

$$\limsup_{x\to\xi,\,x\in\Gamma(\xi,a)} r^{-n}\int_{B(\xi,r)} y_n f_j(y)\,dy \leqq M_7\varepsilon,$$

which implies that the left hand side is equal to zero. Thus we find that $F_j\subset G_j$. Since $H_h(G_j)=0$, we also obtain $H_h(F_j)=0$, and the proof of Theorem 7 is completed.

REMARK. Theorem 7 is best possible as to the size of the exceptional set if we assume (ω_3) instead of (ω_5); in fact, for a compact set $K\subset\partial D$ such that $H_h(K)=0$ we shall construct a nonnegative measurable function f on D satisfying (1) and (2) such that $u(x)=G_1 f(x)=\int_D G_1(x,y)f(y)\,dy$ does not have nontangential limit 0 at any $\xi\in K$, where $G_1(x,y)=|x-y|^{1-n}-|\bar{x}-y|^{1-n}$ with $\bar{x}=(x',-x_n)$ for $x=(x',x_n)$.

For the construction of such f, take a mutually disjoint finite family $\{B(x_{j,1}, r_{j,1})\}$ of balls such that $x_{j,1}\in\partial D$, $\cup_j B(x_{j,1},5r_{j,1})\supset K$ and $\sum_j h(r_{j,1})<1$, and define

$$f_1(y)=a_{j,1}|z_{j,1}-y|^{-1}\,\omega(|z_{j,1}-y|^{-1})^{-1/(n-1)}$$

for $y\in B(z_{j,1},r_{j,1})$, where $z_{j,1}=x_{j,1}+(0,2r_{j,1})$ and $a_{j,1}=\omega^*(r_{j,1}^{-1})^{1/(n-1)}$; set $f_1(y)=0$ otherwise. Letting $\varepsilon_1=\min_j r_{j,1}$, we take a mutually disjoint finite family $\{B(x_{j,2},r_{j,2})\}$ of balls such that $x_{j,2}\in\partial D$, $r_{j,2}<\varepsilon_1/4, \sum_j h(r_{j,2})<2^{-1}$ and $\cup_j B(x_{j,2},5r_{j,2})\supset K$. As above, we define $f_2(y)=a_{j,2}|z_{j,2}-y|^{-1}$ $\omega(|z_{j,2}-y|^{-1})^{-1/(n-1)}$ for $y\in B(z_{j,2},r_{j,2})$, where $z_{j,2}=x_{j,2}+(0,2r_{j,2})$ and $a_{j,2}=\omega^*(r_{j,2}^{-1})^{1/(n-1)}$; define $f_2(y)=0$ otherwise. In the same manner, for each positive integer m we can find a mutually disjoint finite family $\{B(x_{j,m}r_{j,m})\}$ and a function f_m such that $x_{j,m}\in\partial D$, $\sum_j h(r_{j,m})<2^{-m+1}$, $\cup_j B(x_{j,m},5r_{j,m})\supset K$ and $f_m(y)=a_{j,m}|z_{j,m}-y|^{-1}\,\omega(|z_{j,m}-y|^{-1})^{-1/(n-1)}$ for $y\in B(z_{j,m},r_{j,m})$, where $z_{j,m}=x_{j,m}+(0,2r_{j,m})$, $a_{j,m}=\omega^*(r_{j,m}^{-1})^{1/(n-1)}$ and $r_{j,m}<\varepsilon_{m-1}/4$ with $\varepsilon_{m-1}=\min_j r_{j,m-1}$; we set $f_m(y)=0$ outside $\cup_j B(z_{j,m},r_{j,m})$ as above. Then, since $f_m(y)\leqq$

$M_1 |z_{j,m} - y|^{-1}$ on $B(z_{j,m}, r_{j,m})$ with a positive constant M_1, we note

$$\int_D y_n^\alpha f_m(y)^n \omega(f_m(y))\, dy$$

$$\leqq M_2 \sum_j a_{j,m}^n \int_{B(z_{j,m}, r_{j,m})} y_n^\alpha |z_{j,m} - y|^{-n} \omega(|z_{j,m} - y|^{-1})^{1-n/(n-1)} dy$$

$$\leqq M_3 \sum_j a_{j,m}^n r_{j,m}^\alpha \int_0^{r_{j,m}} \omega(t^{-1})^{-1/(n-1)} t^{-1} dt$$

$$= M_3 \sum_j h(r_{j,m}) < M_3 2^{-m+1},$$

$$\int_D y_n f_m(y)\, dy$$

$$\leqq M_4 \sum_j a_{j,m} \int_{B(z_{j,m}, r_{j,m})} y_n |z_{j,m} - y|^{-1} \omega(|z_{j,m} - y|^{-1})^{-1/(n-1)} dy$$

$$\leqq M_5 \sum_j a_{j,m} r_{j,m}^{n-1} \int_0^{r_{j,m}} \omega(t^{-1})^{-1/(n-1)} t^{-1} dt$$

$$= M_5 \sum_j r_{j,m}^{n-1} \leqq M_6 \sum_j h(r_{j,m}) \leqq M_6 2^{-m+1}$$

and

$$G_1 f_m(z_{j,m}) \geqq M_7 \int_{B(z_{j,m}, r_{j,m})} |z_{j,m} - y|^{1-n} f_{j,m}(y)\, dy$$

$$\geqq M_8 a_{j,m} \int_0^{r_{j,m}} \omega(t^{-1})^{-1/(n-1)} t^{-1} dt = M_8,$$

where $M_2, \ldots, M_8$ are positive constants independent of j and m. Consequently, since $\{B(z_{j,m}, r_{j,m})\}$ is mutally disjoint, $f = \sum_{m=1}^{\infty} f_m$ satisfies conditions (1) and (2). Moreover, if $\xi \in K$, then for each m there exists $j(m)$ such that $\xi \in B(x_{j(m),m}, 5r_{j(m),m})$. Then $z_{j(m),m} \in \Gamma(\xi, 5)$, which implies that $\limsup_{x \to \xi, x \in \Gamma(\xi, 5)} u(x) \geqq M_8 > 0$ and hence u does not have nontangential limit zero at ξ.

Finally we prove that

$$\int |\text{grad}\, u(x)|^n \omega(|\text{grad}\, u(x)|) |x_n|^\alpha dx < \infty.$$

First we consider the case $\alpha = 0$. We note by the well known fact in singular integral operators that

$$\lambda(a) \equiv H_n(\{x; |\mathrm{grad}\, u(x)| > a\})$$
$$\leqq M_9 a^{-1} \int_{\{y; f(y) \geqq a/2\}} f(y)\, \mathrm{d}y + M_9 a^{-q} \int_{\{y; f(y) < a/2\}} f(y)^q \, \mathrm{d}y$$
$$= M_9 \mu_1(a) + M_9 \mu_2(a),$$

where $q > n$ and M_9 is a positive constant independent of f and a. Hence, setting $\Phi(r) = r^n \omega(r)$, we have

$$\int \Phi(|\mathrm{grad}\, u(x)|)\, \mathrm{d}x = \int_0^\infty \lambda(a)\, \mathrm{d}\Phi(a)$$
$$\leqq M_9 \int_0^\infty \mu_1(a)\, \mathrm{d}\Phi(a) + M_9 \int_0^\infty \mu_2(a)\, \mathrm{d}\Phi(a)$$
$$\leqq M_9 \int f(y) \left(\int_0^{2f(y)} a^{-1}\, \mathrm{d}\Phi(a) \right) \mathrm{d}y + M_9 \int f(y)^q \left(\int_{2f(y)}^\infty a^{-q}\, \mathrm{d}\Phi(a) \right) \mathrm{d}y$$
$$\leqq M_{10} \int \Phi(f(y))\, \mathrm{d}y < \infty$$

with a positive constant M_{10}.

In the general case, setting $g(y) = |y_n|^{\alpha/p} f(y)$ and $v(x) = \int_D G_1(x, y) g(y)\, \mathrm{d}y$, we note that

$$|\ |x_n|^{\alpha/n}|\ \mathrm{grad}\, u(x)| - |\mathrm{grad}\, v(x)|\ |$$
$$\leqq M_{11} \int K_\alpha(x_n, y_n)(P_{|x_n - y_n|} * g(\bullet, y_n))(x', |x_n - y_n|)\, \mathrm{d}y_n,$$

where $K_\alpha(x_n, y_n) = |1 - |x_n/y_n|^{\alpha/n}| \,/\, |x_n - y_n|$, P denotes the Poisson kernel and M_{11} is a positive constant independent of x and y. Applying Appendix A.3 in Stein's book [10], we see that

$$\lambda(a) \equiv H_n(\{x;\ |\ |x_n|^{\alpha/n}\ |\mathrm{grad}\, u(x)| - |\mathrm{grad}\, v(x)|\ | > a\})$$
$$\leqq M_{12}(\mu_1(a) + \mu_2(a)),$$

where M_{12} is a positive constant independent of a, $\mu_1(a) = a^{-p} \int_{\{y; g(y) \geqq a/2\}} g(y)^p \, \mathrm{d}y$ and $\mu_2(a) = a^{-q} \int_{\{y; g(y) < a/2\}} g/y)^q \, \mathrm{d}y$ with $1 < p < n < q$. Hence, by the above considerations, we obtain

$$\int \Phi(|\,|x_n|^{\alpha/n}|\operatorname{grad} u(x)| - |\operatorname{grad} v(x)|\,|)\,dx \leqq M_{13} \int \Phi(g(y))\,dy.$$

Since f has compact support, it follows that

$$\int \Phi(|x_n|^{\alpha/n}\,|\operatorname{grad} u(x)|)\,dx \leqq M_{14} \int \Phi(g(y))\,dy \leqq M_{15} \int \Phi(f(y))\,|y_n|^{\alpha}\,dy < \infty$$

with positive constants M_{13}, M_{14}, M_{15}. Thus we can establish $\int \Phi(|\operatorname{grad} u(x)|)\,|x_n|^{\alpha}\,dx < \infty$.

Our last aim is to prove the following result concerning global boundary behaviors.

THEOREM 8. *Let u be as in Theorem 6 and $0 \leqq \alpha < n - 1$. Then $[x_n^{\alpha}\,\omega^*(x_n^{-1})]^{1/n} u(x)$ tends to zero as $x \in R^n_+$ tends to the boundary ∂R^n_+.*

PROOF. First we note the following integral representation of u (see the proof of Theorem 6):

$$u(x) = c \sum_{j=1}^{n} \int k_j(x, y)(\partial/\partial y_j)\,\bar{u}(y)\,dy + A,$$

where c, A are constants and $\bar{u}(x)$ is an extension of u to R^n such that $\int |\operatorname{grad} \bar{u}(x)|^n\,\omega(|\operatorname{grad} \bar{u}(x)|)\,|x_n|^{\alpha}\,dx < \infty$. For $x = (x', x_n) \in R^n_+$, we write $u_j(x) = u_j'(x) + u_j''(x)$ as in the proof of Theorem 6.

For simplicity, set $f_j(y) = |(\partial/\partial y_j)\,\bar{u}(y)|$. For $a > 0$ we have by Lemma 1

$$|u_j'(x)| \leqq M_1 a x_n + M_1 \left(\int_{B(x, x_n/2)} f_j(y)^n\,\omega(f_j(y))\,dy \right)^{1/n} \omega^*(a)^{-1/n}$$

with a positive constant M_1 independent of x and a. Letting $a = x_n^{-1}$, we obtain

$$\lim_{x_n \downarrow 0} [x_n^{\alpha}\,\omega^*(x_n^{-1})]^{1/n}\,|u_j'(x)| = 0.$$

We next estimate the function u_j''. For simplicity, set $F = \left(\int f_j(y)^n\,\omega(f_j(y))\,|y_n|^{\alpha}\,dy \right)^{1/n}$. For $\delta > 1$, by Hölder's inequality we have

$$\int_{\{y \in R^n - B(x,x_n/2);\, y_n > x_n/2,\, f_j(y) > |x-y|^{-\delta}\}} |x-y|^{1-n} |f_j(y)|\, \mathrm{d}y$$

$$\leqq \left(\int f_j(y)^n \,\omega\,(f_j(y))\, |y_n|^\alpha \,\mathrm{d}y \right)^{1/n} \left(\int_{\{y \in R^n - B(x,x_n/2);\, y_n > x_n/2\}} |x-y|^{n'(1-n)} \right.$$

$$\left. \times\, \omega(|x-y|^{-\delta})^{-n'/n}\, y_n^{-\alpha n'/n} \,\mathrm{d}y \right)^{1/n'}$$

$$\leqq M_2 F \left(\int_0^{2x_n^{-1}} \omega(r)^{-1/(n-1)}\, r^{\alpha/(n-1)}\, r^{-1} \,\mathrm{d}r \right)^{1/n'}$$

$$\leqq M_3 F[\omega(x_n^{-1})^{-1/(n-1)}\, x_n^{-\alpha/(n-1)}]^{1/n'} \leqq M_4 F x_n^{-\alpha/n}\, \omega^*(x_n^{-1})^{-1/n}$$

and

$$\int_{\{y \in R^n - B(x,x_n/2);\, y_n > x_n/2,\, f_j(y) < |x-y|^{-\delta}\}} |x-y|^{1-n} |f_j(y)|\, \mathrm{d}y \leqq M_5 x_n^{1-\delta}.$$

In the same manner, letting $z = (x', 0)$, we obtain

$$\int_{\{y \in R^n - B(x,x_n/2);\, y_n < x_n/2,\, f_j(y) > |z-y|^{-\delta}\}} |x-y|^{1-n} |f_j(y)|\, \mathrm{d}y$$

$$\leqq \left(\int f_j(y)^n \,\omega(f_j(y))\, |y_n|^\alpha \,\mathrm{d}y \right)^{1/n}$$

$$\times \left(\int_{\{y \in R^n - B(x,x_n/2);\, y_n < x_n/2\}} |x-y|^{n'(1-n)}\, \omega(|z-y|^{-\delta})^{-n'/n}\, |y_n|^{-\alpha n'/n} \,\mathrm{d}y \right)^{1/n'}$$

$$\leqq M_6 F \left(\int_0^\infty (r + x_n)^{-n}\, \omega(r^{-1})^{-1/(n-1)}\, r^{-\alpha/(n-1)}\, r^{n-1} \,\mathrm{d}r \right)^{1/n'}$$

$$\leqq M_7 F[\omega(x_n^{-1})^{-1/(n-1)}\, x_n^{-\alpha/(n-1)}]^{1/n'} \leqq M_8 F[x_n^\alpha\, \omega^*(x_n^{-1})]^{-1/n}$$

and

$$\int_{\{y \in R^n - B(x,x_n/2);\, y_n < x_n/2,\, f(y) < |z-y|^{-\delta}\}} |x-y|^{1-n} |f_j(y)|\, \mathrm{d}y \leqq M_9 x_n^{1-\delta}.$$

Hence $\limsup_{x_n \downarrow 0} [x_n^\alpha\, \omega^*(x_n^{-1})]^{1/n} |u_j''(x)| \leqq M_{10} F$, from which it follows that $\lim_{x_n \downarrow 0} [x_n^\alpha\, \omega^*(x_n^{-1})]^{1/n}\, u_j''(x) = 0$.

To extend our results to the tangential case, we shall need the techniques used in the paper of Aikawa [1], and leave the details to the reader.

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