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DUALITY AND LORENTZ-MARCINKIEWICZ OPERATOR SPACES

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Abstract.

We investigate the dual of the Lorentz-Marcinkiewicz operator space $S_{\varphi,q}$. In particular we determine the dual of $S_{p,q}$ (the space introduced by H. Triebel) for some values of p and q not covered by C. Gapaillard and P. T. Lai.

1. Introduction.

Let H be an arbitrary Hilbert space over the field of complex numbers (H might not be separable), and let $\mathscr{L}(H)$ be the Banach space of all bounded linear operators acting from H into H.

We denote by $S_{\varphi,q}$ the collection of all compact operators $T \in \mathscr{L}(H)$ having a finite quasi-norm

$$\sigma_{\varphi,q}(T) = \left(\sum_{n=1}^{\infty} (\varphi(n)s_n(T))^q n^{-1} \right)^{1/q} \quad \text{if} \quad 0 = q < \infty$$

$$\sigma_{\varphi,q}(T) = \sup_{n \geq 1} (\varphi(n)s_n(T)) \quad \text{if} \quad q = \infty.$$

Here $(s_n(T))$ are the singular numbers of T and φ belongs to the class $\mathscr{B}$, formed by all continuous functions $\varphi: (0, \infty) \to (0, \infty)$ with $\varphi(1) = 1$ and such that

$$\bar{\varphi}(t) = \sup_{s > 0} (\varphi(ts)/\varphi(s)) < \infty \quad \text{for every } t > 0.$$

The space $S_{\varphi,q}$ is the component over H of the Lorentz-Marcinkiewicz operator ideal that we studied in [4] and [5]. Some other additional material on $S_{\varphi,q}$ can be found in [6].

If $0 < p,q \leqq \infty$ and $\varphi(t) = t^{1/p}$ we obtain the operator spaces $(S_{p,q}, \sigma_{p,q})$ introduced in 1967 by H. Triebel [19]. These spaces play an important role

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in some problems on estimates of singular numbers and eigenvalues (see [19] and [2]). The special case $p = q$ gives the Schatten p-class S_p (see [18] and [8]).

For $1 < p < \infty$ and $1 \leqq q < \infty$, the dual of $S_{p,q}$ was described by C. Merucci [11] using interpolation techniques. He showed that $(S_{p,q})' = S_{p',q'}$ with equivalent norms and $1/p + 1/p' = 1/q + 1/q' = 1$. Later on C. Gapaillard and P. T. Lai [7] gave a direct proof of that result. On the other hand, in the case $1 < p < \infty$ and $0 < q < 1$, the dual of $S_{p,q}$ can be determined by means of a result of J. Peetre [15] concerning the dual of the interpolation spaces $(A_0, A_1)_{\theta,q}$ when $0 < q < 1$. In this way, one can derive that $(S_{p,q})' = S_{p',\infty}$. But so far as we know, the dual of $S_{p,q}$ is unknown for $0 < p < 1$ and $0 < q \leqq \infty$, or $p = 1$ and $0 < q < 1$.

In the present article we characterize the dual of the Lorentz-Marcinkiewicz operator space $S_{\varphi,q}$. In particular, we solve the problem mentioned in the preceding paragraph by proving that $(S_{p,q})' = \mathscr{L}(H)$ for $0 < p < 1$ and $0 < q \leqq \infty$, or $p = 1$ and $0 < q < 1$.

The necessary hypothesis on the function parameter φ to obtain our results are expressed by means of the Boyd indices of $\bar{\varphi}$. They are defined by

$$\alpha_{\bar{\varphi}} = \lim_{t\to\infty} (\log \bar{\varphi}(t)/\log t), \quad \beta_{\bar{\varphi}} = \lim_{t\to 0} (\log \bar{\varphi}(t)/\log t)$$

Note that $\sigma_{\varphi,q}(T) = \|(s_n(T))\|_{\lambda^q(\varphi)}$, where $\|\cdot\|_{\lambda^q(\varphi)}$ is the quasi-norm of the Lorentz-Marcinkiewicz sequence space defined by

$$\lambda^q(\varphi) = \left\{\zeta = (\zeta_n) \in l_\infty : \|\zeta\|_{\lambda^q(\zeta)} = \left(\sum_{n=1}^{\infty} (\varphi(n)\zeta_n^*)^q n^{-1}\right)^{1/q} < \infty\right\}$$

where (ζ_n^*) designates the non-increasing rearrangement of the bounded sequence ζ (see [4, §2]).

2. The duality theorem for the case of Banach spaces.

We now investigate the dual of $S_{\varphi,q}$ when $0 < \beta_{\bar{\varphi}} \leqq \alpha_{\bar{\varphi}} < 1$ and $1 \leqq q \leqq \infty$. We shall use interpolation techniques to deal with this case. For this reason, we shall review the definition of the real interpolation space with a function parameter.

Let (A_0, A_1) be a couple of Banach spaces. We equip $A_0 + A_1$ (respectively $A_0 \cap A_1$) with the norm $K(1,x)$ (respectively $J(1,x)$) where $K(t,x) = K(t,x; A_0, A_1)$ and $J(t,x) = J(t,x; A_0, A_1)$ are the functionals of J. Peetre, defined by

$$K(t,x) = \inf\{\|x_0\|_{A_0} + t\|x_1\|_{A_1} : x = x_0 + x_1, x_0 \in A_0, x_1 \in A_1\}$$

and

$$J(t,x) = \max\{\|x\|_{A_0}, t\|x\|_{A_1}\}.$$

The space $(A_0, A_1)_{\varphi,q}$ where $0 < q \leqq \infty$ and $\varphi \in \mathscr{B}$ with $0 < \beta_{\bar{\varphi}} \leqq \alpha_{\bar{\varphi}} < 1$, is defined as the subspace of $A_0 + A_1$ given by the condition $\|x\|_{\varphi,q} < \infty$ where

$$\|x\|_{\varphi,q} = \left(\int_0^\infty (\varphi(t)^{-1}K(t,x))^q dt/t\right)^{1/q} \quad \text{if } 0 < q < \infty$$

$$\|x\|_{\varphi,q} = \sup_{t>0} (\varphi(t)^{-1}K(t,x)) \quad \text{if } q = \infty.$$

The study of this interpolation method was initiated by J. Peetre [14] in 1963. Later on, and mainly in recent years, many papers have appeared concerning it (see, e.g., [10], [9], [12], [13], and [16]).

When $1 \leqq q \leqq \infty$, $((A_0, A_1)_{\varphi,q}, \|\cdot\|_{\varphi,q})$ is a Banach space, but if $0 < q < 1$ in general only a quasi-Banach space. For $\varphi(t) = t^\theta$ $(0 < \theta < 1)$ we get the classical real interpolation space $((A_0, A_1)_{\theta,q}, \|\cdot\|_{\theta,q})$ (see [1]).

In the next statement, we denote by $S^0_{\varphi,\infty}$ the closure of finite rank operators in $S_{\varphi,\infty}$.

THEOREM 2.1. *Assume that* $1 \leqq q < \infty$ *and* $\varphi \in \mathscr{B}$ *with* $0 < \beta_{\bar{\varphi}} \leqq \alpha_{\bar{\varphi}} < 1$. *Then we have, with equivalent norms,*

$$(S_{\varphi,q})' = S_{\psi,q'} \quad \textit{and} \quad (S^0_{\varphi,\infty})' = S_{\psi,1}$$

where $\psi(t) = t/\varphi(t)$ *and* $1/q + 1/q' = 1$.

PROOF. Choose $1 < p_0 < p_1 < \infty$ such that $1/p_1 < \beta_{\bar{\varphi}} \leqq \alpha_{\bar{\varphi}} < 1/p_0$. Put

$$\varrho(t) = t^{p_1/(p_1-p_0)}(\varphi(t^{p_0p_1/(p_1-p_0)}))^{-1},$$

$\mu(t) = t/\varrho(t)$ and $1/p_0 + 1/p_0' = 1/p_1 + 1/p_1' = 1$. According to [4, Theorem 5.1] and [12, Example 3.2.3] (see also [17, Theorem 3.1] and [16, Theorem 2.4]), we have

$$(S_{\varphi,q})' = ((S_{p_0}, S_{p_1})_{\varrho,q})' = ((S_{p_1})', (S_{p_0})')_{\mu,q'}.$$

By [8, Theorem 3.12.3], we know that $(S_{p_j})' = S_{p_j'}$ $(j = 0, 1)$. Consequently, [4, Theorem 5.3] implies that

$$(S_{\varphi,q})' = (S_{p_1'}, S_{p_0'})_{\mu,q'} = S_{\psi,q'}.$$

The case of $S^0_{\varphi,\infty}$ can be carried out in the same way.

3. The duality theorem when $0 < q < 1$.

For the purpose of determining the dual of $S_{\varphi,q}$ when $0 < q < 1$ and $0 < \beta_{\bar{\varphi}} \leqq \alpha_{\bar{\varphi}} < 1$, we shall first extend to the (φ, q)-method a classical result of the (θ, q)-method due to J. Peetre [15].

In what follows, given $(E, \|\cdot\|)$ a quasi-Banach space, $\|\cdot\|^{\#}$ designates the semi-norm defined by

$$\|x\|^{\#} = \inf\left\{\sum_{j=1}^{n} \|x_j\| : x = \sum_{j=1}^{n} x_j\right\}$$

and $(E^{\#}, \|\cdot\|^{\#})$ denotes the completion of the quotient space E/N with the quotient norm induced by $\|\cdot\|^{\#}$. Here

$$N = \{x \in E : \|x\|^{\#} = 0\}.$$

For properties of $E^{\#}$ we refer to [15]. We only recall here that $(E^{\#})' = E'$.

THEOREM 3.1. *Let (A_0, A_1) be a couple of Banach spaces, let $0 < q < 1$ and $\varphi \in \mathscr{B}$ with $0 < \beta_{\bar{\varphi}} \leqq \alpha_{\bar{\varphi}} < 1$. Then*

$$((A_0, A_1)_{\varphi,q})^{\#} = (A_0, A_1)_{\varphi,1}$$

with equivalence of norms.

PROOF. Let $0 < p < \infty$. By the equivalence theorem we know that $(A_0, A_1)_{\varphi,p}$ consists of all $x \in A_0 + A_1$ such that there exists $(u_\nu)_{-\infty}^{\infty} \subset A_0 \cap A_1$ with

$$x = \sum_{\nu=-\infty}^{\infty} u_\nu \quad \text{(convergence in } A_0 + A_1\text{)} \tag{2}$$

and

$$\|(J(2^\nu, u_\nu))\|_{\Omega_{\varphi,p}} = \left(\sum_{\nu=-\infty}^{\infty} (J(2^\nu, u_\nu)/\varphi(2^\nu))^p\right)^{1/p} < \infty. \tag{3}$$

Moreover, $\|\cdot\|_{\varphi,p}$ is equivalent to the quasi-norm

$$\|x\|_{\varphi,p;J} = \inf_{(u_\nu)} \|(J(2^\nu, u_\nu))\|_{\Omega_{\varphi,p}}$$

where the infimum is taken over all sequences (u_ν) satisfying (2) and (3).

As a direct consequence of this representation, we get

$$(A_0, A_1)_{\varphi,q} \hookrightarrow (A_0, A_1)_{\varphi,1}$$

and the last space is a Banach space. Therefore $((A_0, A_1)_{\varphi,q})^{\#}$ is continuously embedded in $(A_0, A_1)_{\varphi,1}$.

Next, we prove the reverse inclusion. Since

$$K(s,x) \leqq \min(1, s/t)J(t,x), \quad x \in A_0 \cap A_1, s,t > 0$$

and

$$\int_0^1 (t\bar{\varphi}(1/t))^q dt/t < \infty, \quad \int_1^\infty \bar{\varphi}(1/t)^q dt/t < \infty \quad \text{(see [3])}$$

it follows that there exists a constant $M = M(\varphi, q)$ such that for any $x \in A_0 \cap A_1$ and any $t > 0$ we have

$$\|x\|_{\varphi,q} \leqq \frac{M}{\varphi(t)} J(t,x).$$

So, if $x \in (A_0, A_1)_{\varphi,1}$ and $x = \sum_{\nu=-\infty}^{\infty} u_\nu$ is a representation of x satisfying (2) and (3), then the series $\sum_{\nu=-\infty}^{\infty} u_\nu$ converges in $((A_0, A_1)_{\varphi,q})^\#$ because

$$\sum_{\nu=N}^{R} \|u_\nu\|_{\varphi,q}^\# \leqq M \sum_{\nu=N}^{R} J(2^\nu, u_\nu)/\varphi(2^\nu)$$

$$\leqq M \sum_{\nu=-\infty}^{\infty} J(2^\nu, u_\nu)/\varphi(2^\nu).$$

Consequently,

$$\|x\|_{\varphi,q}^\# \leqq \sum_{\nu=-\infty}^{\infty} \|u_\nu\|_{\varphi,q}^\# \leqq M\|(J(2^\nu, u_\nu))\|_{\Omega_{\varphi,1}}$$

and the inclusion is established by taking the infimum over all sequences (u_ν) satisfying (2) and (3).

Now we can prove

THEOREM 3.2. *Assume that* $0 < q < 1$ *and* $\varphi \in \mathscr{B}$ *with* $0 < \beta_{\bar{\varphi}} \leqq \alpha_{\bar{\varphi}} < 1$. *Then we have, with equivalent norms,*

$$(S_{\varphi,q})' = S_{\psi,\infty}$$

where $\psi(t) = t/\varphi(t)$.

PROOF. Take p_0, p_1, and ϱ as in the proof of Theorem 2.1. Using [4, Theorem 5.1], we see that

$$S_{\varphi,q} = (S_{p_0}, S_{p_1})_{\varrho,q} \quad \text{and} \quad S_{\varphi,1} = (S_{p_0}, S_{p_1})_{\varrho,1}.$$

Hence Theorems 3.1 and 2.1 give

$$(S_{\varphi,q})' = ((S_{\varphi,q})^\#)' = (((S_{p_0}, S_{p_1})_{\varrho,q})^\#)'$$

$$= (S_{\varphi,1})' = S_{\psi,\infty}.$$

4. The duality theorem in the remaining cases.

Up to now, all spaces $S_{\varphi,q}$ considered have been generated by a sequence space $\lambda^q(\varphi)$ that contains l_1 (see [4, Lemma 2.2]). In this section we shall discuss the converse situation.

THEOREM 4.1. *Let* $0 < q \leqq \infty$ *and* $\varphi \in \mathscr{B}$ *with* $\lambda^q(\varphi)$ *continuously embedded in* l_1. *Then we have, with equivalent norms,*

$$(S_{\varphi,q})' = \mathscr{L}(H).$$

PROOF. We shall prove that $(S_{\varphi,q})^{\#} = S_1$. This gives the result, by [8, Theorem 3.12.1].

The inclusion $(S_{\varphi,q})^{\#} \hookrightarrow S_1$ is trivial. Let us show the opposite one. Let $T \in S_1$ and let

$$T(x) = \sum_{n=1}^{\infty} \lambda_n \langle x, x_n \rangle y_n, \quad \lambda_n = s_n(T)$$

be its Schmidt representation. Since

$$\sigma^{\#}_{\varphi,q}\left(\sum_{j=n}^{m} \lambda_j \langle \cdot, x_j \rangle y_j\right) \leqq \sum_{j=n}^{m} \sigma_{\varphi,q}(\lambda_j \langle \cdot, x_j \rangle y_j) = \sum_{j=n}^{m} \lambda_j \to 0 \quad \text{as } n, m \to \infty$$

it follows that the series $\sum_{n=1}^{\infty} \lambda_n \langle \cdot, x_n \rangle y_n$ is convergent in $(S_{\varphi,q})^{\#}$. Therefore

$$\sigma^{\#}_{\varphi,q}(T) \leqq \sum_{n=1}^{\infty} \lambda_n = \sigma_{1,1}(T).$$

Finally, as an immediate consequence of Theorem 4.1 we can complement the known results on the duals of the spaces $S_{p,q}$ by:

COROLLARY 4.2. *Let* $0 < p < 1$ *and* $0 < q \leqq \infty$, *or* $p = 1$ *and* $0 < q < 1$. *Then*

$$(S_{p,q})' = \mathscr{L}(H).$$

Note added in proof. Theorem 3.1 has been also simultaneously obtained by M. Mastylo, Banach envelopes of some interpolation quasi-Banach spaces, in 'Function Spaces and Applications', Springer L.N.M. 1302, pp. 321–329.

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