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ON PLURIHARMONIC INTERPOLATION

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Introduction.

Given a domain Ω in $\mathbf{C}^n$, a closed subset E of $b\Omega$ is called an *interpolation set* if given a continuous function φ on E, there is an $f \in A(\Omega) = C(\bar{\Omega}) \cap O(\Omega)$, $O(\Omega)$ the space of functions holomorphic on Ω, with $f = \varphi$ on E. The study of these sets has been an active part of the theory of the boundary behavior of holomorphic functions since the characterization, due in independently to Carleson [5] and to Rudin [12] of the interpolation sets in the boundary of the unit disc in the plane. The study of these sets in the case of domains in $\mathbf{C}^n$ is much more complicated than the disc case and leads to some serious questions of a geometric nature. For the theory in the case of the ball, one may consult the book [13]; see also the newer survey [14].

Recently the subject of *pluriharmonic* interpolation has been broached by Bruna and Ortega [3], who show:

If Γ is a smooth simple closed curve in the boundary of the unit ball $\boldsymbol{B}_n$ that is everywhere transverse to the complex directions in $b\boldsymbol{B}_n$, then there is a closed subspace $\mathscr{F} \subset C(\Gamma)$ of finite codimension every element φ of which is of the form $\varphi = u|\Gamma$ for some function u pluriharmonic on $\boldsymbol{B}_n$, continuous $\bar{\boldsymbol{B}}_n$.

(Their result is true also for strongly pseudoconvex domains, as they remark.) It will be convenient to introduce the notation that for a domain Ω in $\mathbf{C}^n$, $\mathrm{Ph}^c(\Omega)$ denotes the space of real-valued functions pluriharmonic on Ω and continuous on $\bar{\Omega}$. Recall that a function is pluriharmonic if locally it is the real part of a holomorphic function. For nonsimply connected domains, this is not equivalent to the condition that $u = \operatorname{Re} f$ for some holomorphic function f for the conjugate of u may very well be multiple-valued.

Two remarks are in order. First, the Bruna-Ortega result treats an essentially multivariate problem, because on the disc or, more generally, on reasonable domains in the plane, the Dirichlet problem is solvable. Secondly, notice that

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the question is not that of interpolating $C(\Gamma)$, to within a finite dimensional subspace, by functions $u = \operatorname{Re} f$, $f \in A(\boldsymbol{B}_n)$: According to [15], if $E \subset b\boldsymbol{B}_n$ is a closed set with $\operatorname{Re} A(\Omega)|E$ closed in $C_{\mathsf{R}}(E)$, then the set E is an interpolation set for $A(\Omega)$.

That one interpolates, in general, only a finite codimensional subspace of $C_{\mathsf{R}}(\Gamma)$ appears in the work of Bruna and Ortega for funtional-analytic reasons: An operator between Banach spaces that is the perturbation of a surjective operator by a compact operator has closed range and the range has finite codimension. In fact, there is a simple geometric explanation for the phenomenon: Let V be a nonsingular one-dimensional complex submanifold of a neighborhood of $\bar{\boldsymbol{B}}_n$ that meets $\boldsymbol{B}_n$ and that meets $b\boldsymbol{B}_n$ transversally and in such a way that $\Gamma = V \cap b\boldsymbol{B}_n$ is a simple closed curve; it will be smooth by the transversality assumption, and it is necessarily transverse to the complex directions in $b\boldsymbol{B}_n$. If each $\varphi \in C_{\mathsf{R}}(\Gamma)$ is the restriction to Γ of a function $u \in \operatorname{Ph}^c(\boldsymbol{B}_n)$, then every $\varphi \in C_{\mathsf{R}}(\Gamma) = C_{\mathsf{R}}(b(V \cap \boldsymbol{B}_n))$ extends to a function harmonic on $V \cap \boldsymbol{B}_n$ that is the real part of a function $f \in O(V \cap \boldsymbol{B}_n)$. This can occur only when V is simply connected. (By Theorem IV.1 of [16], every compact bordered Riemann surface with connected boundary can be realized in the form $V \cap \boldsymbol{B}_3$ contemplated here.)

In the sequel we shall use frequently the observation that for an arbitrary closed set $E \subset b\boldsymbol{B}_n$, if $\operatorname{Ph}^c(\boldsymbol{B}_n)|E$ has finite codimension in $C_{\mathsf{R}}(E)$, then $\operatorname{Ph}^c(\boldsymbol{B}_n)|E$ is closed in $C_{\mathsf{R}}(E)$. To see this, let ϱ be the restriction operator from $\operatorname{Ph}^c(\boldsymbol{B}_n)$ to $C_{\mathsf{R}}(E)$. By hypothesis, there exist $\varphi_1, \ldots, \varphi_d \in C_{\mathsf{R}}(E)$ such that each $u \in C_{\mathsf{R}}(E)$ is of the form

$$u = g|E + \sum_{j=1}^{d} c_j \varphi_j, \quad g \in \operatorname{Ph}^c(\boldsymbol{B}_n).$$

The operator ϱ is continuous, so $T: \operatorname{Ph}^c(\boldsymbol{B}_n) \oplus \mathsf{R}^d \to C_{\mathsf{R}}(E)$ given by

$$T(g, c) = g|E + \sum_{j=1}^{d} c_i \varphi_i$$

is continuous and surjective. If $K: \operatorname{Ph}^c(\boldsymbol{B}_n) \oplus \mathsf{R}^d \to C_{\mathsf{R}}(E)$ is given by

$$K(g, c) = \sum_{j=1}^{d} c_j \varphi_j,$$

then K has finite rank, and so $\varrho = T - K$ has closed range.

The results.

Motivated by the result of Bruna and Ortega, it is natural to extrapolate: Perhaps every totally real smooth submanifold in $b\boldsymbol{B}_n$ that is transverse to the complex directions is a pluriharmonic interpolation set, at least to within a finite dimensional subspace. The thrust of the present paper is that this extrapolation is completely unwarranted.

We fix a bounded domain Ω in $\mathbf{C}^n$, $n \geqq 2$, with $b\Omega$ of class C^1 so that for some real-valued function Q of class C^1 on $\mathbf{C}^n$,

$$\Omega = \{z \in \mathbf{C}^n : Q(z) < 0\},$$

and dQ vanishes at no point of $b\Omega$. At each point $p \in b\Omega$, we have the tangent space

$$T_p(b\Omega) = \{v \in T_p(\mathbf{C}^n) : dQ(v) = 0\};$$

with suitable identifications, this is a real affine hyperplane in $\mathbf{C}^n$ that passes through the point p, and as such, it contains a unique complex affine hyperplane that passes through p. This complex affine plane is denoted by $T_p^{\mathbf{C}}(b\Omega)$. A submanifold M of $b\Omega$ is said to be *complex tangential* if at each point $p \in M$, $T_p(M)$ is contained in $T_p^{\mathbf{C}}(b\Omega)$. If at $p \in M$, $T_p(M)$ is not contained in $T_p^{\mathbf{C}}(b\Omega)$, then M is said to be *transverse to the complex directions at* p or simply *transverse at* p. Those M's that are transverse to the complex directions at each of their points will be called *transverse submanifolds*. We recall (see [4]) complex tangential submanifolds of strongly pseudoconvex boundaries are necessarily totally real.

Our first result is the following fact.

1. THEOREM. *If $M \subset b\Omega$ is a compact C^2 submanifold of $\mathbf{C}^n$, possibly with boundary, M of dimensional at least two, such that* $\mathrm{Ph}^c(\Omega)|M$ *is a closed subspace of $C_{\mathbf{R}}(M)$ of finite codimension, then M is complex tangential.*

It would be more natural to suppose M to be a submanifold of class C^1 rather than of class C^2, but the present arguments do not seem to yield this stronger version. Recall in this connection that the corresponding result for interpolation by $A(\Omega)$ on manifolds of class C^1 (see [13]) requires ideas beyond those used in [10] to treat the class of $C^{1,1}$ manifolds. In contrast with the argument given below, the C^1 interpolation theorem for $A(\Omega)$ uses in an essential way the assumptions that $b\Omega$ is of class C^2 and that the interpolating functions are defined on Ω rather than on certain wedges.

The proof we give for the theorem depends on ideas familiar in the study of C^∞ wave front sets. In this connection, see [2].

For nonsmooth sets, we have the following rather special noninterpolation result.

2. THEOREM. *If X is a compact subset of $b\boldsymbol{B}_2$ such that the Čech cohomology group $H^2(X, \mathbf{C})$ is not zero, than* $\mathrm{Ph}^c(\boldsymbol{B}_2)|X$ *is not a closed subspace of* $C_{\mathbf{R}}(X)$ *of finite codimension.*

As an example, every closed two-dimensional topological submanifold, not necessarily smooth, of $b\boldsymbol{B}_2$, satisfies the hypotheses. In contrast with Theorem 1, Theorem 2 is a global theorem; it is not clear what might be a local version of this theorem, though one might conjecture that, as in the case of $A(\Omega)$ interpolation (see [17]), interpolation sets $E \subset b\boldsymbol{B}_n$ for $\mathrm{Ph}^c(\boldsymbol{B}_n)$ have topological dimension not more than $n-1$, provided $n \geqq 2$.

Let us now turn to the proofs.

PROOF OF THEOREM 1. We will deal first with Theorem 1 under restrictive hypotheses: We assume that

1°. Ω is strongly pseudoconvex, $b\Omega$ of class C^2 and that
2°. M is totally real and real-analytic.

We shall show then that if M is transverse, $\mathrm{Ph}^c(\Omega)|M$ omits a subspace of $C^w(M)$ of finite dimension. This is a very special case of the general theorem, but as its proof is a rather direct application of the edge-of-the-wedge theorem, it seems worth independent treatment.

The proof depends on the edge-of-the-wedge theorem as follows. As M is totally real and real-analytic, M has a complexification M^* in $\mathbf{C}^n$: In some neighborhood W of M in $\mathbf{C}^n$, there is a k-dimensional complex submanifold M^* that contains M, $k = \dim M$. If the neighborhood is chosen correctly, then M^* admits an antiholomorphic involution $\varrho\colon M^* \to M^*$ that leaves M fixed pointwise. We have assumed that the submanifold M of $b\Omega$ is transverse to the complex directions, so the complex manifold M^* meets $b\Omega$ transversally along M. Choose a smoothly bounded strongly pseudoconvex domain $\Omega_0 \subset \Omega$ with the property that $b\Omega_0$ contains M and is otherwise contained in Ω. According to the edge-of-the-wedge theorem, there is a neighborhood U of M in M^* with the property that if f is holomorphic on $\Omega_0 \cap M^*$, if g is holomorphic on the domain $\varrho(\Omega_0 \cap M^*)$, and if the boundary values of f and g along M agree, then for some holomorphic function F on U, F agrees with f on $U \cap \Omega_0 \cap M^*$ and with g on $U \cap \varrho(\Omega_0 \cap M^*)$. (We need not enter into a discussion of precisely how the boundary values of f and g along M are to be assumed. It suffices that they be assumed continuously or, in the event that f and g are bounded or merely have bounded real parts so that the boundary values exists nontangentially at almost every point of M, it suffices that these a.e. existent boundary values agree.) A consequence of this is that if u is a pluriharmonic function on $\Omega_0 \cap M^*$ that assumes continuously the boundary value zero along M, then u continues pluriharmonically into the open set U in M^*. The domain $\Omega_0 \cap M^*$ in M^* is strongly pseudoconvex – at least if we choose Ω_0 correctly, it will be. Thus, there is a function ψ holomorphic on a neighborhood of the closure of $\Omega_0 \cap M^*$ that has a pole at a point $q \in \Omega \cap M^* \cap U$. There is a monic polynomial $P(X)$ in one indeterminant and with complex coefficients such that if $\Phi = P(\psi)$, then the restriction of $\operatorname{Re} \Phi$ to M is of the form $u|M$ for some $u \in \mathrm{Ph}^c(\Omega)|M$. (It is here that we use the hypothesis that $\mathrm{Ph}^c(\Omega)$ interpolate a subspace of $C^\omega(M)$ that has finite codimension.) The function $u_1 = u - \operatorname{Re} \Phi$ is pluriharmonic on Ω_0 and assumes continuously the boundary values zero along M. Thus, there is a pluharmonic function u^* on U that agrees with u_1 near M.

Since the function u is pluriharmonic on all of Ω and since, on the other hand, $\operatorname{Re} \Phi$ has a singularity at the point q, we have a contradiction, and our special case of Theorem 1 is proved.

It is plain that there are certain local variations of this argument.

Let us now take up the proof of Theorem 1 in the general case. We make a preliminary reduction.

Suppose that the manifold $M \subset b\Omega$ is not complex tangential. Thus, at some point, p, which may be chosen not to lie in bM, the tangent space $T_p(M)$ contains two linearly independent vectors ξ' and ξ'' with, say, ξ' not in $T_p^{\mathbf{C}}(bD)$. There is then a C^2 two-dimensional disc with boundary, call it Σ, that is contained in M and that passes through p such that $T_p(\Sigma)$ is spanned by ξ' and ξ''. The disc Σ is totally real at p and so in a neighborhood of p; we may suppose that Σ is totally real at each of its points by shrinking it as required.

Notice that if $\mathrm{Ph}^c(\Omega)|M$ is a closed subspace of finite codimension in $C_{\mathbf{R}}(M)$, then $\mathrm{Ph}^c(\Omega)|\Sigma$ is a closed subspace of finite codimension in $C_{\mathbf{R}}(\Sigma)$. This is more or less evident: Put $E = \mathrm{Ph}^c(\Omega)|M$ so that for some finite dimensional subspace $F \subset C_{\mathbf{R}}(M)$, where $C_{\mathbf{R}}(M) = E \oplus F$. Let $\varrho: C_{\mathbf{R}}(M) \to C_{\mathbf{R}}(\Sigma)$ be the restriction map, and let π_E and π_F be the projections of $C_{\mathbf{R}}(M)$ onto E and F respectively. Thus, if $f \in C_{\mathbf{R}}(M)$,

$$\varrho f = \varrho\pi_E f + \varrho\pi_F f$$

whence

$$\varrho\pi_E f = \varrho f - \varrho\pi_F f.$$

This exhibits the operator $\varrho\pi_E: C_{\mathbf{R}}(M) \to C_{\mathbf{R}}(\Sigma)$ as a finite dimensional perturbation of the surjective map ϱ; as such it has closed range of finite codimension, whence ϱE is a closed subspace of finite dimension in $C_{\mathsf{R}}(\Sigma)$.

Accordingly, we may proceed under the assumption that the M in the statement of the theorem is totally real.

We shall need the following fact:

3. LEMMA. *If* $f \in O(\Omega)$ *has bounded real part, then*

(i) $|f(z)| \leqq \text{const.} \log(\operatorname{dist}(z, b\Omega))^{-1}$, *and*

(ii) $\left|\dfrac{\partial f}{\partial z_j}(z)\right| \leqq \text{const.} \operatorname{dist}(z, b\Omega)^{-1}$.

PROOF. The estimate (i) follows from (ii), and (ii) is a consequence, granted the Cauchy-Riemann equations, of the estimate (see [9, p. 109]) that for a harmonic function u on a domain $D \subset \mathsf{R}^m$,

$$\left|\frac{\partial u}{\partial x_j}\right| \leqq m \operatorname{dist}(x, bD)^{-1} \sup_D |u|.$$

In the sequel, we will not use the full force of (i) but only the estimate that $|f(z)| \leqq \text{const.} \operatorname{dist}(z, b\Omega)^{-1}$.

To prove the theorem, consider an $M \subset b\Omega$ that is a manifold of class C^2, M transverse to the complex directions on $b\Omega$ and totally real. Assume $0 \in M$. As M is totally real, there is a map

$$\Phi : \mathsf{R}^k \to \mathsf{C}^n$$

with $\Phi(0) = 0$, Φ of class C^2, Φ carrying R^k diffeomorphically onto a neighborhood of 0 in M^k. The map Φ admits an extension, again denoted by Φ, to a neighborhood of R^k in C^k in such a way that, near $0 \in \mathsf{C}^k$,

$$|\bar{\partial}\Phi(z)| = o(|y|), \quad z = x + iy, \quad x, y \in \mathsf{R}^k. \tag{1}$$

(For this see Hörmander and Wermer [7].) The hypothesis that M^k be transverse implies that

$$d\Phi_x(T_x\mathsf{C}^k) \not\subset T_{\Phi(x)}(b\Omega) \quad \text{for} \quad x \in \mathsf{R}^k.$$

To see this, note that by hypothesis there is $\xi \in T_x(\mathsf{R}^k)$ such that $d\Phi_x(\xi) \notin T^{\mathsf{C}}_{\Phi(x)}(b\Omega)$. As $\bar{\partial}\Phi = 0$ on R^k,

$$d\Phi_x(J\xi) = J d\Phi_x(\xi)$$

is a vector in $T_{\Phi(x)}(\Phi(\mathsf{C}^k))$ that does not lie in $T_{\Phi(x)}(b\Omega)$.

As $\Phi(\mathsf{C}^k)$ is transverse to $b\Omega$ along M^k, $\Phi^{-1}(b\Omega)$ is a certain real hypersurface through R^k – we work locally along R^k only. There is, then, a purely imaginary vector $iy_0, y_0 = (y_1^0, \ldots, y_k^0) \in \mathbf{R}^k$, such that $d\Phi_0(iy_0)$ points into Ω. (That is, $d\Phi_0(iy_0)$ is not tangent to $b\Omega$ at 0, and the component of it normal to $b\Omega$ at 0 is an inner normal.) If we act on our geometric configuration by elements of $GL(k, \mathbf{R})$, we preserve the essentials of the geometry; we may assume therefore that $u_0 = (1, 0, \ldots, 0)$.

Fix a cone V_0 in $\mathbf{R}^k$ with vertex 0 and axis u_0: For some small $\eta > 0$,

$$V_0 = \{y \in \mathbf{R}^k; |y'| < \eta y_1\}$$

where for $y = (y_1, \ldots, y_k)$, $y' = (y_2, \ldots, y_k) \in \mathbf{R}^{k-1}$. Let V_0^ϱ denote the truncation

$$V_0^\varrho = \{y \in V_0 : |y| < \varrho\}.$$

If ϱ and η are small enough, we shall have that $\Phi(V_0^\varrho) \subset \Omega \cup \{0\}$, and, indeed, for sufficiently small δ_0, the image under Φ of the wedge

$$\mathscr{W}_{\delta_0} = \bigcup \{x + iV_0^\varrho : x \in \mathbf{R}^k, |x| \leqq \delta_0\}$$

will be contained in Ω. We shall have, in addition, $\operatorname{dist}(\Phi(x+iy), b\Omega) \geqq \text{const.}\,|y|$.

Consider now a function $u \in \mathrm{Ph}^c(\Omega)$. Our analysis is entirely local, so there is no loss in assuming Ω simply connected so that $u = \operatorname{Re} f$, f holomorphic on Ω. There is no reason for f to be bounded, but we do have the estimates (i) and (ii) of the lemma for f and its derivatives.

The function $F = f \circ \Phi$ is not holomorphic in the wedge $\mathscr{W}_{\delta_0}$, but we have the estimates that for $x + iy \in \mathscr{W}_{\delta_0}$,

(i′) $$|F(x+iy)| \leqq \text{const.}\,|y|^{-1},$$

and

(ii′) $$\left|\frac{\partial F}{\partial \bar{z}_r}(x - iy)\right| = o(1).$$

The former estimate follows from the estimate (i) for f, the fact that the vector $d\Phi_x(1, 0, \ldots, 0)$ is transverse to $b\Omega$, and the fact that V_0 is a small cone with axis the ray $(t, 0, \ldots, 0)$. The latter estimate comes from

$$\frac{\partial F}{\partial \bar{z}_r} = \sum_{j=1}^{n} (f_j \circ \Phi) \frac{\partial \varphi_j}{\partial \bar{z}_r}$$

where $f_j = \partial f / \partial z_j$. We have then that

$$|f_j(\Phi(x+iy))| \leqq \text{const.}|y|^{-1}$$

by (ii), and we have

$$\left|\frac{\partial \varphi_j}{\partial \bar{z}_k}(x+iy)\right| = o(|y|),$$

by (1).

The estimates (i′) and (ii′) imply that F has boundary values, F^*, along $\mathsf{R}^k \cap \bar{\mathscr{W}}_{\delta_0}$ through $\mathscr{W}_{\delta_0}$ in the sense of distributions.

Denote by χ a C^∞ function on R^k, χ identically one near 0, the support of χ to be a ball of radius less than δ_0. We may extend χ to be a C^∞ function on all of R^k with

$$\bar{\partial}\chi(x+iy) = O(|y|^p) \quad \text{for all } p, y \to 0.$$

We can, in addition, suppose that χ is a supported in a ball of radius less than $\min(\varrho, \delta_0)$ centered at 0.

Fix a vector ξ_0 not in the dual, Γ_0, of the cone V_0 so that $\xi_0 \cdot y_0 < 0$ for some unit vector $y_0 \in V_0$.

Let ξ be a vector so near ξ_0 that $\xi \cdot y_0 < \frac{1}{2}\xi_0 \cdot y_0$. We consider the Fourier transform

$$(\chi F^*)\hat{}(t\xi) = \int_{\mathsf{R}^k} \chi(x) F^*(x) e^{-it\xi \cdot x} dx$$

where the integration is understood to be the pairing of the distribution F^* with the test function $x \to \chi(x)e^{-it\xi \cdot x}$.

Let $\Pi \subset \mathsf{R}^k$ be the $(k-1)$-dimensional subspace orthogonal to the vector y_0, and define

$$\Psi : \mathsf{C} \times \mathsf{R}^{k-1} \to \mathsf{C}^k$$

by

$$\Psi(s_1, \sigma') = s_1 y_0 + T(\sigma') \tag{2}$$

where $T : \mathsf{R}^{h-1} \to \Pi$ is a linear isometry. We take $s_1 = \sigma_1 + i\tau_1$, $\sigma' = (\sigma_2, \ldots, \sigma_k)$. By definition, $\Psi(i, 0) = iy_0$, and the map $\Psi|(\mathsf{R} \times \mathsf{R}^{k-1})$ is a linear isometry. We may write then

$$(\chi F^*)\hat{}(t\xi) = \int_{\mathsf{R} \times \mathsf{R}^{k-1}} \chi(\Psi(s_1, \sigma')) F(\Psi(s_1, \sigma')) e^{it\xi \cdot \Psi(s_1, \sigma')} ds_1 d\sigma'. \tag{3}$$

Notice that

$$\Psi(\sigma_1 + i\tau_1, \sigma') = \Psi(\sigma_1, \sigma') + i\tau_1 y_0,$$

and so when $|\sigma_1|^2+|\sigma'|^2 < \delta_0$ and $0 < \tau_1 < \varrho$, $\Psi(\sigma_1+i\tau_1,\sigma')$ is contained in the wedge $\mathscr{W}_{\delta_0}$. Notice also that

$$\xi\cdot\Psi(\sigma_1+i\tau_1,\sigma') = \xi\cdot\Psi(\sigma_1,\sigma')+i\tau_1\xi\cdot y_0. \tag{4}$$

We apply Stokes's theorem to the integral (3) in which we regard $\mathsf{R}\times\mathsf{R}^k$ as part of the boundary of the domain $\{(s_1,\sigma')\in\mathsf{C}\times\mathsf{R}^k : 0<\tau_1<\varrho\}$. The conclusion is that

$$\begin{aligned}(\chi F^*)\hat{}(t\xi) = &\int_{\substack{\sigma'\in\mathsf{R}^{k-1}\\ s_1\in\mathsf{R}}} \chi(\Psi(s_1+i\varrho,\sigma'))F(\Psi(s_1+i\varrho,\sigma'))e^{-it\xi\cdot\Psi(s_1,\sigma')}e^{\varrho t\xi\cdot y_0}ds_1d\sigma' + \\ &+\int_{\substack{0<r_1<\varrho\\ \sigma'\in\mathsf{R}^{k-1}}} \bar{\partial}_{s_1}\{\chi(\Psi(s_1,\sigma'))F(\Psi(s_1,\sigma'))e^{-it\xi\cdot\Psi(s_1,\sigma')}ds_1\}d\sigma' \\ &= \mathrm{I}+\mathrm{II}.\end{aligned}$$

The integral I is zero, because χ is supported in the ball of radius ϱ around the origin.

For the integrand in II, notice that as the exponential term is holomorphic in s_1, we have

$$\begin{aligned}\bar{\partial}_{s_1}\{\cdots\} = -2i\Big\{&F(\Psi(s_1,\sigma'))\frac{\partial}{\partial\bar{s}_1}\chi(\Psi(s_1+\sigma'))+ \\ &+\chi(\Psi(s_1+\sigma'))\frac{\partial F(\Psi(s_1,\sigma'))}{\partial\bar{s}_1}\Big\}\{e^{-it\xi\cdot\Psi(\sigma_1,\sigma')}e^{t\tau_1\xi\cdot y_0}\}d\sigma_1d\tau_1.\end{aligned}$$

The function χ is bounded by one and by (i′) we have

$$|F(\Psi(s_1,\sigma))| \leqq \text{const.}\,\tau_1^{-1}.$$

Also, by (ii′),

$$\left|\frac{\partial F\circ\Psi(s_1,\sigma')}{\partial\bar{s}_1}\right| = o(1).$$

In addition,

$$\left|\frac{\partial\chi(\Psi(s_1+\sigma'))}{\partial\bar{s}_1}\right| \leqq \text{const.}\ \tau_1^p$$

for all $p = 1,2,\ldots$. As χ is compactly supported, we reach, for t > 0 and large

$$|\mathrm{II}| \leqq \text{const.} \int_0^\infty e^{t\tau_1 \xi \cdot y_0} d\tau_1$$

$$= \text{const.} \frac{-1}{t\xi \cdot y_0} \int_0^\infty e^{-\lambda} d\lambda$$

$$= \text{const.}\ |t|^{-1}.$$

Thus, for $t > 0$

$$|(\chi F^*)\hat{}(t\xi)| \leqq \text{const.}\ |t|^{-1}, \tag{5}$$

where the constant in question is locally uniform in ξ, subject to the condition that $\xi \cdot y_0 < \frac{1}{2}\xi_0 \cdot y_0$.

We can perform the same kind of analysis, starting with the function $\bar{f}$, the complex conjugate of f, rather than with f, and show that for vectors ξ_0 not in the negative, $-\Gamma_0$, of the dual cone of V_0, there is an estimate of the form

$$|(\chi \bar{F}^*)\hat{}(t\xi)| \leqq \text{const.}\ |t|^{-1} \tag{6}$$

uniformly in ξ, ξ near ξ_0, for $t > 0$ large.

The stimates (5) and (6) combine to yield the estimate

$$[\chi(u \circ \Phi)]\hat{}(t\xi) \leqq \text{const.}|t|^{-1} \tag{7}$$

when $\xi \notin -\Gamma_0 \cup \Gamma_0$.

As Γ_0 is the dual of the cone V_0, we have that

$$\Gamma_0 = \left\{ y \in \mathbf{R}^k : |y'| \leqq \frac{1}{\eta} y_1 \right\},$$

where, as before, $y = (y_1, y')$, $y_1 \in \mathbf{R}$, $y' \in \mathbf{R}^{k-1}$. Thus,

$$-\Gamma_0 = \left\{ y \in \mathbf{R}^k : |y'| \leqq \frac{-1}{\eta} y_1 \right\}.$$

In particular, $-\Gamma_0 \cup \Gamma_0$ omits certain vectors, ξ. For these vectors ξ, (7) imposes a genuine condition on the function u.

This condition precludes the possibility that $\mathrm{Ph}^c(\Omega)|M = C_{\mathbf{R}}(M)$ or even, the obstruction being local, that $\mathrm{Ph}^c(\Omega)|M$ be a closed subspace of finite codimension in $C_{\mathbf{R}}(M)$: As is known from the Riemann-Lebesgue lemma, for every continuous function h, $(\chi h)\hat{}(t\xi) = o(1)$, $t \to \infty$, provided $\xi \neq 0$. But it is also known that the Fourier transform of an arbitrary function need not decay to zero at any particular rate, and it certainly need not be $O(1/t)$.

There are various other statements that can be formulated on the basis of what we have done. For example, the analysis shows that when M is transverse, it is not possible to realize every $\varphi \in C_{\mathbb{R}}(M)$ as the a.e. $[dM]$ nontangential limit of a bounded, pluriharmonic function on Ω.

Finally, let us observe that this approach yields another proof, independent of the theory of peak sets, of the result that interpolation manifolds for $A(\Omega)$ are necessarily complex tangential. Since in this case we do not have to treat unbounded functions, the smoothness requirement on M can be reduced from C^2 to C^1. Moreover, if we use the almost analytic extension Φ constructed in [11, p. 334] by Nagel and Wainger we can treat the case of curves (for $A(\Omega)$ interpolation) as well as the case of higher dimensional manifolds.

PROOF OF THEOREM 2. To begin with, we need the following simple fact. (We denote by $\hat{X}$ the polynomially convex hull of the set X.)

4. LEMMA. *If $X \subset b\boldsymbol{B}_n$ is a closed set, if $f \in O(\bar{X})$ and if $u \in \mathrm{Ph}^c(\boldsymbol{B}_n)$ satisfies $u = \mathrm{Re} f$ on X, then $u = \mathrm{Re} f$ on $\bar{X}$.*

PROOF. As $u \in \mathrm{Ph}^c(\boldsymbol{B}_n)$, there is a sequence $\{f_n\}_{n=1}^{\infty}$ of functions, each holomorphic on a neighborhood of the closed ball, $\bar{\boldsymbol{B}}_n$, with $\{\mathrm{Re} f_n\}_{n=1}^{\infty}$ converging uniformly to u on $\boldsymbol{B}_n$. We have that

$$\begin{aligned} |e^{f_n - f}|_{\hat{X}} &= |e^{f_n - f}|_X \\ &= |e^{u_n - \mathrm{Re} f}|_X \to 1, \end{aligned}$$

so $\overline{\lim}\, \mathrm{Re}(f_n - f)(x) = 0$ for $x \in \hat{X}$. Considering in a similar way $e^{f - f_n}$, we find $\underline{\lim}\, \mathrm{Re}(f_n - f)(x) = 0$ for $x \in \hat{X}$, so $\mathrm{Re} f_n \to f$ on $\hat{X}$. But as $\mathrm{Re} f_n \to u$ on $\overline{\boldsymbol{B}_n}$, we have $u = \mathrm{Re} f$ on $\hat{X}$, as we wished to show.

5. LEMMA. *If $X \subset b\boldsymbol{B}_n$ is a compact set with the property that $\mathrm{Ph}^c(\boldsymbol{B}_n)|X$ has finite codimension in $C_{\mathbb{R}}(X)$, then $\hat{X} \setminus X$ contains the germ of no real-analytic, totally real n-dimensional submanifold of $\mathbb{C}^n$.*

PROOF. Assume the lemma false, and let $X \subset b\boldsymbol{B}_n$ be a compact set such that $\mathrm{Ph}^c(\boldsymbol{B}_n)|X$ has finite codimension in $C_{\mathbb{R}}(X)$ and such that $\hat{X} \setminus X$ contains M, an n-dimensional, real-analytic totally real closed submanifold of an open subset of $\mathbb{C}^n$. We may suppose $0 \in M$.

The hypotheses imply the existence of a biholomorphic map ψ from the open unit polydisc U^n in $\mathbb{C}^n$ onto an open subset of $\mathbb{C}^n$ with $\psi(0) = 0$ such that $\psi(\mathbb{R}^n \cap U^n)$ is a neighborhood of 0 in M.

If P is a holomorphic polynomial, then $\mathrm{Re}\, P|X$ and $\mathrm{Im}\, P|X$ belong to $\mathrm{Ph}^c(\boldsymbol{B}_n)|X$, so as the latter space is closed, the open mapping theorem yields a constant C, independent of P, such that there are $u, v \in \mathrm{Ph}^c(\boldsymbol{B}_n)$ that match $\mathrm{Re}\, P$ and $\mathrm{Im}\, P$, respectively, on X and that satisfy

$$|u|_{B_n} \leqq C|\operatorname{Re} P|_X \quad \text{and} \quad |v|_{B_n} \leqq C|\operatorname{Im} P|_X.$$

The functions $u \circ \psi$ and $v \circ \psi$ are pluriharmonic in U^n and are bounded there by $C|\operatorname{Re} P|_X$ and $C|\operatorname{Im} P|_X$, respectively. On $\mathbf{R}^n$ we have Taylor expansions

$$u \circ \psi = \sum \alpha_J x^J$$
$$v \circ \psi = \sum \beta_J x^J,$$

which are valid for all x with $\max|x_j| < 1$. Consequently, for every $\varrho \in (0, 1)$, there is a constant C_p such that

$$\sum |\alpha_J| \varrho^{|J|} \leqq C_\varrho C|P|_X$$

and

$$\sum |\beta_J| \varrho^{|J|} \leqq C_\varrho C|P|_X.$$

On U^n, we have

$$P \circ \psi(z) = \sum (\alpha_J + i\beta_J) z^J.$$

Thus, for any fixed $z \in U^n$,

$$|P \circ \psi(z)| \leqq C_z |P|_X,$$

where the constant C_z is independent of the choice of P. Applying this to P^k, $k = 1, \ldots$ and taking kth roots shows that C_z may be taken to be one: For every $z \in U^n$, $\psi(z) \in \hat{X}$, and $\hat{X}$ is seen to contain an open set.

This, however, is impossible: Choose a point $z_0 \in B_n \setminus \hat{X}$. There is a function F holomorphic on a neighborhood of $\hat{X}$, meromorphic on $\bar{B}_n$ with a pole at the point z_0. As $\operatorname{Ph}^c(B_n)|X$ has finite codimension, there is a positive integral d such that for some choice if $\alpha_0, \ldots, \alpha_{d-1} \in \mathbf{C}$, the function

$$\operatorname{Re}(\alpha_0 + \alpha_1 F + \ldots + \alpha_{d-1} F^{d-1} + F^d) = u_0$$

satisfies $u_0|X = u|X$ for some $u \in \operatorname{Ph}^c(B_n)$. By Lemma 6, $u_0 = u$ on $\hat{X}$, whence $u_0 = u$ on all of B_n off the singular set of $\alpha_0 + \alpha_1 F + \ldots + F^d$, for we have that $\hat{X}$ contains an open set in $\mathbf{C}^n$. As u is pluriharmonic throughout B_n but u_0 has singularities, we have reached a contradiction, and the lemma is proved.

The proof of the Theorem itself now goes as follows. Let $X \subset bB_2$ be a compact set such the Čech cohomology group $H^2(X, \mathbf{C})$ is not zero, and suppose that $\operatorname{Ph}^c(B_n)|X$ has finite codimension in $C_{\mathbf{R}}(X)$.

The hypothesis that $H^2(X, \mathbf{C}) \neq 0$ implies that $\dim(\hat{X} \setminus X) > 2$, dimension taken in the topological sense. This is a result of Alexander [1]. If $\dim(\hat{X} \setminus X) = 4$, then (see [8, p. 44]) the set $\hat{X} \setminus X$ contains an open subset of $\mathbf{C}^2$, and this is impossible, as we saw in the proof of the last lemma. Thus, $\dim \hat{X} \setminus X = 3$.

There is another way to see the set $\hat{X}$ has dimension three, at least in certain cases. Granted that $\mathrm{Ph}^c(\boldsymbol{B}_2)$ interpolates all of $C(X)$, the algebra $\mathscr{P}(X)$ is a Dirichlet algebra, and so its Gleason parts are all points or discs. Thus, $\hat{X}$ cannot contain an open set in $\mathbf{C}^2$. If $\mathrm{Ph}^c(\boldsymbol{B}_2)$ only interpolates a subspace of $C(X)$ of finite codimension, then, provided there are invertible elements $h_1, \ldots, h_r$ of $P(X)$, such that the functions $\log|h_j|$ together with $\mathrm{Ph}^c(\boldsymbol{B}_2)$ span all of $C_{\mathbf{R}}(X)$ so that the algebra $\mathscr{P}(X)$ is a hypo-Dirichlet algebra, the result of [6] on the structure of the Gleason parts of hypo-Dirichlet algebras can be applied to conclude as above that $\hat{X}$ cannot contain an open set. If we merely suppose that $\mathrm{Ph}^c(\boldsymbol{B}_2)|X$ has finite codimension, we are not assured that $\mathscr{P}(X)$ is a hypo-Dirichlet algebra; it would be necessary to extend the results of [6] to deal with this general case.

Let q be a point in $\boldsymbol{B}_2 \setminus \hat{X}$, and let the function f be meromorphic on C^2, holomorphic on $\hat{X}$ with a pole at q. As above, our hypotheses imply the existence of a positive integer d such that for suitable constants $\alpha_0, \ldots, \alpha_{d-1}$, if

$$f_0 = \alpha_0 + \alpha_1 f + \ldots + \alpha_{d-1} f^{d-1} + d^{\,d},$$

then there is a function F_0 holomorphic on the ball such that $\mathrm{Re}\, F_0 \in \mathrm{Ph}^c(\boldsymbol{B}_2)$ and $\mathrm{Re} F_0|X = \mathrm{Re} f_0|X$. Set $u = \mathrm{Re}(F_0 - f_0)$. This function vanishes on X whence on $\hat{X}$, by Lemma 6. The set $\{u = 0\} \cap \boldsymbol{B}_2$ is a real-analytic subset of $\boldsymbol{B}_2$ and so is generically a three-dimensional analytic manifold. If contains the three-dimensional set $\hat{X} \setminus X$, and so there is an open set Ω of manifold points of $\{u = 0\}$ contained in $\hat{X}$ (see [8, p. 44]). The three-dimensional manifold Ω contains a two-dimensional real-analytic, totally real submanifold, whence a contradiction to Lemma 5.

The theorem is proved.

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