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TOPOLOGICAL FROBENIUS PROPERTIES FOR NILPOTENT GROUPS

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Introduction.

Let G be a locally compact group and H a closed subgroup of G, and suppose that π and τ are irreducible representations of G and H, respectively. If G is compact, then according to the classical Frobenius reciprocity theorem, the restriction $\pi | H$ contains τ if and only if π is contained in the induced representation $\mathrm{ind}_H^G \tau$.

Fell [6] first introduced and studied topological (or weak) Frobenius properties which a general locally compact group G may or may not possess and which are defined by replacing containment by weak containment. Thus, G is said to have property (FP) if for any closed subgroup H of G and irreducible representations π and τ of G and H, respectively, π is weakly contained in $\mathrm{ind}_H^G \tau$ if and only if $\pi | H$ weakly contains τ. It is reasonable to treat separately the if and the only if part of (FP) which will be called (FP1) and (FP2), respectively. For connected groups it is also of interest to consider the versions (FPC1) and (FPC2) of (FP1) and (FP2) that are obtained by restricting H to connected subgroups of G.

These topological Frobenius properties have been studied by several authors [3], [5], [6], [8], [11], [17], [18], [19]. For instance, Felix, Henrichs and Skudlarek [5], [8] proved that for an amenable group G with open connected component, (FP2) implies that G is an $[FC]^-$ group, i.e. every conjugacy class in G is relatively compact. On the other hand, $[FC]^-$ groups satisfy (FP)(see [11]). Nielsen [18] has shown that if G is a connected and simply connected nilpotent Lie group and H any nonnormal connected subgroup of G, then there exists an irreducible representation π of G such that the trivial representation of H is not weakly contained in $\pi | H$ and yet π is weakly contained in the quasi-regular representation of G on $L^2(G/H)$.

The purpose of this paper is to contribute to the investigation of the properties (FP1) and (FPC1) for nilpotent groups. In the first two sections we are concerned with connected, simply connected nilpotent Lie groups. Let G be such a group

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and $\mathfrak{g}$ its Lie algebra. We first show that (FP1) holds for a fixed irreducible representation π of G provided that the Kirillov orbit in $\mathfrak{g}^*$ corresponding to π is a linear variety (Theorem 1.3). There is some evidence that the converse is also true. We were only able to prove the following special result (Theorem 1.5): If G satisfies (FP1) and is, in addition, a semi-direct product of R with some R^n, then G is 2-step nilpotent. On the other hand, we verify (FPC1) for G when G is of the form $\mathsf{R} \ltimes \mathsf{R}^n$ or of dimension $\leqq 5$ (Corollary 1.7).

Section 2 is devoted to the study of hereditary properties of (FP1) and (FPC1). We first prove that the direct product $G \times G$ satisfies (FP1) (or (FPC1)) if and only if all the Kirillov orbits are linear varieties (Theorem 2.1). Thus, in general, (FPC1) is not inherited by direct products. However, (FP1) (respectively (FPC1)) holds for a direct product $G_1 \times G_2$ if, say, G_1 has the corresponding property and all the Kirillov orbits of G_2 are linear varieties (Proposition 2.3). Moreover, we fgive an example showing that (FP1) (or (FPC1)) also fails to be inherited by connected normal subgroups (Example 2.4).

The second author conjectured in [11] that every 2-step nilpotent locally compact group has property (FP1). The main object of Section 3 is to prove this conjecture for pro-Lie, in particular for compactly generated, 2-step nilpotent groups (Theorem 3.4). Finally, extending a result of [17], we observe that nilpotent pro-Lie groups satisfy (FP) as far as only normal subgroups are considered (Proposition 3.5).

1. On (FP1) and (FPC1) for simply connected nilpotent Lie groups.

We first fix some notation. Let G be a locally compact group. We will use the same letter to denote a unitary representation of G and the corresponding $*$-representation of the group C*-algebra $C^*(G)$, and ker π always means the kernel of π in $C^*(G)$. If S and T are sets of unitary representations of G, then S is weakly contained in T $(S \prec T)$ if

$$\bigcap_{\sigma \in S} \ker \sigma \supseteqq \bigcap_{\tau \in T} \ker \tau,$$

and S and T are said to be weakly equivalent $(S \sim T)$ if $S \prec T$ and $T \prec S$. If H is a closed subgroup of G and τ a representation of H, then $\mathrm{ind}_H^G \tau$ denotes the representation of G induced by τ. We will frequently use that $\pi \prec \mathrm{ind}_H^G \pi \mid H$ for every representation π of G and every closed subgroup H of G if G is amenable [7].

The dual space $\hat{G}$ of G is the set of equivalence classes of irreducible unitary representations of G, and the primitive ideal space $\mathrm{Prim}\, C^*(G)$ consists of all $\ker \pi$, $\pi \in \hat{G}$. $\mathrm{Prim}\, C^*(G)$ carries the hull-kernel topology and $\hat{G}$ its inverse image with respect to the canonical mapping $\pi \to \ker \pi$ from $\hat{G}$ onto Prim $C^*(G)$. Finally, for any unitary representation π of G, the support of π is the closed subset $\operatorname{supp} \pi = \{\rho \in \hat{G}; \rho \prec \pi\}$ of $\hat{G}$.

We will almost exclusively be concerned with the following Frobenius properties. G is said to satisfy (FP1) (respectively (FPC1)) if for all closed (respectively closed and connected) subgroups H of G and all $\tau \in \hat{H}$ and $\pi \in \hat{G}$, $\tau \prec \pi \mid H$ implies $\pi \prec \operatorname{ind}_H^G \tau$.

The following simple lemma will play a fundamental role.

LEMMA 1.1. *Let G be a connected and simply connected nilpotent Lie group, and let H and N be closed subgroups of G. If N is connected and normal, then HN is closed in G.*

PROOF. Recall that if $\mathfrak{g}$ is the Lie algebra of G, then the exponential mapping exp: $\mathfrak{g} \to G$ is a diffeomorphism whose inverse is denoted by log.

Now, H is contained in some connected subgroup L of G such that L/H is compact and the connected component H_0 of H is normal in L. Consider the simply connected nilpotent Lie group $K = L/H_0$ and the discrete cocompact subgroup $\Gamma = H/H_0$ of K. Denote by $\mathfrak{k}$ the Lie algebra of K. By [15, Theorem 2] there exists a discrete cocompact subgroup Δ of K with the following properties: $\log \Delta$ is a subgroup of $\mathfrak{k}$, $\Delta \subseteqq \Gamma$, and Δ has finite index in Γ. It follows that if $M = \{x \in L; xH_0 \in \Delta\}$, then $\log M$ is a closed subgroup of $\mathfrak{g}$.

It is well-known that if V is a vector group, W a vector subgroup and S any closed subgroup of V, then $S + W$ is closed in V. Hence $\log M + \log N$ is closed in $\mathfrak{g}$. Now, using the Baker-Campbell-Hausdorff formula and the fact that $\log N$ is an ideal in $\mathfrak{g}$, it is easily verified that

$$MN = \exp(\log \mathrm{M} + \log N).$$

Thus MN is closed, and since M has finite index in H, HN is closed in G.

LEMMA 1.2. *Let G be an amenable group, Z the center of G and H a closed subgroup of G. Suppose that HZ is closed in G and that either H or Z is σ-compact. Let $\pi \in \hat{G}$ be such that $\{\ker \pi\}$ is closed in* Prim $C^*(G)$ *and $\pi \sim \operatorname{ind}_Z^G \pi \mid Z$. Then for any $\tau \in \hat{H}$, $\tau \prec \pi \mid H$ implies that $\pi \prec \operatorname{ind}_H^G \tau$.*

PROOF. Let $\lambda \in \hat{Z}$ be such that $\pi \mid Z \sim \lambda$. Then $\tau \mid H \cap Z \sim \lambda \mid H \cap Z$, since $\tau \prec \pi \mid H$. It follows from the assumptions made on H and Z that the continuous homomorphism $(x, z) \to xz$ from $H \times Z$ onto HZ is open. Therefore, the formula

$$\sigma(xz) = \lambda(z)\tau(x), \qquad x \in H, \ z \in Z,$$

defines a unitary representation σ of HZ in the Hilbert space of τ. G being amenable, we obtain

$$\operatorname{ind}_{HZ}^G \sigma \prec \operatorname{ind}_{HZ}^G(\operatorname{ind}_Z^{HZ} \sigma \mid Z) \sim \mathit{ind}_{HZ}^G(\operatorname{ind}_Z^{HZ} \lambda) = \operatorname{ind}_Z^G \lambda \sim \pi.$$

Thus $\pi \sim \mathrm{ind}_{HZ}^{G} \sigma$, since $\{\ker \pi\}$ is closed. As $\sigma | H = \tau$, we have

$$\mathrm{ind}_{HZ}^{G} \sigma \prec \mathrm{ind}_{HZ}^{G}(\mathrm{ind}_{H}^{HZ} \tau) = \mathrm{ind}_{H}^{G} \tau.$$

This completes the proof.

Now, we are ready to prove the main result of this section. Before doing this, let us briefly review Kirillov's theory. Let G be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Denote by Ad* the coadjoint representation of G on $\mathfrak{g}^*$, the dual of $\mathfrak{g}$. Kirillov [12] established a bijection between $\hat{G}$ and the orbit space $\mathfrak{g}^*/\mathrm{Ad}^*$. If $\mathfrak{g}^*/\mathrm{Ad}^*$ is endowed with the quotient topology, then the Kirillov correspondence $\mathfrak{g}^*/\mathrm{Ad}^* \to \hat{G}$ is a homeomorphism (see [2]). Moreover, since G is CCR (see [12, Theorem 7.3]), points in $\hat{G}$ are closed, and the mapping $\hat{G} \to \mathrm{Prim}\, C^*(G)$ is a bijection.

Recall also that an irreducible representation π of G is square integrable modulo its kernel C in G (i.e. every coordinate function associated to π is square integrable on G/C) if and only if the corresponding Kirillov orbit in $\mathfrak{g}^*$ is a linear variety [1, Theorem 1.1].

THEOREM 1.3. *Let G be a connected nilpotent group and $\pi \in \hat{G}$, and suppose that π is square integrable modulo its kernel. Then, for any closed subgroup H of G and $\tau \in \hat{H}$, $\tau \prec \pi | H$ implies $\pi \prec \mathrm{ind}_{H}^{G} \tau$. In particular, this conclusion holds if G is a connected, simply connected nilpotent Lie group and the Kirillov orbit corresponding to π is a linear variety.*

PROOF. It is well-known that the connected group G is a projective limit of Lie groups G_ι, $\iota \in I$, and that $\hat{G} = \bigcup_{\iota \in I} \hat{G}_\iota$. Thereby, it is straightforward to reduce to the Lie group case (compare the proof of [11, Lemma 1.3]). Next let G' be a simply connected covering group of G and p: $G' \to G$ a covering homomorphism. Let $H' = p^{-1}(H)$, and consider $\pi \circ p \in \hat{G}'$ and $\tau \circ p \in \hat{H}'$. Then $\tau \circ p \prec \pi \circ p | H'$, and once we have shown that

$$\pi \circ p \prec \mathrm{ind}_{H'}^{G'}(\tau \circ p) = (\mathrm{ind}_{H}^{G} \tau) \circ p,$$

it follows that $\pi \prec \mathrm{ind}_{H}^{G} \tau$. Thus we are reduced to the case that G is a simply connected Lie group.

Let K denote the connected component of the kernel of π. Clearly, $\tau | H \cap K = 1$ since $\tau | H \cap K \prec \pi | H \cap K$. By Lemma 1.1, HK is closed in G, hence $HK/K = H/H \cap K$. Let $\dot{G} = G/K$ and $\dot{H} = HK/K$, and define $\dot{\pi} \in \hat{\dot{G}}$ and $\dot{\tau} \in \hat{\dot{H}}$ by

$$\dot{\pi}(xK) = \pi(x) \quad \text{and} \quad \dot{\tau}(yK) = \tau(y), \qquad x \in G,\ y \in H.$$

Then $\dot{\tau} \prec \dot{\pi} | \dot{H}$. Suppose that we already know $\dot{\pi} \prec \mathrm{ind}_{\dot{H}}^{\dot{G}} \dot{\tau}$. Denoting by p the canonical projection $G \to \dot{G}$, we have $\dot{\tau} \circ p \,|\, H = \tau$, and hence

$$\pi = \dot{\pi} \circ p \prec \operatorname{ind}_{\dot{H}}^{\dot{G}} \dot{\tau} \circ p \prec \operatorname{ind}_{HK}^{G}(\dot{\tau} \circ p) \prec \operatorname{ind}_{HK}^{G}(\operatorname{ind}_{H}^{HK} \tau) = \operatorname{ind}_{H}^{G} \tau.$$

Therefore, if suffices to show the assertion of the theorem under the additional assumption that the kernel of π is discrete and hence contained in the center Z of G. Thus π is square ingrable modulo Z. But then

$$\pi \sim \operatorname{ind}_{Z}^{G} \pi \,|\, Z$$

([see [16, Theorem 1]), and an application of Lemma 1.2 finishes the proof.

COROLLARY 1.4. *Let G be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$, and assume that all the Kirillov orbits in $\mathfrak{g}^*$ are linear varieties (this is, for instance, the case if G is 2-step nilpotent, that is $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0$). Then* (FP1) *holds for G.*

We next prove the converse of the above result for a special class of nilpotent Lie groups.

THEOREM 1.5. *Let G be a nilpotent Lie group that is also a semi-direct product $\mathsf{R} \rtimes \mathsf{R}^n$. If* (FP1) *holds for G, then G is 2-step nilpotent.*

PROOF. Assume that G is m-step nilpotent where $m \geqq 3$. The Lie algebra $\mathfrak{g}$ of G has the form $\mathfrak{g} = \mathsf{R}X \oplus V$, where V is an n-dimensional abelian ideal. Looking at the Jordan canonical decomposition of the nilpotent endomorphism $\operatorname{ad}(X) \,|\, V$ and using the fact that $\operatorname{ad}(X)^2 \neq 0$, we see that V decomposes into a direct sum $V = V_1 \oplus V_2$ with the following properties: V_1 and V_2 are ideals in $\mathfrak{g}$, V_1 has a basis $X_1, \ldots, X_d$, $d \geqq 3$, such that $\operatorname{ad}(X)X_i = X_{i-1}$ for $2 \leqq i \leqq d$ and $\operatorname{ad}(X)X_1 = 0$.

Since it suffices to show that (FP1) fails to hold for G/V_2, we can assume that $V = V_1$, that is $\mathfrak{g}$ has a basis $X, X_1, \ldots, X_n$ $(n \geqq 3)$ with nontrivial commutators $[X, X_i] = X_{i-1}$, $2 \leqq i \leqq n$. Let $X^*, X_1^*, \ldots, X_n^*$ be the dual basis of $\mathfrak{g}^*$, and consider for each $s \in \mathsf{R}$ the functional $f_s = sX_1^* + X_n^*$. It is easily seen that for $t, s \in \mathsf{R}$,

$$\operatorname{Ad}^*(\exp tX)f_s = \left(s + \frac{t^{n-1}}{(n-1)!}\right)X_1^* + \frac{t^{n-2}}{(n-2)!}X_2^* + \ldots + tX_{n-1}^* + X_n^*.$$

Denote by π_s the irreducible representation of G corresponding to f_s. Since $n \geqq 3$, it is obvious that $\operatorname{Ad}^*(G)f_s \neq \operatorname{Ad}^*(G)f_{s'}$ and hence $\pi_s \neq \pi_{s'}$ for $s \neq s'$. For $g \in \mathfrak{g}^*$ denote by χ_g the character of V defined by

$$\chi_g(y) = e^{2\pi i (g, y)}, \quad y \in V.$$

It is clear that for each $s \in \mathsf{R}$

$$\pi_s \,|\, V \sim \{\chi_g; g \in \operatorname{Ad}^*(\exp \mathsf{R}X)f_s\}.$$

Now, consider the lattice $N = \mathbf{Z}X_1 + \mathbf{Z}X_2 + \ldots + \mathbf{Z}X_n$ of V and, for fixed $s \in \mathbf{R}$, the sequence of real numbers

$$t_k = (n-1)!k - s/(n-1)((n-1)!)^{n-3}k^{n-2}, \quad k \in \mathbf{N}$$

One checks that

$$\lim_{k \to \infty} \mathrm{Ad}^*(\exp t_k X) f_s \mid N = 0 \pmod{\mathbf{Z}}.$$

It follows that $1_N \prec \pi_s \mid N$ for each $s \in \mathbf{R}$. It is easily verified (writing down explicitly the multiplication in G) that

$$\Gamma = \exp(n!\mathbf{Z}X \oplus N) = n!\,\mathbf{Z} \ltimes N$$

(is a discrete) subgroup of G. Moreover, it is clear that Γ and V are regularly related and hence by [6, Theorem 5.3]

$$\pi_s \mid \Gamma = (\mathrm{ind}_V^G \chi_{f_s}) \mid \Gamma \sim \{\mathrm{ind}_{V^x \cap \Gamma}^{\Gamma} \chi_{f_s} \mid V^x \cap \Gamma;\ x \in G\} = \{\mathrm{ind}_N^{\Gamma} \pi_s \mid N\}$$

for each $s \in \mathbf{R}$. It follows that

$$1_\Gamma \prec \mathrm{ind}_N^{\Gamma} 1_N \prec \mathrm{ind}_N^{\Gamma} \pi_s \mid N \sim \pi_s \mid \Gamma$$

for each $s \in \mathbf{R}$. Since G/Γ is compact, $\mathrm{ind}_\Gamma^G 1_\Gamma$ is a CCR representation, and hence has a discrete support. On the other hand, $s \to \pi_s$ is a continuous and injective mapping of $\mathbf{R}$ into $\hat{G}$. Thus, it is impossible that $\pi_s \prec \mathrm{ind}_\Gamma^G 1_\Gamma$ for all $s \in \mathbf{R}$. Consequently, (FP1) cannot hold for G.

In contrast to the above result, we are going to show now that (FPC1) always holds for nilpotent semi-direct products $\mathbf{R} \ltimes \mathbf{R}^n$. But first, we have to mention one further fact. Let G be a connected simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Let $f \in \mathfrak{g}^*$ and $\Omega = \mathrm{Ad}^*(G)f$. To Ω one can associate an ideal $\mathfrak{m}(\Omega)$ in the following way (cf. [21]). Define F to be the largest subspace of $\mathfrak{g}^*$ saturating Ω in the sense that $\Omega + F = \Omega$, and let $\mathfrak{m}(\Omega) = F^\perp$. It turns out that $\mathfrak{m}(\Omega)$ is the ideal in $\mathfrak{g}$ generated by the radical

$$\mathfrak{g}(f) = \{X \in \mathfrak{g};\ (f, [X, \mathfrak{g}]) = 0\}$$

(see [21, proof of Theorem 1]). In what follows, for a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, we will denote by $p_{\mathfrak{h}}$ the projection $\mathfrak{g}^* \to \mathfrak{h}^*$, $f \to f \mid \mathfrak{h}$.

LEMMA. 1.6. *Let G be a connected simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Let $f \in \mathfrak{g}^*$, $\Omega = \mathrm{Ad}^*(G)f$, and let $\mathfrak{m} = \mathfrak{m}(\Omega)$ be the ideal associated to Ω in $\mathfrak{g}$. Suppose that $\dim p_{\mathfrak{m}}(\Omega) \leq 1$, and denote by π the irreducible representation of G corresponding to f. Then, for any closed connected subgroup H of G and $\tau \in \hat{H}$, $\tau \prec \pi \mid H$ implies $\pi \prec \mathrm{ind}_H^G \tau$.*

PROOF. By [3, 2.1 Lemma], it suffices to show that, if $\mathfrak{h}$ is any subalgebra of $\mathfrak{g}$, then $p_{\mathfrak{h}}(\Omega)$ is closed in $\mathfrak{h}^*$. This is obvious if $\dim p_{\mathfrak{m}}(\Omega) = 0$, since then Ω is a linear variety. Thus, we can assume that $\dim p_{\mathfrak{m}}(\Omega) = 1$. Choose a basis $\{Y_1, \ldots, Y_n, X_1, \ldots, X_m\}$ of $\mathfrak{g}$ such that

$$\mathfrak{h} \cap \mathfrak{m} = \langle Y_1, \ldots, Y_r \rangle, \; \mathfrak{h} = \langle Y_1, \ldots, Y_r, X_1, \ldots, X_s \rangle \quad \text{and}$$
$$\mathfrak{m} = \langle Y_1, \ldots, Y_n \rangle$$

for some $r \leqq n$ and $s \leqq m$. Denote by $\{Y_1^*, \ldots, Y_n^*, X_1^*, \ldots, X_m^*\}$ the corresponding dual basis of $\mathfrak{g}^*$. Since $\dim p_{\mathfrak{m}}(\Omega) = 1$ and $\Omega = p_{\mathfrak{m}}^{-1}(p_{\mathfrak{m}}(\Omega))$, we have

$$\Omega = \{t_1 X_1^* + \ldots + t_m X_m^* + p_1(t) Y_1^* + \ldots + p_n(t) Y_n^*; t_1, \ldots, t_m, \; t \in \mathsf{R}\}$$

for certain real polynomials $p_1, \ldots, p_n$. Thus

$$p_{\mathfrak{h}}(\Omega) = \{t_1 X_1^* + \ldots + t_s X_s^* + p_1(t) Y_1^* + \ldots + p_r(t) Y_r^*; t_1, \ldots, t_s, \; t \in \mathsf{R}\}$$

is closed in $\mathfrak{h}^*$. Indeed, this is a consequence of the following fact. Let p be a nonconstant polynomial of a real variable, and suppose that, for some sequence $(t_k)_k$, $(p(t_k))_k$ converges. Then $(t_k)_k$ is bounded and, passing to a subsequence, we can assume that $\lim t_k = t$. Hence, $\lim p(t_k) = p(t)$.

COROLLARY 1.7. (i) *Let G be a connected, simply connected nilpotent Lie group. Let $\pi \in \hat{G}$, and denote by Ω the Kirillov orbit corresponding to π. Suppose that* $\dim \Omega = 2$. *Then, for any closed and connected subgroup H of G and $\tau \in \hat{H}$, $\tau \prec \pi | H$ implies $\pi \prec \mathrm{ind}_H^G \tau$.*

(ii) *Let G be a nilpotent Lie group that is also a semi-direct product* $\mathsf{R} \ltimes \mathsf{R}^n$. *Then G satisfies* (FPC1).

(iii) (FPC1) *holds for all connected, simply connected nilpotent Lie groups G with* $\dim G \leqq 5$ (*for the classification, see* [20]).

PROOF. (i) Take $f \in \Omega$. Then $\operatorname{codim} \mathfrak{g}(f) = \dim \Omega = 2$. Since $\mathfrak{g}$ is nilpotent, there is an ideal $\mathfrak{n}$ of $\mathfrak{g}$ with $\mathfrak{g}(f) \subseteqq \mathfrak{n}$ and $\operatorname{codim} \mathfrak{n} = 1$. $\mathfrak{m}(\Omega)$ being the ideal generated by $\mathfrak{g}(f)$, it follows that either $\mathfrak{m}(\Omega) = \mathfrak{g}(f)$ or $\mathfrak{m}(\Omega) = \mathfrak{n}$. Thus $\dim p_{\mathfrak{m}(\Omega)}(\Omega) = 0$ or $\dim p_{\mathfrak{m}(\Omega)}(\Omega) = 1$.

(ii) It is easily seen that, if $G = \mathsf{R} \ltimes \mathsf{R}^n$, then $\dim \Omega = 0$ or $\dim \Omega = 2$ for any $\mathrm{Ad}^*(G)$-orbit Ω.

(iii) Let Ω be an Ad^* (G)-orbit. If $\dim \Omega = 4$, then $\dim G = 5$ and therefore $\dim \mathfrak{g}(f) = 1$ for any $f \in \Omega$. Since the center of $\mathfrak{g}$ is contained in $\mathfrak{g}(f)$, it follow that $\mathfrak{m}(\Omega) = \mathfrak{g}(f)$.

REMARK 1.8. Felix [3, (2.3)] has shown that (FPC1) does not hold for the 6-dimensional, 3-step nilpotent group of all upper triangular real 4×4 matrices with ones on the diagonal.

2. On hereditary properties of (FP1) and (FPC1).

Let G be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. We consider the direct product $G \times G$ and raise the following question: When does (FP1) (respectively (FPC1)) hold for $G \times G$? Before stating the result, recall that, if G_1 and G_2 are separable groups of type I, the mapping

$$\hat{G}_1 \times \hat{G}_2 \to (G_1 \times G_2)^\wedge, \quad (\pi, \rho) \to \pi \times \rho,$$

where $\pi \times \rho$ denotes the outer tensor product of π and ρ, identifies $\hat{G}_1 \times \hat{G}_2$ and $(G_1 \times G_2)^\wedge$.

THEOREM 2.1. *The following conditions are equivalent:*
(i) (FP1) *holds for* $G \times G$.
(ii) (FPC1) *holds for* $G \times G$.
(iii) *All the Kirillov orbits in* $\mathfrak{g}^*$ *are linear varieties.*

PROOF. (iii) ⇒ (i) follows from Corollary 1.4, and (i) ⇒ (ii) is trivial. The implication (ii) ⇒ (iii) will be proved by induction on the dimension of G. The case $\dim G = 1$ being obvious, let $\dim G > 1$ and suppose that (FPC1) holds for $G \times G$. We first show that the $\mathrm{Ad}^*(G)$-orbits in general position (as defined in [12, Proposition 2]) are linear varieties. Let Ω be such an orbit, and let $\pi \in \hat{G}$ be the corresponding representation. Suppose that Ω is not a linear variety. Then, by a result of Felix [4, Theorem], there exists $\rho \in \hat{G}$, $\rho \neq \pi$, such that the trivial 1-dimensional representation 1_G is weakly contained in the (inner) tensor product $\pi \otimes \bar{\rho}$, where $\bar{\rho}$ is the conjugate of ρ. Thus, denoting by H the diagonal subgroup $\{(x, x);\ x \in G\}$ of $G \times G$, we have

$$1_H \prec \pi \times \bar{\rho} \,|\, H,$$

and therefore

$$\pi \times \bar{\rho} \prec \mathrm{ind}_H^{G \times G} 1_H.$$

Now, it is easily deduced from [3, 2.1 Lemma] (compare also [14, Theorem 3.2]), that

$$\mathrm{ind}_H^{G \times G} 1_H \sim \{\sigma \times \bar{\sigma}; \sigma \in \hat{G}\}.$$

Hence there exists a sequence $(\sigma_n)_n$ in $\hat{G}$ with $\lim \sigma_n = \pi$ and $\lim \sigma_n = \rho$. It follows from this that if Ω' denotes the $\mathrm{Ad}^*(G)$-orbit corresponding to ρ, then $p(\Omega) = p(\Omega')$ for every $\mathrm{Ad}^*(G)$-invariant polynomial p on $\mathfrak{g}^*$. Since the orbit Ω in general position is the zero-set of certain invariant polynomials, we obtain $\Omega = \Omega'$, and hence $\pi = \rho$. This contradiction shows that the orbits in general position in $\mathfrak{g}^*$ are linear varieties.

Now, let Ω be an arbitrary $\mathrm{Ad}^*(G)$-orbit in $\mathfrak{g}^*$, π the corresponding representation and N the connected component of the kernel of π. Then $\pi \in (G/N)^\wedge$ and $\Omega \subseteqq \mathfrak{n}^\perp = (\mathfrak{g}/\mathfrak{n})^*$, where $\mathfrak{n}$ is the ideal of $\mathfrak{g}$ corresponding to N. Clearly, (FPC1) is inherited by $(G/N) \times (G/N)$. Thus, if $N \neq 0$, then Ω is a linear variety by induction hypothesis.

Suppose now that $N = 0$. Then the center $\mathfrak{z}$ of $\mathfrak{g}$ has dimension one, and since $\pi \mid \exp \mathfrak{z} \neq I$, $\Omega \mid \mathfrak{z} \neq 0$. Let Ω' be an orbit in general position with $\Omega' \mid \mathfrak{z} \neq 0$. The corresponding representation has a discrete kernel. Hence $\Omega' = f' + \mathfrak{z}^\perp$ for some $f \in \mathfrak{g}^*$, since Ω' is a linear variety. Let $f \in \Omega$. Then $f' \mid \mathfrak{z} = s(f \mid \mathfrak{z}) = sf \mid \mathfrak{z}$ for some $s \in \mathsf{R}$, $s \neq 0$. Thus, $sf \in \Omega'$. Therefore $\mathrm{Ad}^*(G)sf = s\mathrm{Ad}^*(G)f = s\Omega = \Omega'$, that is $\Omega = s^{-1}\Omega'$, and Ω is a linear variety. This completes the proof of the theorem.

REMARK 2.2. It follows from Theorem 2.1 that, in general, (FPC1) is not inherited by direct products. Indeed, let $\mathfrak{g}_4$ be the Lie algebra with basis $\{X_1, X_2, X_3, X_4\}$ and nontrivial commutators $[X_4, X_3] = X_2$, $[X_4, X_2] = X_1$. The corresponding nilpotent Lie group G_4 satisfies (FPC1) (see Corollary 1.7), but not all $\mathrm{Ad}^*(G_4)$-orbits in $\mathfrak{g}_4^*$ are linear varieties. Thus, (FPC1) does not hold for $G_4 \times G_4$. We do not know, whether (FP1) is inherited by direct products. Observe that, if this were the case, then Theorem 2.1 would give a characterization of nilpotent Lie groups with property (FP1). However, (FP1) and (FPC1) are inherited by direct products in a certain special situation:

PROPOSITION 2.3. *Let G_1 and G_2 be connected, simply connected nilpotent Lie groups, and assume that all the Kirillov orbits of G_2 are linear varieties. If G_1 satisfies* (FP1) (*respectively* (FPC1)), *then so does $G = G_1 \times G_2$.*

PROOF. We first prove the assertion in the case $G_2 = \mathsf{R}$. To this end, let H be a (connected) closed subgroup of G, $\tau \in \hat{H}$, and $\pi = \rho \cdot \alpha \in \hat{G}_1 \times \hat{\mathsf{R}}$ such that $\tau \prec \pi \mid H$. Let $\sigma = \bar{\alpha} \mid H \cdot \tau$, then $\sigma \prec \pi \mid H \cdot \bar{\alpha} \mid H = \rho \mid H$. Suppose we have shown that this implies $\rho \prec \mathrm{ind}_H^G \sigma$. Then

$$\pi = \rho \cdot \alpha \prec (\mathrm{ind}_H^G \sigma) \cdot \alpha = \mathrm{ind}_H^G(\sigma \cdot \alpha \mid H) = \mathrm{ind}_H^G \tau.$$

Since $\rho \mid \mathsf{R} = 1$, we are thus reduced to the case $\pi \in \hat{G}_1$. By Lemma 1.1, $H\mathsf{R}$ is a closed (connected) subgroup of G, and $H\mathsf{R}/\mathsf{R} = H/H \cap \mathsf{R}$. Moreover, $\tau \mid H \cap \mathsf{R} = 1$, so that $\dot{\tau} \in (H\mathsf{R}/\mathsf{R})^\wedge$ can be defined as usual, and $\dot{\tau} \prec \dot{\pi} \mid \dot{H}$. Thus, $\pi \prec \mathrm{ind}_H^G \tau$ (compare the proof of Theorem 1.3).

We now turn to the general case. H, τ being as above, let $\pi = \pi_1 \times \pi_2 \in \hat{G}_1 \times \hat{G}_2$ such that $\tau \prec \pi \mid H$. We can assume that π_2 has a discrete kernel (cf. the proof of Theorem 1.3). Thus $\pi_2 \sim \mathrm{ind}_Z^{G_2} \lambda$, where Z denotes the 1-dimensional center of G_2 and $\lambda \in \hat{Z}$. Set $N = G_1 \times Z$ and $M = H \cap N$, then $\tau \mid M \prec (\pi \mid N) \mid M$ and $\pi \mid N \sim \pi_1 \times \lambda \in \hat{N}$. Since $N = G_1 \times \mathsf{R}$, (FP1) (respectively (FPC1)) holds for N. H and N are regularly related since HN is closed (Lemma 1.1) and $HgN = HNg$, $g \in G$. Therefore, by [6, Theorem 5.3]

$$(\mathrm{ind}_H^G \tau) \mid N \sim \{\mathrm{ind}_{H^x \cap N}^N (\tau^x \mid H^x \cap N); x \in G\} \succ \mathrm{ind}_M^N \tau \mid M.$$

Hence there exist $\rho_n \in \operatorname{supp}(\operatorname{ind}_H^G \tau)$ such that $\lim \rho_n | N = \pi_1 \times \lambda$. Now, if

$$\rho_n = \rho_n' \times \rho_n'' \in \hat{G}_1 \times \hat{G}_2 \quad \text{and} \quad \rho_n'' | Z \sim \lambda_n \in \hat{Z},$$

then $\lim \rho_n' = \pi_1$ and $\lim \lambda_n = \lambda$. Let V be the set of all $\sigma \in \hat{G}_2$ that are square integrable modulo Z. Then V is open (see [16, Theorem 2]) and $\pi_2 \in V$. It follows that $\rho_n'' \in V$, that is $\rho_n'' \sim \operatorname{ind}_Z^{G_2} \lambda_n$, for large n and hence $\lim \rho_n = \pi_1 \times \operatorname{ind}_Z^{G_2} \lambda \sim \pi$, that is $\pi \prec \operatorname{ind}_H^G \tau$.

We now present an example showing that neither (FP1) nor (FPC1) is inherited by normal subgroups.

EXAMPLE 2.4. Let $\mathfrak{g}$ be the Lie algebra of dimension 7 with basis $X_1, X_2, \ldots, X_7$ and nontrivial commutators

$$[X_7, X_5] = X_1, \quad [X_6, X_2] = X_1, \quad [X_6, X_5] = X_3,$$

$$[X_5, X_4] = -X_2 \quad \text{and} \quad [X_4, X_3] = X_1.$$

Let G be the associated nilpotent Lie group. Consider the ideal $\mathfrak{n}$ in $\mathfrak{g}$ generated by $X_1, X_2, \ldots, X_6$. It is readily verified that $\mathfrak{n}$ is isomorphic to the Lie algebra consisting of all strictly upper triangular real 4×4 matrices. Hence, by [3, (2.3)], (FPC1) does not hold for the corresponding connected normal subgroup of G. On the other hand, a straightforward computation shows that, if $X_1^*, \ldots, X_7^*$ denotes the dual basis of $\mathfrak{g}^*$, then the $\mathrm{Ad}^*(G)$-orbit of $f = \sum_{i=1}^7 f_i X_i^*$ is given by

$$\begin{aligned}\Omega_f = \{ & f_1 X_1^* + (t_6 f_1 + f_2) X_2^* + (t_4 f_1 + f_3) X_3^* + (t_3 f_1 + t_5 f_2 + f_4) X_4^* + \\ & + (t_7 f_1 + t_4 f_2 + t_6 f_3 + t_6 t_4 f_1 + f_5) X_5^* + (t_5 t_4 f_1 + t_5 f_3 + t_2 f_1 + \\ & + f_6) X_6^* + (t_5 f_1 + f_7) X_7^*; t_2, \ldots, t_7 \in \mathrm{R}).\end{aligned}$$

It is easily seen that Ω_f is a linear variety for every $f \in \mathfrak{g}^*$. Therefore, by Corollary 1.4, G satisfies (FP1).

3. (FP1) for 2-step nilpotent groups.

As already mentioned (Remark 1.8), a 3-step nilpotent simply connected group need not satisfy (FPC1). On the other hand, (FP1) holds for 2-step nilpotent connected groups (Theorem 1.3). It has been conjectured in [11] that every 2-step nilpotent locally compact group has property (FP1). The main purpose of this section is to prove this conjecture in the case of a pro-Lie group. Recall that, by definition, a locally compact group G is a Lie group if the connected component G_0 is open in G and real analytic.

We have to introduce some more notation. For subsets A and B of the group G we denote by $[A, B]$ the set of all commutators $[a, b] = aba^{-1}b^{-1}$, $a \in A$, $b \in B$. If G is locally compact and N a closed normal subgroup of G, then G acts in the usual way on the set of unitary representations of N, in particular on $\hat{N}$. The G-orbit of the representation σ of N will be denoted by $G(\sigma)$. We will frequently use the fact that $(\mathrm{ind}_N^G \sigma) \mid N \sim G(\sigma)$.

LEMMA 3.1. *Let G be a 2-step nilpotent locally compact group and Z a closed central subgroup such that G/Z is abelian. For $\lambda \in \hat{Z}$, let*

$$Z_\lambda = \{z \in Z; \lambda(z) = 1\} \quad \text{and} \quad L_\lambda = \{x \in G; [x, G] \subseteq Z_\lambda\},$$

that is L_λ/Z_λ is the center of G/Z_λ.

(i) *If $\pi \in \hat{G}$ and $\pi \mid Z \sim \lambda \in \hat{Z}$, then $\pi \sim \mathrm{ind}_{L_\lambda}^G \alpha$ for some (G-invariant) character α of L_λ.*
(ii) *If N is an abelian closed subgroup of G containing Z and $\varphi \in \hat{N}$, then*

$$\overline{G(\varphi)} = \varphi \cdot (N/N \cap L_{\varphi \mid Z})^{\wedge}.$$

PROOF. The first part is a slight extension of [10, Lemma 2], and the proof requires only minor modifications. In fact, analyzing the proof of [10, Lemma 2] shows that if Z is a central subgroup of G such that G/Z is abelian and if $\pi \in \hat{G}$ such that $\pi \mid Z$ is faithful, then π is weakly equivalent to the representation induced by some character of the center of G. From this (i) follows easily be reducing to G/Z_λ.

To prove (ii), let $\lambda = \varphi \mid Z$ and choose $\pi \in \hat{G}$ such that $\pi \mid N \sim G(\varphi)$. By (i) $\pi \sim \mathrm{ind}_{L_\lambda}^G \alpha$ for some G-invariant character α of L_λ, and $\alpha \mid Z = \lambda$. Now

$$\pi \sim \mathrm{ind}_{L_\lambda}^G \pi \mid L_\lambda = \pi \otimes \mathrm{ind}_{L_\lambda}^G 1_{L_\lambda} \sim \pi \otimes (G/L_\lambda)^{\wedge}$$

and hence $\pi \mid N \sim \pi \mid N \otimes (G/L_\lambda)^{\wedge} \mid N$. The subgroup $(G/L_\lambda)^{\wedge} \mid N$ separates the points of $N/L_\lambda \cap N$ and is therefore dense in $(N/L_\lambda \cap N)^{\wedge}$. It follows that

$$\pi \mid N \sim \pi \mid N \otimes (N/L_\lambda \cap N)^{\wedge} \sim \mathrm{ind}_{L_\lambda \cap N}^N (\alpha \mid L_\lambda \cap N).$$

Clearly, $\varphi \mid N \cap L_\lambda = \alpha \mid N \cap L_\lambda$, so that

$$\mathrm{ind}_{L_\lambda \cap N}^N (\alpha \mid L_\lambda \cap N) \sim \varphi \cdot (N/L_\lambda \cap N)^{\wedge}.$$

This shows (ii).

LEMMA 3.2. *Let G be a 2-step nilpotent group, and assume that* (FP1) *holds for all pairs (A, B), where A is a quotient of G and B an abelian subgroup of A. Then G satisfies* (FP1).

PROOF. Let H be a closed subgroup of G, $\tau \in \hat{H}$ and $\pi \in \hat{G}$ such that $\pi \mid H \succ \tau$. Let Z be the center of G, and $\pi \mid Z \sim \lambda \in \hat{Z}$. By Lemma 3.1 there exist closed normal subgroups K and L of H and $\sigma \in (L/K)^{\wedge}$ with the following properties:

$$\tau \in (H/K)^{\wedge}, \quad L/K \text{ is the center of } H/K, \text{ and } \tau \sim \mathrm{ind}_L^H \sigma.$$

Now $\tau | K \cap Z = 1$ and

$$\tau | K \cap Z \prec \pi | K \cap Z \sim \lambda | K \cap Z,$$

so that $\lambda | K \cap Z = 1$ and $\pi \in (G/K \cap Z)^\wedge$. Moreover, $L/K \cap Z$ is abelian. Now, using the assumption that (FP1) holds for the pair $(G/K \cap Z, L/K \cap Z)$, we obtain as in the proof of Theorem 1.3

$$\pi \prec \mathrm{ind}_L^G \sigma = \mathrm{ind}_H^G(\mathrm{ind}_L^H \sigma) \sim \mathrm{ind}_H^G \tau.$$

LEMMA 3.3. *Let G be an abelian Lie group, and suppose that A and B are closed subgroups of G such that AB is dense in G. Let $\alpha \in \hat{A}$ and $\beta \in \hat{B}$, and let $(\gamma_\iota)_{\iota \in I}$ be a net in $\hat{G}$ such that $\gamma_\iota | A \to \alpha$ and $\gamma_\iota | B \to \beta$. If $(x_\iota)_{\iota \in I}$ is a net in G converging to $x \in B$, then*

$$\gamma_\iota(x_\iota) \to \beta(x).$$

PROOF. G contains an open subgroup H of the form $H = \mathsf{R}^p \times \mathsf{T}^q$, and $D = AB \cap H$ is dense in H. Now, $x_\iota = y_\iota x$ with $y_\iota \in H$ for $\iota \geqq \iota_0$, $\gamma_\iota(x) \to \beta(x)$ and

$$\gamma_\iota(y_\iota) = \gamma_\iota(x_\iota)\overline{\gamma_\iota(x)}.$$

It therefore suffices to show that if $x_\iota \in H$ and $x_\iota \to e$ then $\gamma_\iota(x_\iota) \to 1$. Let

$$\varphi\colon \mathsf{R}^r = \mathsf{R}^p \times \mathsf{R}^q \to \mathsf{R}^p \times \mathsf{T}^q$$

denote the canonical covering homomorphism, and set $\lambda_\iota = \gamma_\iota | H \circ \varphi \in (\mathsf{R}^r)^\wedge$.

$\varphi^{-1}(D)$ is a dense subgroup of R^r, hence contains a vector space basis $\{v_1, \dots, v_r\}$ of R^r. If $\varphi(v_j) = a_j b_j$, $a_j \in A$, $b_j \in B$, $1 \leqq j \leqq r$, then

$$\lambda_\iota(v_j) = \gamma_\iota(\varphi(v_j)) = \gamma_\iota(a_j)\gamma_\iota(b_j) \to \alpha(a_j)\beta(b_j),$$

that is $(\lambda_\iota)_{\iota \in I}$ converges on a basis of R^r. It follows that $\lambda_\iota \to \lambda \in (\mathsf{R}^r)^\wedge$ and $\lambda = \delta \circ \varphi$ for some $\delta \in \hat{H}$. Thus $\gamma_\iota | H \to \delta$, and since the mapping

$$H \times \hat{H} \to \mathsf{C}, \quad (y, \chi) \to \chi(y)$$

is continuous, we obtain $\gamma_\iota(x_\iota) \to \delta(e) = 1$.

We will use several times that if G is a projective limit of groups $G_\iota = G/K_\iota$, $\iota \in I$, and all G_ι satisfy (FP1) (or (FP1) with respect to normal subgroups), then so does G. This can be seen by standard arguments (compare the proof of [11, Proposition 1.3]), the crucial fact being that given $\pi \in \hat{G}$, there exists $\iota \in I$ such that $\pi(K_\iota) = I$, that is $\pi \in \hat{G}_\iota$. The analogue holds for (FP2). In fact, this is proved very similarly by applying the following result of [8]. Let H be a closed subgroup of G, $\tau \in \hat{H}$ and $\pi \in \hat{G}$ such that $\pi \prec \mathrm{ind}_H^G \tau$. If K is a compact normal subgroup of G such that $\pi(K) = I$, then $\tau(H \cap K) = I$, and if $\dot{\pi}$ and $\dot{\tau}$ denote the corresponding representations of G/K and $HK/K = H/H \cap K$, then $\dot{\pi} \prec \mathrm{ind}_{HK/K}^{G/K} \dot{\tau}$.

THEOREM 3.4. *Every 2-step nilpotent pro-Lie group (in particular every compactly generated 2-step nilpotent group) has the Frobenius property* (FP1).

PROOF. By [9, 9. Theorem] every compactly generated nilpotent group is pro-Lie. Thus, by the above remark, we are reduced to the case of a Lie group G. According to Lemma 3.2, it suffices to show that if H is an abelian closed subgroup of G, $\tau \in \hat{H}$ and $\pi \in \hat{G}$ such that $\tau \prec \pi \mid H$, then $\pi \prec \operatorname{ind}_H^G \tau$. To this end let Z denote the center of G and consider the abelian normal subgroup $N = \overline{HZ}$ of G. Once we have proved $\pi \mid N \prec G(\operatorname{ind}_H^N \tau)$, it follows that

$$\pi \prec \operatorname{ind}_N^G \pi \mid N \prec \{\operatorname{ind}_N^G(\operatorname{ind}_H^N \tau)^x; x \in G\} = \{\operatorname{ind}_H^G \tau\}.$$

By Lemma 3.1 there exist a closed subgroup L of G with $Z \subseteqq L \subseteqq N$ and a G-invariant character λ on L such that $\pi \mid N \sim \operatorname{ind}_L^N \lambda$. In fact, if $\pi \mid Z \sim \mu \in \hat{Z}$, then $L = \{x \in N; \mu \mid [x, G] = 1\}$. Let now

$$\Gamma = \{\gamma \in \hat{N}; \gamma \mid H = \tau\} = \operatorname{supp}(\operatorname{ind}_H^N \tau).$$

For every $\gamma \in \Gamma$, $\gamma \mid H \cap L = \tau \mid H \cap L = \lambda \mid H \cap L$, and

$$\Gamma \mid L = \gamma \mid L \cdot (N/H)^\wedge \mid L \sim \gamma \mid L \cdot (L/L \cap H)^\wedge.$$

Therefore

$$\Gamma \mid L \sim \lambda \cdot (L/L \cap H)^\wedge.$$

Choose a net $(\gamma_\iota)_{\iota \in I} \subseteqq \Gamma$ such that $\gamma_\iota \mid L \to \lambda$. For $\gamma \in \hat{N}$,

$$G(\gamma) \sim \gamma \cdot (N/N_\gamma)^\wedge,$$

where

$$N_\gamma = \{x \in N; \gamma \mid [x, G] = 1\}$$

(Lemma 3.1). Let now $\mathfrak{X}(N)$ denote the set of all closed subgroups of N endowed with the compact-open topology (see [6, p. 427]). As $\mathfrak{X}(N)$ is compact, we can assume that $N_{\gamma_\iota} \to N_0$ for some $N_0 \in \mathfrak{X}(N)$. Using the description of N_γ and L and $\gamma_\iota \mid Z \to \lambda \mid Z$, it is easily verified that $N_0 \subseteqq L$. Indeed, if $x_\iota \in N_{\gamma_\iota}$ and $x_\iota \to x \in N_0$, then for every $y \in G$,

$$1 = \gamma_\iota([x_\iota, y]) \to \lambda([x, y]),$$

that is $x \in L$. We claim that $(N_{\gamma_\iota}, \gamma_\iota | N_{\gamma_\iota}) \to (N_0, \lambda \mid N_0)$ in Fell's subgroup representation topology [6]. We have to show that if $x_\iota \in N_{\gamma_\iota}$ and $x_\iota \to x$, then $\gamma_\iota(x_\iota) \to \lambda(x)$. But this follows by applying Lemma 3.3 to N, the closed subgroups H and $N_0 \supseteqq Z$, the characters $\tau \in \hat{H}$ and $\lambda \mid N_0 \in \hat{N}_0$ and the net $(\gamma_\iota)_{\iota \in I} \subseteqq \hat{N}$. Inducing being continuous [6], we obtain

$$\operatorname{ind}_{N_{\gamma_\iota}}^N(\gamma_\iota \mid N_{\gamma_\iota}) \to \operatorname{ind}_{N_0}^N \lambda \mid N_0.$$

Finally, recall that

$$\pi \mid N \sim \operatorname{ind}_L^N \lambda \prec \operatorname{ind}_{N_0}^N \lambda \mid N_0 \text{ and } \operatorname{ind}_{N_{\gamma_t}}^N(\gamma_t \mid N_{\gamma_t}) \sim \gamma_t(N/N_{\gamma_t})^\wedge \sim G(\gamma_t).$$

Thus we have proved $\pi \mid N \prec G(\operatorname{ind}_H^N \tau)$.

It is worth mentioning that Theorem 3.4 could be shown for arbitrary 2-step nilpotent groups provided that some substitute of Lemma 3.3 is available in the non-Lie group case.

Moscovici [17, Theorem 2] proved that the Frobenius property (FP) holds for pairs (G, N) where G is a simply connected nilpotent Lie group and N a closed connected normal subgroup. We conclude the paper by showing that the results and methods of [13] yield the following generalization.

PROPOSITION 3.5. *Let G be a nilpotent pro-Lie group and N a closed normal subgroup of G. Then* (FP) *holds for* (G, N).

PROOF. By the remark preceding Theorem 3.4 we can assume that G is a Lie group. Using the terminology of [13], it suffices to recognize that if $\pi \in \hat{G}$ and $\tau \in \hat{N}$, then $\ker \pi \mid N$ and $\ker G(\tau)$ are G-maximal ideals in $C^*(N)$. Indeed, for $\tau \in \operatorname{supp} \pi \mid N$, we then obtain $\pi \mid N \sim G(\tau)$ and therefore

$$\pi \prec \operatorname{ind}_N^G \pi \mid N \sim \operatorname{ind}_N^G \tau.$$

Conversely, if $\pi \prec \operatorname{ind}_N^G \tau$, then

$$\pi \mid N \prec (\operatorname{ind}_N^G \tau) \mid N \sim G(\tau),$$

that is $\pi \mid N \sim G(\tau)$.

The connected component N_0 of N is normal in G. Choose a sequence N_j of closed normal subgroups of G, $1 \leqq j \leqq r$, such that $N_0 \subseteqq \ldots \subseteqq N_r = N$ and N_j/N_{j-1} is central in G/N_{j-1}. Now, for any closed normal subgroup M of G contained in N, $\ker \pi \mid M$ and $\ker G(\tau \mid M)$ are G-prime ideals in $C^*(M)$ [13, 1. Lemma]. On the other hand the first part of the proof of [13, Theorem] shows that every G-prime ideal in $C^*(N_0)$ is G-maximal. Finally, [13, 6. Lemma] and induction on the sequence N_j give that every G-prime ideal in $C^*(N)$ is G-maximal.

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