MATH. SCAND 63 (1988), 297–304

THE MAXIMUM UNITARY RANK OF SOME C*-ALGEBRAS

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Abstract.

In this paper we show that not all elements in the unit ball of certain C*-algebras (infinite dimensional separable abelian C*-algebras, infinite dimensional AF-algebras, irrational rotation C*-algebras, and the reduced group C*-algebra of the free group of n generators) can be written as the mean of two unitary elements in the algebra. Thus the maximum unitary rank of these algebras is either 3 or ∞.

1. Introduction

Over the past three years, convex combinations of unitary operators in a C*-algebra have been studied in several articles [1, 2, 4, 9]. Let A be a unital C*-algebra, let $(A)_1$ be $\{a \in A: \ \|a\| \leqslant 1\}$, the unit ball in A. Let $U(A)$ denote the group of unitary elements in A, and let A_{inv} denote the group of invertible elements in A. Kadison and Pedersen [2] define the *unitary rank* $u(a)$ of a in A as the least integer n for which a has a representation as a convex combination $\alpha_1 u_1 + \ldots + \alpha_n u_n$ ($\alpha_j \geqq 0$, $\alpha_1 + \ldots + \alpha_n = 1$) with u_j in $U(A)$. (If a has no such representation, e.g. if $\|a\| > 1$, set $u(a) = \infty$). They show that $u(a) \leqq n$ if $\|a\| < 1 - 2n^{-1}$, and that a is the *mean* of n unitaries in A if $u(a) \leqq n$. In [9] it is proved that $u(a) \leqq n$ if $a \in (A)_1$ and $\mathrm{dist}(a, A_{\mathrm{inv}}) < 1 - 2n^{-1}$; and $\mathrm{dist}(a, A_{\mathrm{inv}}) \leqq 1 - 2n^{-1}$ if $u(a) \leqq n$. Moreover, if $a \in (A)_1$ and $\mathrm{dist}(a, A_{\mathrm{inv}}) = 1 - 2n^{-1}$, where n is an integer, then $u(a) \leqq n + 1$ and for each $\varepsilon > 0$ there is a convex combination $\alpha_1 u_1 + \ldots + \alpha_{n+1} u_{n+1}$ equalling a with u_j in $U(A)$ and $\alpha_{n+1} < \varepsilon$. In particular, if the invertibles are dense in A, then $u(a) \leqq 3$ for all a in $(A)_1$. In this case, the *maximum unitary rank*, $u(A)$ $(= \sup\{u(a): a \in (A)_1\})$, of A is 2 or 3. If the invertibles in A are not dense, then, from [9], there is an

* Supported by NSF (USA).

** Supported by the Danish Natural Science Research Council and NSF (USA).

Received May 13, 1987; in revised form September 21, 1987.

element a in A with $\|a\| = \operatorname{dist}(a, A_{\text{inv}}) = 1$, and $u(a) = \infty$. Thus, for all C*-algebras A, $u(A)$ is 2, 3 or ∞. Kadison and Pedersen remark in [2] that $u(a) \leq 2$ for all a in $(A)_1$ when A is a finite von Neumann algebra, and they show that $u(A) = 3$ when $A = C(\mathsf{N} \cup \{\infty\})$. However, the maximum unitary rank for specific C*-algebras of interest has not been computed in the papers mentioned above, and techniques for doing so have been absent.

In this paper we show that the maximum unitary rank of A is greater than or equal to 3 when A is an infinite dimensional separable abelian C*-algebra, an infinite dimensional AF-algebra, an irrational rotation C*-algebra or the reduced group C*-algebra of the free group of n generators.

We show that any pair, u and v, of unitary operators on a Hilbert space $\mathscr{H}$, for which $u + v$ is positive, commute (Proposition 3). We conclude that if a is a normal operator in a C*-algebra A, and $a = u + v$ where u and v are in $\mathrm{U}(A)$, then u and v commute with $|a|$ (Corollary 4). Hence, if C is a maximal abelian subalgebra (= masa) in A, $a \in A$, a is not a sum of two unitaries in C, and the commutant of $|a|$ in A, $\{|a|\}' \cap A$, is C, then a is not a sum of two unitaries in A.

We know of no examples of separable infinite dimensional C*-algebras A where $u(A) = 2$. We give below an example of a (non-separable) C*-algebra A which is not a von Neumann algebra and for which $u(A) = 2$. Let $C_0(\mathsf{N})$ be the set of all sequences vanishing at ∞, let $B(\mathsf{N})$ ($\simeq C(\beta\mathsf{N})$) be the set of all bounded sequences, the multiplier algebra of $C_0(\mathsf{N})$, and let $Q(\mathsf{N})$ be the quotient $B(\mathsf{N})/C_0(\mathsf{N})$, $Q(\mathsf{N}) \simeq C(\beta\mathsf{N} - \mathsf{N})$. We have a short exact sequence

$$(0) \to C_0(\mathsf{N}) \to B(\mathsf{N}) \to Q(\mathsf{N}) \to (0).$$

It is easy to check that $u(Q(\mathsf{N})) = 2$ and that $Q(\mathsf{N})$ is not a von Neumann algebra.

We would like to thank Gert Pedersen for suggesting the present much simpler proof of Proposition 3.

2. Abelian C*-algebras

In this section, we show that the maximum unitary rank of a separable, infinite dimensional abelian C*-algebra is at least 3. Our argument is essentially taken from Kadison and Pedersen [2].

LEMMA 1. *Let X be a compact Hausdorff space and suppose that $\{x_n\}_1^\infty$ is a sequence of distinct points converging to x in X. If f is a continuous function on X such that $f(x) = 0$ and*

$$f(x_n) = \begin{cases} r_n & n \text{ odd} \\ ir_n & n \text{ even} \end{cases}$$

for each n, where $\{r_n\}_1^\infty$ is a sequence of non-zero real numbers converging to zero, then f cannot be written as the mean of two unitaries in $C(X)$.

PROOF. Let us assume the contrary, that is $f = \frac{1}{2}(u + v)$, where u and v are continuous functions on X taking values in the unit circle $\mathsf{T} \subset \mathsf{C}$. A simple geometric argument then shows that for fixed odd n, $\operatorname{Re}(u(x_n)) = r_n$ $(= \operatorname{Re}(v(x_n))$. From the continuity of u, we see that $\operatorname{Re}(u(x)) = 0$. An analogous argument using the even integers then shows $\operatorname{Im}(u(x)) = 0$. This is a contradiction to the assumption that u takes values in T.

For an abelian C*-algebra $C(X)$, the question of whether the invertible elements are dense may be answered in terms of the dimension of X, denoted $\dim(X)$ (see for example Pears [5] for a definition). We refer the reader to Rieffel [8] for a complete discussion. We may summarize the situation for the maximum unitary rank of separable abelian C*-algebras as follows.

THEOREM 2. *Let X be a second countable, compact Hausdorff space. Then*
(i) $u(C(X)) = 2$ *if and only if X is finite.*
(ii) $u(C(X)) = 3$ *if and only if X is infinite and* $\dim(X) \leqq 1$.
(iii) $u(C(X)) = \infty$ *if and only if* $\dim(X) > 1$.

PROOF. First of all, if X is finite it is clear that $u(C(X)) = 2$. If X is infinite, then there is a sequence, $\{x_n\}_1^\infty$, in X converging to x in X with all x_n's distinct. (Here we use the fact that X is second countable). By Tietze's extension theorem we may find a function f in $(C(X))_1$ satisfying the conditions of Lemma 1, and so $u(C(X)) \geqq 3$. The distinction between (ii) and (iii) now follows from the fact that the invertibles in $C(X)$ are dense if and only if $\dim(X) \leqq 1$ [5, 8], and from the results of [9] mentioned in the introduction.

3. Reducing to the abelian case

In the following proposition we show that the C*-algebra generated by two unitary operators with positive sum is abelian.

PROPOSITION 3. *Let A be a C*-algebra, let h be a positive element in A of norm 2 or less, and let u and v be unitary elements in A such that $u + v = h$. Then u, v and h all commute.*

PROOF. Write $u = a + ic$ and $v = b - ic$ with a, b and c selfadjoint. ($\operatorname{Im}(u) = -\operatorname{Im}(v)$ because $u + v$ is self-adjoint). Since u and v are unitary, a and b commute with c and $a^2 = b^2$ $(= 1 - c^2)$. Thus

$$ah = a(a + b) = a^2 + ab = b^2 + ab = (a + b)b = hb,$$

and similary, $bh = ha$. Hence $ah^2 = hbh = h^2a$, so a commutes with h^2 and consequently with $h(=(h^2)^{\frac{1}{2}})$. Now, h commutes with a, b $(= h - a)$ and c (because $h = a + b$), so h commutes with u and v.

We remark that the condition in Proposition 3 that h be positive cannot be relaxed to, say, h be self-adjoint as the following example shows:

$$(h =)\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

COROLLARY 4. *Let A be a C*-algebra. Let a be a normal element in A of norm 2 or less and suppose that u and v are unitary elements in A such that $u + v = a$. Then u and v commute with $|a|$.*

PROOF. Upon choosing a faithful representation of A on a Hilbert space $\mathscr{H}$ we may suppose that $A \subset \mathscr{B}(\mathscr{H})$, the algebra of all bounded operators on $\mathscr{H}$. Since a is normal there is a unitary operator w on $\mathscr{H}$ such that $a = w|a|$, and w and $|a|$ commute. Now $|a| = w^*u + w^*v$, so, by Proposition 3, w, w^*u and w^*v commute with $|a|$. Hence $|a|$ commutes with u and v.

4. Main results

In this section we show that three is a lower bound for the maximum unitary rank of infinite dimensional AF-algebras, irrational rotation C*-algebras and the reduced C*-algebra of the free group of n generations. The basic strategy, as outlined in the introduction, is to reduce the computations to a masa, using Section 3, and then use the abelian results of Section 2. The specific techniques, however, vary in the three cases.

While we concentrate on these specific examples, we will state some of the technical results in greater generality (Lemma 5 and Lemma 7).

LEMMA 5. *Suppose that A is a separable C*-algebra containing a* masa, C, *whose spectrum, X, is totally disconnected and infinite. Then $u(A) \geqq 3$.*

REMARK. If A is a C*-algebra containing a masa whose spectrum is finite, then A is finite dimensional (see [3, Exercise 4.6.12]).

PROOF OF LEMMA 5. We will identify C and $C(X)$. Since X is infinite, we may find x in X which is not isolated. Using the fact that X is second countable and that x is not isolated, we may find neighborhoods U_n, $n = 0, 1, \dots$, of x such that:

(i) each U_n is closed and open,
(ii) $U_{n+1} \subset U_n$ and $U_{n+1} \neq U_n$,
(iii) $\cap_n U_n = \{x\}$,
(iv) $U_0 = X$.

Then letting $X_n = U_n - U_{n+1}$, for $n = 1, 2, \dots$, we obtain a sequence $\{X_n\}_1^\infty$ of pairwise disjoint subsets of X such that each is non-empty, closed and open and their union is $X - \{x\}$. For each n, we may find an embedding $f_n\colon X_n \to (1, 2)$. (For this, note first that X_n is totally disconnected and second countable. Let

$\{V_k\}_1^\infty$ be a countable basis of clopen sets for the topology on X_n. Then the function $1 + \sum 2^{-k} 1_{V_k}$ generates $C(X_n)$). We define f in $C(X)$ as follows: for y in X_n, set

$$f(y) = \begin{cases} 2^{-n} f_n(y) & n \text{ odd} \\ i\, 2^{-n} f_n(y) & n \text{ even} \end{cases},$$

and let $f(x) = 0$.

It is easy to verify that $f \in (C(X))_1$, that $|f|$ separates the points of X and that f satifies the conditions of Lemma 1 (with x_n chosen arbitrarily from X_n).

We claim that $f \in C \subset A$ cannot be written as the mean of two unitaries in A. Suppose the contrary; that is $f = \frac{1}{2}(u + v)$ with u and v in $\mathrm{U}(A)$. Then by Corollary 4, u and v both commute with $|f|$. Since $|f|$ separates the points of X, it generates C (as a C*-algebra), and so u and v commute with all of C. Since C is maximal abelian, u and v must both lie in C, which is impossible by Lemma 1.

THEOREM 6. *If A is an infinite dimensional* AF-*algebra, then the maximum unitary rank of A is* 3.

PROOF. It is well-known that the invertibles of an AF-algebra A are dense and so $u(A) \leqq 3$. The reverse inequality follows from the fact that every AF-algebra has a "diagonal" subalgebra satisfying the hypothesis of Lemma 5 (see Stratila and Voiculescu [10]).

We now turn our attention to certain transformation group C*-algebras. We suppose we have an action of the group of integers, Z, on a compact metrizable space, X, which is induced by the homeomorphism, φ, on X. We consider the crossed product or transformation group C*-algebra $C(X) \times_\varphi \mathsf{Z}$. This C*-algebra is generated by $C(X)$ and a unitary operator u such that $ufu^* = f \circ \varphi^{-1}$ for all f in $C(X)$. The reader is referred to Pedersen [6] for a complete treatment.

First we wish to show that, under certain conditions, although X may have large dimension so that $C(X)$ is not singly generated, there are single elements of $C(X)$ whose commutant in $C(X) \times_\varphi \mathsf{Z}$ is just $C(X)$.

LEMMA 7. *Suppose $h \in C(X)$ is such that, for each integer $n \neq 0$, the set*

$$X_n = \{x \in X\colon h(x) \neq h(\varphi^{-n}(x))\}$$

is dense in X. Then $\{h\}' \cap C(X) \times_\varphi \mathsf{Z} = C(X)$.

PROOF. For each integer n, let $E_n\colon C(X) \times_\varphi \mathsf{Z} \to C(X)$ be defined by

$$E_n(a) = \int_T \hat{\varphi}_z(au^{-n})\, dz = \int_T \hat{\varphi}_z(a) z^{-n} u^{-n}\, dz, \qquad a \in C(X) \times_\varphi \mathsf{Z},$$

where $\hat{\varphi}$ is the dual action of the circle group T on $C(X) \times_\varphi \mathsf{Z}$ (see [6]). Then $a \in C(X)$ if and only if $E_n(a) = 0$ for all $n \neq 0$.

Let us suppose that a in $C(X) \times_{\varphi} \mathbf{Z}$ commutes with h. We wish to show that $E_n(a) = 0$ for all $n \neq 0$. It is easily checked that $E_n(ah) = E_n(a) \cdot u^n h u^{-n} = E_n(a) \cdot h \circ \varphi^{-n}$, and that $E_n(ha) = h \cdot E_n(a)$. Now if $E_n(a) \in C(X)$ is not zero, we may find $x \in X_n$ with $E_n(a)(x) \neq 0$. But this implies $h(x) = h(\varphi^{-n}(x))$, which is a contradiction to x being in X_n.

Although Lemma 7 may be used to deal with various transformation group C*-algebras, we will restrict our attention to the irrational rotation C*-algebras. That is, we fix an irrational number θ and let X be $\mathbf{T}$. The homeomorphism φ is rotation through angle $2\pi\theta$. In this case, the crossed product $C(T) \times_{\varphi} \mathbf{Z}$ is denoted A_θ.

THEOREM 8. *For any irrational number θ, the irrational rotation* C**-algebra A_θ has maximum unitary rank* 3 *or* ∞.

REMARK. Whether $u(A_\theta)$ is 3 or ∞ depends on whether the invertibles in A_θ are dense or not. The best partial result on that question is found in Riedel [7], where it is determined that the invertibles in A_θ are dense when θ takes certain diophantine values.

PROOF OF THEOREM 8. Define f in $C(T)$ by

$$f(z) = \begin{cases} \mathrm{Re}(z) & \text{if } \mathrm{Re}(z) \geqq 0 \\ -i\,\mathrm{Re}(z) & \text{if } \mathrm{Re}(z) \leqq 0 \end{cases}$$

It is simple to verify that $|f|$ satisfies the hypothesis of Lemma 7. So if f can be written as the mean of two unitaries in A_θ, then these unitaries must commute with $|f|$ by Corollary 4, and then, by Lemma 7, they lie in $C(T)$. Finally, upon choosing a suitable sequence $\{x_n\}_1^\infty$ in $\mathbf{T}$ converging to $x = i$, we may apply Lemma 1 to assert $u(f) \geqq 3$.

Finally, we turn our attention to $C_r^*(F_n)$, the reduced C*-algebra of the free group of n generators ($n \geqq 2$) (see 7.2 of Pedersen [6]). We denote the generators by $p_1, \ldots, p_n$ and for each g in F_n we let $u(g)$ denote the corresponding unitary operator in $C_r^*(F_n)$.

THEOREM 9. *For any $n \geqq 2$, the maximum unitary rank of $C_r^*(F_n)$ is either* 3 *or* ∞.

PROOF. Here we will make use of the masa $C^*(u(p_1))$ in $C_r^*(F_n)$. Letting f be as in the proof of Theorem 8, we define $a = f(u(p_1))$. Note that

$$|a|^2 = 4^{-1}(2 + u(p_1)^2 + u(p_1)^{-2}).$$

First we wish to show that if $b \in C_r^*(F_n)$ and b commutes with $|a|$, then $b \in C^*(u(p_1))$.

As in Kadison and Ringrose [3, Section 6.7], we may write

$$b = \sum_{g \in F_n} \alpha(g)u(g)$$

where $\alpha: F_n \to \mathbf{C}$ is square-summable. It suffices to show that $\alpha(g) = 0$, except for $g \in \{p_1{}^k\colon k \in \mathbf{Z}\}$. Suppose the contrary: that is, there is $g_0 \notin \{p_1{}^k\colon k \in \mathbf{Z}\}$ with $\alpha(g_0) \neq 0$. Set $d_m = u(p_1)^m + u(p_1)^{-m}$. If b commutes with $|a|$, then b commutes with $|a|^2$ and consequently with d_2. Moreover, if b commutes with d_m, then b commutes with $d_m{}^2$ $(= 2 + u(p_1)^{2m} + u(p_1)^{-2m})$, and so b commutes with d_{2m}. Thus b commutes with all d_m where m is of the form 2^s for some natural number s. A brief computation shows that $bd_m = d_m b$ is equivalent to

$$\alpha(p_1{}^{-m}g) + \alpha(p_1{}^m g) = \alpha(gp_1{}^{-m}) + \alpha(gp_1{}^m) \qquad (g \in F_n).$$

Letting $g = p_1{}^m g_0$, we see that

$$\alpha(g_0) = -\alpha(p_1{}^{2m}g_0) + \alpha(p_1{}^m g_0 p_1{}^{-m}) + \alpha(p_1{}^m g_0 p_1{}^m)$$

for all $m = 2^s$. Thus $|\alpha(q_s)|$ is greater than or equal to $3^{-1}\,|\alpha(g_0)|$ where q_s is one of $p_1{}^{2m}g_0$, $p_1{}^m g_0 p_1{}^{-m}$, $p_1{}^m g_0 p_1{}^m$. Using the fact that $g_0 \notin \{p_1{}^k\colon k \in \mathbf{Z}\}$ and standard facts about the free group, we obtain

$$p_1{}^j g_0 p_1{}^k = g_0 \quad \text{implies } j = k = 0.$$

We conclude that $q_s \neq q_t$ when $s \neq t$, $s, t \in \mathbf{N}$, which is a contradiction to α being square-summable.

To show that a may not be written as the mean of two unitary elements in $C_r^*(F_n)$ now follows the same routine as in the proof of Theorem 8.

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