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COMPARISONS OF IDEAL STRUCTURES IN ALGEBRAS OF ANALYTIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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Abstract:

For a commutative unital Banach algebra, an ideal theory is a characterization of its closed ideals and the corresponding quotient algebras. We investigate to what extent an ideal theory of an algebra like $A(W) = H^\infty(W) \cap C(\bar{W})$ for a domain $W \subset \mathbf{C}^n$ carries over to a closed subalgebra consisting of functions holomorphic on a larger domain than W.

§0. Introduction.

The present paper continues my theme from [Hed 2] to $\mathbf{C}^n$, $n > 1$.

For a domain (nonempty open connected set) W in $\mathbf{C}^n$, let $H^\infty(W)$ be the Banach algebra of bounded analytic functions on W, endowed with the uniform norm on W, and put $A(W) = C(\bar{W}) \cap H^\infty(W)$. It is convenient to consider these algebras only when W is the natural domain of definition for them. For instance, W should be pseudoconvex, if we want to avoid the Hartogs phenomenon. Very little appears to be known about the ideal structure and in particular, about the structure of closed ideals, in such basic algebras as $A(W)$ and $H^\infty(W)$ if $n > 1$, even when W is the ball or the polydisc. Yngve Domar [Dom] has described the closed primary ideals (that is, closed ideals contained is only one maximal ideal) at interior points for a certain class of algebras, namely the rationally generated ones. And at least for well-behaved bounded domains $W \subset \mathbf{C}^n$, $A(W)$ is rationally generated. More recently, Joaquim Bruna and Joaquin Ortega [BrO] studied the closed finitely generated ideals in $A(W)$ and $A_k(W) \equiv C^k(\bar{W})$, $k \geqq 1$, for bounded, strictly pseudoconvex domains W with C^∞-boundary.

For a commutative unital Banach algebra, let an *ideal theory* mean a characterization of its closed ideals and the corresponding quotient algebras. The object of this paper is to investigate to what extent an ideal theory of an algebra like

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$A(W)$ is related to that of a closed subalgebra consisting of functions holomorphic on a larger domain than W. We obtain results analogous to those in [Hed2], but our methods are partially different.

The reader interested in these questions should also consult [Hed1,3].

§1. Basic concepts.

All Banach algebras are assumed complex and commutative, but not necessarily unital. Recall that a uniform algebra is a Banach algebra with a norm equivalent to the supremum norm of the Gelfand transform. The bilinear form linking any Banach space A with its dial Banach space A^* will be denoted by $\langle \cdot, \cdot \rangle$.

For any Banach algebra B, we write $\mathscr{M}(B)$ for its Gelfand (or carrier) space, equipped with the Gelfand topology. If B has a unit, this is its maximal ideal space. The hull of a B-ideal I is the set

$$h(I, B) = \{m \in \mathscr{M}(B): \ x(m) = 0 \text{ for all } x \in I\},$$

which is a closed subset of $\mathscr{M}(B)$. It is well known that if I is closed, one can identify $h(I, B)$ and $\mathscr{M}(B/I)$ (see [Sto, p. 27]). Let B have a unit. Then for any element $x \in B$,

$$\sigma(x, B) = \{\lambda \in \mathbf{C}: \ \lambda - x \text{ is not invertible}\}$$

is its spectrum, and for finitely many $y_1, \ldots, y_n \in B$, their joint spectrum $\sigma(y, B) = \sigma(y_1, \ldots, y_n; B)$ (where $y = (y_1, \ldots, y_n) \in B^n$) is the set of all $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n$ such that the ideal generated by the elements $\lambda_1 - y_1, \ldots, \lambda_n - y_n$ is proper in B. Here, complex numbers are identified with the corresponding multiples of the unit. Let $\hat{y}$: $\mathscr{M}(B) \to \mathbf{C}^n$ be the mapping $\hat{y}(m) = (\hat{y}(m), \ldots, \hat{y}_n(m))$, $m \in \mathscr{M}(B)$. It is well known and easy to check that if I is a closed B-ideal,

$$\hat{y}(h(I, B)) = \sigma(y + I, B/I) \equiv \sigma(y_1 + I, \ldots, y_n + I; B/I).$$

This holds in particular for $I = \{0\}$, making $\hat{y}(\mathscr{M}(B)) = \sigma(y, B)$.

A subalgebra A of B is said to be a Banach subalgebra if it is equipped with a norm stronger than that of B and which makes A a Banach algebra. By the closed graph theorem, a subalgebra can have (within equivalence) at most one Banach subalgebra norm.

Let K be a compact subset of $\mathbf{C}^n$. The polynomially convex hull of K is the set

$$\hat{K} = \{\zeta \in \mathbf{C}^n : |p(\zeta)| \leqq \sup_K |p| \text{ for all } p \in \mathscr{P}(\mathbf{C}^n)\},$$

where $\mathscr{P}(\mathbf{C}^n)$ denotes the set of all complex-valued (holomorphic) polynomials in $\zeta = (\zeta_1, \ldots, \zeta_n)$. K is polynomially convex if $K = \hat{K}$.

Let z_j be the coordinate function $z_j(\zeta) = \zeta_j$ for $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbf{C}^n$, $1 \leqq j \leqq n$, and let $z = (z_1, \ldots, z_n)$ be the identity mapping $\mathbf{C}^n \to \mathbf{C}^n$. It will be clear from the context when we use the symbols z and z_j to denote functions or points in $\mathbf{C}^n$ and $\mathbf{C}$, respectively.

W will for the time being be an arbitrary bounded domain in $\mathbf{C}^n$.

DEFINITION 1.1. B is an *acceptable* algebra on W if
(a) B is a Banach subalgebra of $H^\infty(W)$ containing the unit 1,
(b) $z_j \in B$ for $j = 1, \ldots, n$, and
(c) $\sigma(z, B) = \bar{W}$.

REMARKS 1.2. (a) Observe that because of the Hartogs phenomenon, there are bounded domains $W \subset \mathbf{C}^n$ which have no acceptable algebras on them if $n > 1$. There are even pseudoconvex W that carry no acceptable algebras. See [Ber] for a survey on natural domains of definition for analytic functions.

(b) Since point evaluations in W define complex homomorphisms, every acceptable algebra on W is semisimple.

(c) An acceptable algebra B on W contains $\mathcal{O}(\bar{W})$ because $\sigma(z, B) = \bar{W}$, by the holomorphic functional calculus. In a sense, the algebra B is "sandwiched" between $\mathcal{O}(\bar{W})$ and $H^\infty(W)$.

For ease of notation, we shall write $Z(I, B)$ instead of $\hat{z}(h(I, B))$ for B-ideals I.

§2. A Lemma.

The following lemma, which will prove useful later on, is probably known. The author has however been unable to find a suitable reference.

LEMMA 2.1. *Let B be a Banach algebra with unit* 1, *and let A be a closed subalgebra of B containing the unit. Pick n elements $a_1, \ldots, a_n \in A^n$. If the joint spectrum $\sigma(a, B)$ is polynomially convex, $\sigma(a, A) = \sigma(a, B)$.*

PROOF. Clearly, $\sigma(a, B) \subset \sigma(a, A)$, so it suffices t prove the opposite inclusion. Pick a $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbf{C}^n \setminus \sigma(a, B)$. Since $\sigma(a, B)$ is polynomially convex, there exists by the spectral radius formula a polynomial p such that $|p(\zeta)| > \|p(a)\|$, so that $p(a) - p(\zeta)$ is invertible in A. By the Taylor expansion of $p(z)$ around the point ζ, one can find polynomials $q_1, \ldots, q_n$ such that

$$p(a) - p(\zeta) = (a_1 - \zeta_1)\, q_1(a) + \ldots + (a_n - \zeta_n)\, q_n(a)$$

Since $q_j(a) \in A$ for all $j = 1, \ldots, n$, $\zeta \in \mathbf{C}^n \setminus \sigma(a, A)$, and the assertion follows.

REMARK 2.2. The one-dimensional variant of this result is well known (see [Rud], Theorem 10.18) modulo the observation that a compact subset of $\mathbf{C}$ is polynomially convex if and only if its complement is connected.

§3. The problem and its solution.

Fix the dimension $n \geqq 1$. Let W be a bounded domain in $\mathbf{C}^n$. B will be an acceptable algebra on W; recall that this puts certain restrictions on W if $n > 1$. Let two domains W_1 *and* W_2 be given, of which W_1 is bounded, such that $W_1 \cap W_2 = W$. It is instructive for the reader to keep the example $W = (\mathbf{D} \setminus r\bar{\mathbf{D}}) \times \mathbf{D}$, $W_1 = \mathbf{D} \times \mathbf{D}$, and $W_2 = (\mathbf{C} \setminus r\bar{\mathbf{D}}) \times \mathbf{D}$ in mind; here r is a fixed number in the interval $(0, 1)$, and $\mathbf{D}$ is the open unit disc, as usual. Set $B_1 = B \cap H^\infty(W_1)$, and assume that this algebra is acceptable on W_1. Moreover, we assume there is a closed subalgebra B_2 of $B \cap H^\infty(W_2)$ such that $B = B_1 \oplus B_2$, meaning $B = B_1 + B_2$ and $B_1 \cap B_2 = \{0\}$. By the closed graph theorem, this implies that B_1 and B_2 are *closed* subalgebras of B. Of course, the interesting case is when $B_2 \neq \{0\}$. We denote by P_1 and P_2 the continuous projections onto B_1 and B_2, respectively, which add up to identity.

We plan to compare the structure of closed ideals in B_1 with that of B. If I is a closed ideal in B_1, and J is a closed ideal in B, we can form the *extension* $\overline{I \cdot B}$ of I, which is the closure of the B-ideal generated by I, and the *contraction* $J \cap B_1$ of J, which is a closed B_1-ideal. A basic question arises arises naturally: for which I is $I = \overline{I \cdot B} \cap B_1$, and for which J is $J = \overline{(J \cap B_1) \cdot B}$? One would tend to guess that the pertinent conditions are $Z(I, B_1) \subset W_2$ and $Z(J, B) \subset W_2$. Later, we will show that under some conditions on B, $I = \overline{I \cdot B} \cap B_1$ does indeed hold if $Z(I, B_1) \subset W_2$, but we have only been able to obtain the relation $J = \overline{(J \cap B_1) \cdot B}$ under the additional condition that $Z(J, B)$ is polynomially convex.

An essential ingredient of our proof is the construction of a continuous epimorphism (surjective homomorphism) $B \to B_1/I$ that is canonical on B_1/I that is canonical on B_1 for proper closed B_1-ideals I with $Z(I, B_1) \subset W_2$.

The holomorphic functional calculus (abbreviated HFC) provides us with a morphism (a continuous homomorphism mapping unit onto unit)

$$\mathcal{O}(Z(I, B_1)) \to B_1/I,$$

which takes z_j onto $z_j + I, j = 1, \ldots, n$ (see [Bou], pp. 31–46, or [Wael, 2]). For an open set $\Omega \subset \mathbf{C}$, $\mathcal{O}(\Omega)$ denotes the Fréchet algebra of all holomorphic functions on Ω, and if K is a compact subset of $\mathbf{C}$, $\mathcal{O}(K)$ denotes the algebra of germs of functions analytic in neighborhoods of K, endowed with its natural inductive limit topology. Assuming $Z(I, B_1) \subset W_2$, the HFC morphism gives us a continuous homomorphism $B_2 \to B_1/I$ when composed with the injection mapping $B_2 \to \mathcal{O}(Z(I, B_1))$. Denote by L_I the linear mapping $B = B_1 \oplus B_2 \to B_1/I$ defined to be the canonical epimorphism on B_1 and the HFC morphism on B_2. L_I is continuous by the closed graph theorem.

PROPOSITION 3.1. *Let I be a proper closed B_1-ideal such that $Z(I, B_1) \subset W_2$. If $\mathcal{O}(\bar{W}_1)$ is dense in B_1, L_I is a continuous epimorphism $B \to B_1/I$.*

PROOF. Since L_I is continuous and canonical on B_1, the assertion will follow as soon as we have shown that L_I is a homomorphism.

It is implicit in the assumption that $\mathcal{O}(\bar{W}_1)$ is a subalgebra of B_1. Let us now try to see why this is so. The holomorphic functional calculus (see [Bou], p. 44, or [Wae2], p. 522) defines a morphism $\mathcal{O}(\bar{W}_1) \to B_1$ which takes z_j onto z_j for $j = 1, \ldots, n$, and since the Gelfand transform of the image of an $f \in \mathcal{O}(\bar{W}_1)$ equals $f \circ \hat{z}$, we realize that it is just the obvious injection mapping; just check on those complex homomorphisms which are point evaluations in W_1. On the other hand, if we compose the morphism $\mathcal{O}(\bar{W}_1) \to B_1$ with the canonical epimorphism $B_1 \to B_1/I$, we arrive at the restriction of the HFC morphism $\mathcal{O}(Z(I, B_1)) \to B_1/I$ to $\mathcal{O}(\bar{W}_1)$.

Hence

$$L_I(f \cdot g) = L_I(f)\, L_I(g) \text{ for } f \in \mathcal{O}(\bar{W}_1),\ g \in B_2,$$

and since L_I is continuous, the assertion is immediate.

For proper closed B-ideals J with $Z(J, B) \subset W_2$, let $\Lambda_J : B \to B/J$ be the linear mapping defined to be the canonical quotient mapping $B_1 \to (B_1 + J)/J$ on B_1, and the HFC morphism

$$B_2 \to \mathcal{O}(Z(J, B)) \to B/J$$

on B_2; Λ_J is continuous by the closed graph theorem.

PROPOSITION. *Let J be a proper closed B-ideal such that $Z(J, B) \subset W_2$. If $B_2 \cap \mathcal{O}(\bar{W})$ is dense in B_2,* Λ_J coincides with the canonical epimorphism $B \to B/J$.

PROOF. Just as in the proof of the previous proposition, the restriction to $\mathcal{O}(\bar{W})$ of the HFC morphism $\mathcal{O}(Z(J, B)) \to B/J$ coincides with the canonical quotient mapping $\mathcal{O}(\bar{W}) \to (\mathcal{O}(\bar{W}) + J)/J$, and hence Λ_J is the canonical epimorphism $B \to B/J$ at least on a dense subspace of B. The assertion is now immediate, because Λ_J is continuous.

PROPOSITION 3.3. *Let J be a proper B-ideal with $Z(J, B) \subset W_2$, such that $Z(J, B)$ is polynomially convex, and assume Λ_J coincides with the canonical epimorphism $B \to B/J$. Then there exists a constant $C = C(J)$ such that*

$$\|g\| \leqq C \cdot \|g|_{B_1}\| \text{ for all } g \in J^\perp.$$

PROOF. We argue by contradiction. So, assume there is a sequence $\{g_k\}_0^\infty \subset J^\perp$ such that $\|g_k\| = 1$ for all $k \in \mathbf{N} = \{0, 1, 2, \ldots\}$, and $\|g_k|_{B_1}\| \to 0$ as $k \to \infty$. Then there is a sequence $\{f_k\}_0^\infty \subset B$, $\|f_k\| = 1$, $k \in \mathbf{N}$, such that $\langle f_k, g_k \rangle \to 1$ and $\langle P_1 f_k, g_k \rangle \to 0$ as $k \to \infty$.

We now intend to see how our assumptions on $Z(J, B)$ and Λ_J come into play; in order to do so, we shall have to take a closer look at how the HFC morphism is defined in [Bou] and [Wae1,2].

Since $Z(J, B) = \sigma(z + J, B/J)$ is polynomially convex, we can find two polynomial polyhedra,

$$\Delta_0 = \{\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbf{C}^n : |p_j(\zeta)| < r_j \text{ for all } j = 1, \ldots, N\} \text{ and}$$

$$\Delta_1 = \{\zeta \in \mathbf{C}^n : |p_j(\zeta)| < R_j \text{ for all } j = 1, \ldots, N\},$$

where $N \geqq n, R_j > r_j > \|p_j(z) + J\|_{B/J}$ for all $j = 1, \ldots, N, p_j(\zeta) = \zeta_j$ for $1 \leqq j \leqq n$, and where $p_j(\zeta)$ *is a* (holomorphic) polynomial for $n < j \leqq N$, such that

$$Z(J, B) \subset \Delta_0 \subset \bar{\Delta}_0 \subset \Delta_1 \subset \bar{\Delta}_1 \subset W_2.$$

For ease of notation, write p for the mapping $(p_1, \ldots, p_N) : \mathbf{C}^n \to \mathbf{C}^N$. Put

$$\mathscr{D}_0 = \{w = (w_1, \ldots, w_N) \in \mathbf{C}^N : |w_j| < r_j \text{ for all } j = 1, \ldots, N\} \text{ and}$$

$$\mathscr{D}_1 = \{w \in \mathbf{C}^N : |w_j| < R_j \text{ for all } j = 1, \ldots, N\};$$

these are two open polydiscs in $\mathbf{C}^N$, for which

$$\Delta_0 = \{\zeta \in \mathbf{C}^n : p(\zeta) \in \mathscr{D}_0\} \text{ and}$$

$$\Delta_1 = \{\zeta \in \mathbf{C}^n : p(\zeta) \in \mathscr{D}_1\}.$$

Since $\mathscr{D}_1$ is a Stein domain and $p(\Delta_1)$ is a closed complex submanifold of $\mathscr{D}_1$, the restriction mapping $\mathcal{O}(\mathscr{D}_1) \to \mathcal{O}(p(\Delta_1))$ is surjective (see [GuR], p. 245, or p. 41 for the particular case we are interested in. Now since $\mathcal{O}(\Delta_1)$ and $\mathcal{O}(p(\Delta_1))$ can obviously be identified, it follows that the mapping $\varphi : \mathcal{O}(\mathscr{D}_1) \to \mathcal{O}(\Delta_1)$ defined by the relation $\varphi f \equiv f \circ p$ is a continuous epimorphism, and by the open mapping theorem, φ maps open sets onto open sets, or, in short, φ is open, because $\mathcal{O}(\mathscr{D}_1)$ and $\mathcal{O}(\Delta_1)$ are both Fréchet spaces. Let U be the open set $\{f \in \mathcal{O}(\mathscr{D}_1) : \|f\|_{H^\infty(\mathscr{D}_0)} < 1\}$. By the way the topology on $\mathcal{O}(\Delta_1)$ is defined , $\varphi(U)$ is open implies that there is a $\delta > 0$ such that all functions in $H^\infty(\Delta_1)$ with norm $< \delta$ belong to $\varphi(U)$. Expressed differently, to every function $f \in H^\infty(\Delta_1)$ with norm $< \delta$, we can find a function $F \in \mathcal{O}(\mathscr{D}_1)$ with $\|F\|_{H^\infty(\mathscr{D})} < 1$ such that $F \circ p = f$. For $k \in \mathbf{N}$, choose an $F_k \in \mathcal{O}(\mathscr{D}_1)$ such that $F_k \circ p = P_2 f_k$ on Δ_1; since $\|f_k\| = 1$ and P_2 is continuous, we may thus choose the F_k's so that for some constant C, $\|F_k\|_{H^\infty(\mathscr{D}_0)} \leqq C$ for all $k \in \mathbf{N}$.

By the definition of the HFC morphism (see [Wae2], pp. 189–191, or use the standard proof of Taylor's formula together with Proposition 1 [Bou], p. 41, and Théorème 2 [Bou], p. 46), the image of $P_2 f_k \in B_2^0 \subset H^\infty(W_2)$ is

$$P_2 f_k[z + J] = (2\pi i)^{-N} \int_{\check{S}(\mathscr{D}_0)} \prod_{j=1}^{N} (w_j - p_j(z) + J)^{-1} F_k(w)\, dw,$$

where $\check{S}(\mathscr{D}_0) = \{w = (w_1, \ldots, w_N) \in \mathbf{C}^N : |w_j| = r_j \text{ for all } j\}$ is the distinguished boundary of $\mathscr{D}_0$, and $dw = dw_1 \wedge \ldots \wedge dw_N$. The expression $w_j - p_j(z) + J$ is invertible in B/J because $|w_j| = r_j > \|p_j(z) + J\|$.

For $k \in \mathbf{N}$, introduce the functions

$$\mathscr{G}_k(w) = \left\langle \prod_{j=1}^{N} (w_j - p_j(z) + J)^{-1}, g_k \right\rangle, \quad w = (w_1, \ldots, w_N) \in \Omega,$$

where Ω is the connected domain $\{w \in \mathbf{C}^n : |w_j| > \|p_j(z) + J\| \text{ for all } j = 1, \ldots, N\}$, which are holomorphic is Ω because the expression $\prod_{j=1}^{N} (w_j - p_j(z) + J)^{-1}$ has a convergent power series expansion locally arond every $w \in \Omega$, and which satisfy the estimate

$$|\mathscr{G}_k(w)| \leqq \prod_{j=1}^{N} \|(w_j - p_j(z) + J)^{-1}\|, \quad w \in \Omega.$$

Replacing $\{\mathscr{G}_k\}_0^\infty$ by a subsequence (this is actually not necessary), we can assume by normality $\{\mathscr{G}_k\}_0^\infty$ converges uniformly on compact subsets of Ω to some function in $\mathscr{O}(\Omega)$. Since $(w_j - p_j(z))^{-1} \in B_1$ if $|w_j| > \|p_j(z)\|$ and $\|g_k|_{B_1}\| \to 0$ as $k \to \infty$, $\mathscr{G}_k(w) \to 0$ as $k \to \infty$ on the set

$$\{w = (w_1, \ldots, w_N) \in \mathbf{C}^N : |w_j| > \|p_j(z)\| \text{ for all } j\}.$$

Hence $\mathscr{G}_k(w) \to 0$ as $k \to \infty$ uniformly on compact subset of Ω is connected, and in particular on $\check{S}(\mathscr{D}_0)$.

Since by assumption, Λ_J coincides with the canonical epimorphism $B \to B/J$,

$$\langle P_2 f_k, g_k \rangle = (2\pi i)^{-N} \int_{\check{S}(\mathscr{D}_0)} \mathscr{G}_k(w) F_k(w)\, dw.$$

Now because we know that $\|F_k\|_{H^\infty(\mathscr{D}_0)} \leqq C$ for all k, we may conclude that

$$\langle P_2 f_k, g_k \rangle \to 0 \text{ as } k \to \infty,$$

which gives us our desired contradiction.

REMARK 3.4. In connection with the proof of the previous proposition, we would like to mention the following result by G. M. Henkin and P. L. Polyakov [HeP]. If M is an analytic variety in the open unit polydisc $\mathbf{D}^n$ satisfying certain regularity conditions, there exists a continuous linear operator $E: H^\infty(M) \to H^\infty(\mathbf{D}^n)$ such that $Ef|_M = f$ for all $f \in H^\infty(M)$.

We now state our main result, which is formulated in two theorems, Theorems 3.5 and 3.6.

THEOREM 3.5. *Assume $\mathscr{O}(\bar{W}_1)$ is dense in B_1 and $\mathscr{O}(\bar{W}) \cap B_2$ is dense in B_2. Let I be a closed B_1-ideal such that $Z(I, B_1) \subset W_2$. Then*

(a) *$Z(\overline{I \cdot B}, B) = Z(I, B_1)$ and $\overline{I \cdot B} \cap B_1 = I$.*

(b) *L_I is a continuous epimorphism with kernel $\overline{I \cdot B}$.*

(c) *The quotient algebras B_1/I and $B/\overline{I \cdot B}$ are canonically isomorphic.*

PROOF. Let us first check (b). By Proposition 3.1, L_I is a continuous epimorphism, so it remains to show that $\ker L_I = \overline{I \cdot B}$, since L_I is canonical on B_1. Let m be arbitrary in $h(\overline{I \cdot B}, B)$. Then $m|_{B^1} \in h(I, B_1)$, and consequently

$$Z(\overline{I \cdot B}, B) \equiv \hat{z}(h(\overline{I \cdot B}, B)) \subset \hat{z}(h(I, B_1)) \equiv Z(I, B_1).$$

Hence $\Lambda_{\overline{I \cdot B}}$ is well defined. More or less by the definitions of L_I and $\Lambda_{\overline{I \cdot B}}$, the composition of L_I and canonical homomorphism $B_1/I \to B/\overline{I \cdot B}$ equals $\Lambda_{\overline{I \cdot B}}$. By Proposition 3.2, $\Lambda_{\overline{I \cdot B}}$ coincides with the canonical epimorphism $B \to B/\overline{I \cdot B}$, and it is now immediate that $\ker L_I = \overline{I \cdot B}$, which verifies (b).

Now we turn our attention to (c). The mapping L_I induces a Banach isomorphism

$$\tilde{L}_I : B/\ker L_I = B/\overline{I \cdot B} \to B_1/I.$$

Since L_I is canonical on B_1, $(\tilde{L}_I)^{-1}$ must coincide with the canonical homomorphism $B_1/I \to B/\overline{I \cdot B}$.

We proceed with (a). First we show that $Z(\overline{I \cdot B}, B) = Z(I, B_1)$. We already know that $Z(\overline{I \cdot B}, B) \subset Z(I, B_1)$, so it suffices to obtain the opposite inclusion. To this end, let $m_1 \in h(I, B_1) (\subset I^\perp)$ be arbitrary. Then $m \equiv L_I^*(m_1) \in (\overline{I \cdot B})^\perp$ is a complex homomorphism in $h(\overline{I \cdot B}, B) = (\overline{I \cdot B})^\perp \cap \mathscr{M}(B)$, whose restriction to B_1 is m_1, since L_I is canonical on B_1. Here, $L_I^* : I^\perp \to B^*$ is the adjoint mapping of L_I. Since $\hat{z}(m) \equiv (m(z_1), \ldots, m(z_n)) = (m_1(z_1), \ldots, m_1(z_n))$, we conclude that $Z(\overline{I \cdot B}, B) = Z(I, B_1)$. To finish the verification of (a), we need to show that $\overline{I \cdot B} \cap B_1 = I$. This follows immediately from the facts that L_I is canocial on B_1 and that its kernel is $\overline{I \cdot B}$, by (b). The proof is complete.

THEOREM 3.6. *Assume $\mathcal{O}(\bar{W}_1)$ is dense in B_1 and $\mathcal{O}(\bar{W}) \cap B_2$ is dense in B_2. Let J be a closed ideal in B. Then*

(a) *J is of the form $\overline{I \cdot B}$ for some closed B_1-ideal I with $Z(I, B_1) \subset W_2$ if and only if $Z(J \cap B_1, B_1) \subset W_2$; an I that works is $I = J \cap B_1$.*

(b) *If $Z(J, B)$ is a polynomially convex subset of W_2, then $Z(J \cap B_1, B_1) \subset W_2$, so by (a), $J = \overline{(J \cap B_1) \cdot B}$. In particular, $J \cap B_1 \neq \{0\}$.*

PROOF. Let us first check (a). It will be sufficient to prove that $J = \overline{(J \cap B_1) \cdot B}$. For ease of notation, we write $J_0 = \overline{(J \cap B_1) \cdot B}$, and observe that $J_0 \subset J$. By Theorem 3.5(c), the quotient algebras $B_1/J \cap B_1$ and B/J_0 are canonically isomorphic. By some elementary algebra, this implies that $B = B_1 + J_0$. Then for an arbitrary $f \in J$, there exists a $g \in B_1$ such that $f - g \in J_0$. Since $J_0 \subset J$, we conclude that $g \in J \cap B_1$, and consequently, $f \in J_0$; hence $J = J_0$.

We proceed with (b). Let ν be the canonical monomorphism $B_1/J \cap B_1 \to B/J$. Its adjoint mapping $\nu^* : J^\perp \to (J \cap B_1)^\perp$ restricts the functionals in $J^\perp$ to B_1. By Proposition 3.3, $\operatorname{im} \nu^* = J^\perp|_{B_1}$ is norm closed, so an application of Theorem 4.14

[Rud] shows that im v is a closed subalgebra of B/J. Certainly, this subalgebra is canonically isomorphic to $B_1/J \cap B_1$, and

$$\sigma(z + J, \operatorname{im} v) = \sigma(z + J \cap B_1, B_1/J \cap B_1) = Z(J \cap B_1, B_1).$$

Since $Z(J, B) = \sigma(z + J, B/J)$ is polynomially convex, Lemma 2.1 shows that

$$Z(J \cap B_1, B_1) = \sigma(z + J, \operatorname{im} v) = Z(J, B),$$

and consequently, $Z(J \cap B_1, B_1) \subset W_2$. The proof of the theorem is complete.

REMARKS 3.6. (a) The above two theorems apply to the algebra $A(W)$ for some domains W; we refer to [Ran], pp. 280–282, 303–307, and 360–361 for details on when $\mathcal{O}(\bar{W})$ is dense in $A(W)$.

(b) If the polynomials are dense in B_1, it is well known (see [Sto], p. 25) that the set $\sigma(z, B_1) = \hat{z}(\mathcal{M}(B_1))$ is polynomially convex. The same result applied to the quotient algebra B_1/I shows that under the same assumption, $Z(I, B_1)$ is polynomially convex for all closed B_1-ideals I.

(c) There are many domains W to which Theorem 3.5 applies which are not of product type, that is, not of the form a domain in $\mathbf{C}^1$ times a domain in $\mathbf{C}^{n-1}$. Simple examples are provided by tilting product domains. A nontrivial example is

$$W = \{z = (z_1, z_2) \in \mathbf{C}^2 : |z_1|^2 + |z_2|^2 < 1, |z_2| < 1/2, \text{ and } |z_1| > 1/2\}.$$

Here, W_1 is the set

$$\{z \in \mathbf{C}^2 : |z_1|^2 + |z_2|^2 < 1, |z_2| < 1/2\},$$

and W_2 is

$$\{z \in \mathbf{C}^2 : |z_2| < 1/2, |z_1| > 1/2\}.$$

The point is that the distinguished boundaries of W_1 and W_2 do not intersect, so that we can get a decomposition $B = B_1 \oplus B_2^0$ for the algebra $A(W)$.

(d) If $w : \mathbf{Z}^n \to (0, \infty)$ is a submultiplicatieve weight function when $\mathbf{Z}^n$ is given its standard additive group structure, we can introduce the Beurling algebra $l^1(w, \mathbf{Z}^n)$ as consisting of those functions f: $\mathbf{Z}^n \to \mathbf{C}$ for which

$$\|f\| = \sum_{\alpha \in Z^n} |f(\alpha)| \, w(\alpha) < \infty,$$

supplied with convolution multiplication. For a class of weight functions, which is the one-dimensional case $n = 1$ consists of those which are of what one calls analytic type, Theorems 3.5 and 3.6 gives us information about the relation between the ideal theories of $l^1(w, \mathbf{Z}^n)$ and its closed subalgebra $l^1(w, \mathbf{N} \times \mathbf{Z}^n)$.

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