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ON TEICHMÜLLER'S MODULUS PROBLEM IN R^n

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1. Introduction.

For $x \in R^n \setminus \{0, e_1\}$, $n \geqq 2$, $e_1 = (1, 0, \ldots, 0)$, define

$$p(x) = \inf_{E,F} M(\Delta(E, F; R^n)) \tag{1.1}$$

where E and F are continua with 0, $e_1 \in E$ and x, $\infty \in F$ (see Section 2 for notation). O. Teichmüller has suggested the problem of evaluating $p(x)$ in terms of well-known functions when $n = 2$. Making use of the geometric method of symmetrization, he also solved this problem in the particular case $x = te_1, t > 1$, when the extremal continua E and F are linear and constitute the boundary components of a ring which is conformally equivalent to the so-called Teichmüller ring.

M. Schiffer [S] gave a qualitative solution of the general case of this problem in 1946 and a quantitative expression for $p(x)$ was found by H. Wittich in 1949 [W]. G. V. Kuz'mina's book [K] contains a complete account of this extremal problem of conformal geometry with several applications to univalent functions (pp. 187–217). See also J. G. Krzyż [Kr].

Generalizing Teichmüller's work on symmetrization to the multidimensional case F. W. Gehring [G1] proved in 1961 that the conformal capacity of a ring decreases under symmetrization. By performing spherical symmetrizations with centers at 0 and e_1 we see by [G1] that for $x \in R^n \setminus \{0, e_1\}$

$$p(x) \geqq \max\{\tau(|x|), \tau(|x - e_1|)\} \tag{1.2}$$

where $\tau(s) = \tau_n(s)$ is the capacity of the Teichmüller ring in R^n (see Section 2). Equality holds in (2.1) if $x = se_1$ and $s < 0$ or $s > 1$. Therefore Gehring's work provides the answer to Teichmüller's problem in the particular case $x = te_1$, $t > 1$. Finding a multidimensional analogue of the general case, i.e. generalizing

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Schiffer's and Wittich's work to R^n seems extremely difficult and no results of this kind are known.

In the present paper we shall prove the following results in the direction opposite to (1.2).

1.3. THEOREM. *For* $x \in \mathsf{R}^n \setminus \{0, e_1\}$, $|x - e_1| \leqq |x|$

(1) $\quad p(x) \leqq 2\tau(|x - e_1|)$, *when* $|x + e_1| \geqq 2$,

(2) $\quad p(x) \leqq 4\tau(|x - e_1|)$, *when* $|x| \geqq 1$,

(3) $\quad p(x) \leqq 2^{n+1}\tau(|x - e_1|)$.

This theorem enables one to find some estimates for a conformal invariant introduced by J. Lelong-Ferrand in [LF] and studied by the present author in [Vu]. If $G \subset \bar{\mathsf{R}}^n$ is a domain with $\operatorname{card}(\bar{\mathsf{R}}^n \setminus G) \geqq 2$ set ([LF], [Vu])

$$\lambda_G(x, y) = \inf_{C_x, C_y} M(\Delta(C_x, C_y; G)) \tag{1.4}$$

where $x, y \in G$, $x \neq y$, and where C_x and C_y are disjoint curves in G with $x \in C_x$, $y \in C_y$ and $\bar{C}_x \cap \partial G \neq \varnothing \neq \bar{C}_y \cap \partial G$. The conformal invariant $\lambda_G(x, y)$ has found recent applications to the theory of manifolds of negative curvature, due to P. Pansu [P]. The main result of this paper is the following theorem.

1.5. THEOREM. $1 \leqq \lambda_{\mathsf{R}^n \setminus \{0\}}(x, y)/\tau(|x - y|/\min\{|x|, |y|\}) \leqq 4$ *for* $x, y \in \mathsf{R}^n \setminus \{0\}$.

An immediate application of Theorem 1.5 is the next result.

1.6. THEOREM. *If* $f\colon \mathsf{R}^n \to \mathsf{R}^n$ *is a* K*-quasiconformal mapping with* $f(0) = 0$, *then for* $x, y \in \mathsf{R}^n \setminus \{0\}$

$$\frac{|f(x) - f(y)|}{\min\{|f(x)|, |f(y)|\}} \leqq \tau^{-1}\left(\frac{1}{4K}\tau\left(\frac{|x - y|}{\min\{|x|, |y|\}}\right)\right).$$

We shall give several applications of the above results to the distortion theory of quasiconformal mappings. Because the special function $t \mapsto \tau^{-1}\tau(t)/(4K)$ admits a dimension-free hölderian majorant (cf. [AVV1] and Theorem 2.15 below), Theorem 1.6 provides a dimension-free distortion theorem. For further results of this kind, see [AVV2]. An application of Theorem 1.6 is contained in [AVV2, Section 4].

For a general domain G there is no counterpart of Theorem 1.5. A simple counterexample for $n = 2$ is the unit disk minus a radius. If, however, $\mathsf{R}^n \setminus G$ is a null set for extremal distances in the sense of Ahlfors and Beurling, then there is a counterpart of Theorem 1.5 for G. More generally, this holds if $\mathsf{R}^n \setminus G$ is a QED-set in the sense of Gehring and Martio [GM], as we shall prove.

In Section 2 we prove some functional inequalities for $\tau(s)$ which are crucial for the sequel and are perhaps of independent interest. Sections 3 and 4 contain the proofs of Theorems 1.3 and 1.5 together with some applications. In Section 5 we prove a distortion theorem for Möbius transformations and give a conformally invariant formulation of Theorem 1.5 in terms of the absolute ratio.

The functional inequalities in Section 2 are particular cases of more general results, which are given in [AVV3]. In the two-dimensional particular case one can improve the results of this paper by using the methods of [K]. Such results are given in [LeVu].

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2. Some functional inequalities for $\tau(s)$

2.1. *Notation.* We shall adopt the relatively standard notation and terminology of [V1]. The coordinate unit vectors in R^n are $e_1, \dots, e_n$. If $x, y \in R^n$, then we denote $[x, y] = \{ty + (1-t)x\colon 0 \leqq t \leqq 1\}$ and similarly for open or half-open segments. If $x \in R^n \setminus \{0\}$, then $[x, \infty) = \{ux\colon u \geqq 1\}$. For $x \in R^n$ and $r > 0$ let $B^n(x,r) = \{z \in R^n\colon |z - x| < r\}$, $S^{n-1}(x,r) = \partial B^n(x,r)$, $B^n(r) = B^n(0,r)$, $S^{n-1}(r) = \partial B^n(r)$, $B^n = B^n(1)$, and $S^{n-1} = \partial B^n$. If $\emptyset \neq A \subset R^n$ set $d(A) = \sup\{|x - y|\colon x, y \in A\}$ and $d(A,B) = \inf\{|x - y|\colon x \in A, y \in B\}$ for $\emptyset \neq A, B \subset R^n$.

For the definition and some properties of the modulus $M(\Gamma)$ of a curve family Γ the reader is referred to [V1]. If E, F, G are subsets of R^n or $\bar{R}^n = R^n \cup \{\infty\}$, then $\Delta(E, F; G)$ stands for the family of all curves joining E to F in G; see [V1, p. 21]. If $G = R^n$ or $G = \bar{R}^n$, we denote $\Delta(E, F; G) = \Delta(E, F)$. For the definition and some properties of K-quasiconformal, K-quasiregular, and K-quasimeromorphic mappings the reader is referred to [V1], [MRV], and [R].

A ring in $\bar{R}^n$ is a domain such that its complement has exactly two components. By definition, the complementary components of the Teichmüller ring $R_T(s) = R_{T,n}(s)$ in R^n are $[-e_1, 0]$ and $[se_1, \infty)$, $s \in (0, \infty)$ while those of the Grötzsch ring $R_G(s) = R_{G,n}(s)$ are $\bar{B}^n$ and $[se_1, \infty)$, $s \in (1, \infty)$. The capacities of these rings are denoted by

$$\tau(s) = \tau_n(s) = \operatorname{cap} R_{T,n}(s) \tag{2.2}$$
$$\gamma(s) = \gamma_n(s) = \operatorname{cap} R_{G,n}(s).$$

The following identity holds

$$\gamma(s) = 2^{n-1}\,\tau(s^2 - 1), \quad s > 1. \tag{2.3}$$

It is well-known that $\tau(s)$ is a strictly decreasing function.

In the special case $n = 2$, $\gamma_2(t)$ has an explicit expression (2.4). No formula like (2.4) is known for $n \geqq 3$. For $r \in (0, 1)$ (see [LV])

$$\gamma_2(1/r) = 2\pi/\mu(r); \quad \mu(r) = \frac{\pi K(\sqrt{1 - r^2})}{2K(r)} \tag{2.4}$$

where

$$K(r) = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - r^2x^2)}}.$$

The function μ satisfies the functional identities

$$\mu(r) = 2\mu\left(\frac{2\sqrt{r}}{1 + r}\right) = \frac{\pi^2}{4\mu(\sqrt{1 - r^2})} = \frac{\pi^2}{2\mu\left(\frac{1 - r}{1 + r}\right)}. \tag{2.5}$$

By (2.3), (2.4), and (2.5) we get

$$\tau_2(t) = \pi/\mu(1/\sqrt{1 + t}) = 2\pi/\mu((\sqrt{1 + t} - \sqrt{t})^2). \tag{2.6}$$

By performing inversions we get

$$\tau_n\left(\frac{t - s}{s(1 - t)}\right) = M(\Delta([0, se_1], [te_1, e_1])); \quad 0 < s < t < 1 \tag{2.7}$$

$$\tau_n(s) = M(\Delta([-e_1, -ae_1], [ae_1, e_1])), \quad s > 0 \tag{2.8}$$

where $a = 1 + \frac{2}{s}(1 - \sqrt{1 + s}) \in (0,1)$.

2.9. THEOREM. *The following inequalities hold*

(1) $\tau(s) \leqq \gamma(1 + 2s) = 2^{n-1}\tau(4s^2 + 4s), \quad s > 0,$

(2) $\tau(s) \leqq 2\tau(2s + 2s\sqrt{1 + 1/s}), \quad s > 0,$

(3) $\tau(s) \leqq \tau(t) + \tau\left(\frac{s(1 + t)}{t - s}\right), \quad 0 < s < t < \infty,$

(4) $\tau(u) \leqq \tau\left(\frac{uv}{u + v + 1}\right) \leqq \tau(u) + \tau(v), \quad u, v > 0.$

PROOF. (1) Let $\Gamma = \Delta(S^{n-1}(-e_1/2, 1/2), [se_1, \infty))$. Then by (2.3)

$$M(\Gamma) = \gamma(1 + 2s) = 2^{n-1}\tau(4s^2 + 4s)$$

while by [V1, 6.4] $\tau(s) \leqq M(\Gamma)$ and the desired inequality follows.

(2) We can map $R_T(s)$ by a Möbius transformation onto a ring in $\bar{R}^n$ with complementary components $[-e_1, e_1]$ and $[be_1, \infty) \cup \{\infty\} \cup [-be_1, \infty)$, $b = 1 + 2s(1 + \sqrt{1 + 1/s})$. A symmetry property [GV, Lemma 3.3] of the modulus shows that

$$\begin{aligned}\tau(s) &= 2M(\Delta([0, e_1], [be_1, \infty); \{x: x_1 > 0\}))\\ &\leqq 2\tau(b-1),\end{aligned}$$

as desired.

(3) Let $\Gamma_1 = \Delta[-e_1, 0], [se_1, te_1]), \Gamma_2 = \Delta([-e_1, 0], [le_1, \infty))$. Then by (2.7)

$$\tau(s) \leqq M(\Gamma_1 \cup \Gamma_2) \leqq M(\Gamma_1) + M(\Gamma_2) = \tau\left(\frac{s(1+t)}{t-s}\right) + \tau(t).$$

(4) After a change of variables the right inequalities in (4) follows from (3). The left inequality follows from the fact that τ is decreasing.

2.10. COROLLARY. $\tau(s) \leqq 2\tau(\sqrt{s}), \quad s > 0.$

PROOF. The left inequality follows from 2.9 (2) because τ is decreasing. The right inequality follows from 2.9 (1).

2.11. REMARK. For $n = 2$ 2.10 and (2.6) yield the following result for the function μ

$$\mu(1/\sqrt{1+t}) \leqq 2\mu(1/\sqrt{1+\sqrt{t}}) \leqq 4\mu(1/\sqrt{1+t}), t > 0.$$

For what follows we require the well-known inequalities [LV, p. 61]

$$\log\frac{1}{s} < \mu(s) < \log\frac{4}{s}, \quad 0 < s < 1, \tag{2.12}$$

$$e^{-u} < \mu^{-1}(u) < 4e^{-u}, \quad 0 < u < \infty. \tag{2.13}$$

Note that (2.13) follows from (2.12). From [Vu, 2.14(2), 5.20] we recall that for all $s > 0$

$$c_n \log a \leqq \tau(s) \leqq c_n \mu(1/a); \ a = 1 + \frac{2}{s}(1 + \sqrt{1+s}). \tag{2.14}$$

2.15. THEOREM. *For $n \geqq 2$, $K \geqq 1$, and $t \in (0, 2^{2-3K})$*

$$\tau^{-1}(\tau(t)/K) \leqq 4^{3-1/K} t^{1/K}.$$

PROOF. Let $x = \tau^{-1}(\tau(t)/K)$ and $b = \log\left(1 + \frac{2}{t}(1 + \sqrt{1+t})\right)$.

By (2.14) we obtain

$$c_n b \leqq c_n K\mu\left(1 + \frac{2}{x}(1 - \sqrt{1+x})\right)$$

and further

$$x \leqq 4\mu^{-1}(b/K)/(1 - \mu^{-1}(b/K))^2.$$

It follows from (2.13) that $\mu^{-1}(b/K) < 1/2$ for $t \in (0, 2^{2-3K})$. From (2.13) and the above inequality we obtain

$$x \leqq 4^3\left(\frac{t}{t + 2(1 + \sqrt{1+t})}\right)^{1/K} \leqq 4^{3-1/K} t^{1/K}$$

for $t \in (0, 2^{2-3K})$, which is the desired inequality.

2.16. Remark. Note that the upper bound in Theorem 2.15 is independent of the dimension n. Some other dimension-free estimates of this type were given in [AVV1, 3.9].

3. Bounds for $p(x)$

It follows from the definition (1.1) that the values of $p(x)$ are completely determined by its values in the set

(3.1) $$D_1 = \{(x_1, 0, \ldots, 0, x_n) : x_1 \geqq 1/2, x_n \geqq 0\} \setminus \{e_1\}.$$

We shall need the following elementary lemma.

3.2. Lemma. *If* $x \in \mathbf{R}^n \setminus B^n(-2e_1, 3)$, *then*

$$4(|x| - 1) \geqq \min\{|x - e_1|, |x - e_1|^2\}.$$

Proof. Write $x = x + 2e_1 - 2e_1$ and $x - e_1 = x + 2e_1 - 3e_1$. Then

$$|x|^2 = |x + 2e_1|^2 + 4 - 4(x + 2e_1)\cdot e_1,$$

$$|x - e_1|^2 = |x + 2e_1|^2 + 9 - 6(x + 2e_1)\cdot e_1,$$

$$3|x|^2 - 2|x - e_1|^2 = |x + 2e_1|^2 - 6 \geqq 9 - 6 = 3.$$

Hence $|x| \geqq \sqrt{1 + \frac{2}{3}|x - e_1|^2}$, so that

$$|x| - 1 \geqq \frac{\frac{2}{3}|x - e_1|^2}{1 + \sqrt{1 + \frac{2}{3}|x - e_1|^2}}.$$

Case A. $|x - e_1| \leqq 1$. Then

$$|x| - 1 \geqq \frac{\frac{2}{3}|x - e_1|}{1 + \sqrt{1 + \frac{2}{3}}} \geqq \frac{1}{4}|x - e_1|^2. \tag{3.3a}$$

Case B. $|x - e_1| > 1$. Then

$$|x| - 1 \geqq \frac{\frac{2}{3}|x - e_1|\,|x - e_1|}{1 + \sqrt{1 + \frac{2}{3}|x - e_1|^2}} \geqq \frac{2/3}{1 + \sqrt{1 + \frac{2}{3}}}|x - e_1| > \frac{1}{4}|x - e_1| \tag{3.3b}$$

since $t/(1 + \sqrt{1 + \frac{2}{3}t^2})$ is increasing on $(0, \infty)$.

The proof follows from (3.3a) and (3.3b), respectively.

All the upper bounds that we shall prove for $p(x)$ rely on the following lemma, which is based on Lemma 3.2 and on a lemma of F. W. Gehring [Vu, 2.58].

3.4. LEMMA. *Let* $E = [0, e_1]$ *and* $F = [x, \infty)$ *for* $x \in R^n \setminus B^n$. *Then*

(1) $\quad p(x) \leqq M(\Delta(E, F)) \leqq \tau(|x| - 1)$.

If $x \in R^n \setminus B^n(-2e_1, 3)$, *then*

(2) $\quad p(x) \leqq M(\Delta(E, F)) \leqq 2\tau(|x - e_1|)$.

PROOF. (1) was proved in [Vu, 2.58]. It follows from 2.9(2) that $\tau(u) \leqq 2\tau(2u + 2\sqrt{u}) < 2\tau(2\sqrt{u})$ and hence $\tau(s^2/4) \leqq 2\tau(s)$. From 2.9 (2) it also follows that $\tau(s/4) < 2\tau(s)$. In conclusion, for $s > 0$ the following inequality holds

$$\tau(\min\{s, s^2\}/4) < 2\tau(s)$$

The proof of (2) follows from part (1), the above inequality, and Lemma 3.2.

3.5. PROOF OF THEOREM 1.3 (1). Let $Y = \{x \in S^{n-1}(-e_1, 2): x_1 = 1/2\}$. Note that $d(e_1, Y) = \sqrt{2}$. It suffices to prove the assertion for $x \in D_1 \setminus B^n(-e_1, 2)$.

Case A. $|x - e_1| \leqq \sqrt{2}$.

Choose $\bar{x} \in S^{n-1}(-e_1, 2) \cap D_1$ with $|\bar{x} - e_1| = |x - e_1|$. Then $|\bar{x} - e_1| = 4\sin\frac{\beta}{2}$ where β is the acute angle between the segments $[-e_1, e_1]$ and $[-e_1, \bar{x}]$. Let $x_0 = (e_1 - e_n)/2$. Because $x \in D_1 \setminus B^n(-e_1, 2)$ we obtain

$$\begin{aligned}|x - x_0|^2 \geqq |\bar{x} - x_0|^2 &= \left(2\sin\beta + \frac{1}{2}\right)^2 + \left(\frac{1}{2} - 4\sin^2\frac{\beta}{2}\right)^2 \\ &= \frac{1}{2} + 12\sin^2\frac{\beta}{2} + 2\sin\beta = \frac{1}{2}(1 + A),\end{aligned}$$

where $A = 24\sin^2\frac{\beta}{2} + 4\sin\beta$. Because $|x_0 - e_1| = 1/\sqrt{2}$ an elementary but lengthy computation shows that

$$\frac{|\bar{x}-x_0|}{|x_0-e_1|}-1=\frac{A}{1+\sqrt{1+A}}\geqq|\bar{x}-e_1|=4\sin\frac{\beta}{2}$$

holds for all $x\in D_1\setminus B^n(-e_1,2)$. Let $E_1=[x_0,e_1]$, $E_2=[0,x_0]$, and $F=\{x_0+t(x-x_0):t\geqq 1\}$. By 3.4 (1) and the above inequality

$$M(\Delta(E_j,F))\leqq\tau\left(\frac{|x-x_0|}{|x_0-e_1|}-1\right)\leqq\tau\left(\frac{|\bar{x}-x_0|}{|x_0-e_1|}-1\right)\leqq\tau(|\bar{x}-e_1|)$$

for $j=1,2$. Because $|\bar{x}-e_1|=|x-e_1|$, we obtain

$$p(x)\leqq M(\Delta(E_1\cup E_2,F))<2\tau(|x-e_1|)$$

as desired. (Note that the condition $|x-e_1|\leqq\sqrt{2}$ was used only for the construction of $\bar{x}$).

Case B. $|x-e_1|>\sqrt{2}$.

It is easy to see that in the Case B we have

$$\frac{|x|-1}{|x-e_1|}\geqq 1-1/\sqrt{2}>1/4$$

and hence by 3.4 (1)

$$p(x)<\tau(|x|-1)\leqq\tau(|x-e_1|/4).$$

Finally as $\tau(s)<2\tau(4s)$ by 2.9 (2), we obtain

$$p(x)\leqq 2\tau(|x-e_1|).$$

3.6. THEOREM. *For* $x\in D_1\setminus B^n(-(e_1+3e_n/(\tan\alpha))/2,\ 3/(2\sin\alpha))$ $0<\alpha<\pi/2$, *the following inequality holds*

$$p(x)\leqq 4\tau(2(\sin\alpha)|x-e_1|).$$

PROOF. Let $x_0=(e_1-e_n/\tan\alpha)/2$. Let $E_1=[x_0,e_1]$, $E_2=[0,x_0]$, $F=\{x_0+t(x-x_0):t\geqq 1\}$, $\Gamma_j=\Delta(E_j,F)$. It follows from 3.4 (1) that

$$M(\Gamma_j)\leqq\tau\left(\frac{|x-x_0|}{|x_0-e_1|}-1\right)$$

for $j=1,2$. Because of the choice of x, lemma 3.4 yields

$$\tau\left(\frac{|x-x_0|}{|x_0-e_1|}-1\right)\leqq 2\tau\left(\frac{|x-e_1|}{|x_0-e_1|}\right)=2\tau((2\sin\alpha)|x-e_1|).$$

These inequalities yield

$$p(x) \leqq M(\Delta(E_1 \cup E_2, F)) \leqq 4\tau((2\sin\alpha)|x - e_1|)$$

as desired.

3.7. PROOF OF THEOREM 1.3 (2). Choose $\alpha = \pi/4$ in Theorem 3.6. May assume $x \in D_1 \setminus B^n$. The proof follows now from Theorem 3.6.

3.8. PROOF OF THEOREM 1.3 (3). We may assume $x \in D_1$. If $x_n \geqq \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} x_1$, then Theorem 3.6 with $\alpha = \pi/6$ yields

$$p(x) \leqq 4\tau(|x - e_1|). \tag{3.9}$$

If $x_n < \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} x_1$, choose $\bar{x} \in D_1$ with $\bar{x}_1 = x_1$ and $\bar{x}_n = \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} \bar{x}_1$. Let $x_0 = (e_1 - \sqrt{3}\, e_n)/2$, $E = [0, x_0] \cup [x_0, e_1]$, and $F = [x, \bar{x}] \cup \{x_0 + (\bar{x} - x_0) t : t \geqq 1\}$. Since $|\bar{x} - e_1| > |x - e_1|$, we obtain by (3.9) and (2.3)

$$\begin{aligned} p(x) \leqq M(\Delta(E, F)) &\leqq 4\tau(|\bar{x} - e_1|) + M(\Delta([x, \bar{x}], E)) \\ &\leqq 4\tau(|x - e_1|) + M(\Gamma) \\ &\leqq 4\tau(|x - e_1|) + \gamma(2) = 4\tau(|x - e_1|) + 2^{n-1}\tau(3) \\ &\leqq (4 + 2^{n-1})\tau(|x - e_1|) \leqq 2^{n+1}\tau(|x - e_1|) \end{aligned}$$

where $\Gamma = \Delta([x, \bar{x}], S^{n-1}(\bar{x}, |\bar{x} - e_1|))$.

3.10. REMARK. For $n = 2$ the shape of the extremal ring for $p(x)$ has been studied by G. V. Kuz'mina [K, Chapter 5]. If $x = (1/2, x_2) \in D_1$, then the extremal ring is Mori's ring, i.e.

$$p(x) = M(\Delta(E, F));$$

where E is an arc of a circle with center at x joining 0 to e_1 and $F = \{(\tfrac{1}{2}, t) : t \geqq x_2\}$. It follows from [LV, (1.11), p. 58] that

$$p(x) = 2\pi/\mu(\tfrac{1}{2}\sqrt{2 - \sqrt{4 - t^2}}),$$

where $t = 1/|x|$. Choosing $x = (\tfrac{1}{2}, 0)$ yields

$$p((\tfrac{1}{2}, 0)) = 2\pi/\mu(1/\sqrt{2})$$

while Theorem 1.3 (3) yields in view of (2.6)

$$p((\tfrac{1}{2}, 0)) \leqq 8\tau(\tfrac{1}{2}); \quad \tau(\tfrac{1}{2}) = \pi/\mu(\sqrt{\tfrac{2}{3}}).$$

Therefore the least constant c in 1.3 (3) for $n = 2$,

$$c_2 = \inf\{d: p(x) \leqq d\tau(|x - e_1|)\ \forall x, |x - e_1| \leqq |x|\}$$

must satisfy

$$c_2 \geqq p((\tfrac{1}{2}, 0))/\tau(\tfrac{1}{2}) = 2\mu(\sqrt{\tfrac{2}{3}})/\mu(1/\sqrt{2}) = 1.71\ldots.$$

4. Bounds for λ_G

In this section we shall prove some inequalities for the conformal invariant $\lambda_{R^n\setminus\{0\}}(x, y)$ which was defined in (1.4). Previously the exact expression for $\lambda_{B^n}(x, y)$ was found in [Vu]. We shall give also some applications of these inequalities to quasiconformal mappings.

For $x \in R^n \setminus \{0\}$ we denote by r_x a similarity with $r_x(x) = e_1$ and $|r_x(y) - e_1| = |x - y|/|x|$. It follows immediately from the definitions (1.1) and (1.4) that

$$\text{(4.1)} \qquad \lambda_{R^n\setminus\{0\}}(x, y) = \min\{p(r_x(y)), p(r_y(x))\}.$$

In particular, in the two-dimensional case one can find an explicit expression for $\lambda_{R^2\setminus\{0\}}$ applying (4.1) and the expression for $p(x)$ obtainable from [K, Theorem 5.2 p. 192].

4.2. Proof of Theorem 1.5. We may assume $|x| \leqq |y|$.
We shall first prove the lower bound. Because $|r_x(y) - e_1| = |x - y|/|x|$ and $|r_y(x) - e_1| = |x - y|/|y|$ and τ is decreasing the lower bound follows from (1.2) and (4.1).

For the proof of the upper bound let V be the $(n - 1)$-dimensional plane orthogonal to $[0, x]$ at $x/2$ and let H_0, H_x be the components of $R^n \setminus V$, $x \in H_x$. Consider two cases.

Case A. $y \in \bar{H}_x$. Because $|y| \geqq |x|$ it follows from 1.3 (2) that

$$\lambda_{R^n\setminus\{0\}}(x, y) \leqq 4\tau(|x - y|/|x|).$$

Case B. $y \in H_0$. Let $E_1 = [0, x/2]$, $E_2 = [x/2, x]$, and $F = \{x/2 + t(y - x/2): t \geqq 1\}$, $\Gamma_j = \Delta(E_j, F)$, $j = 1, 2$. Then by 3.4

$$M(\Gamma_j) \leqq \tau(2|y - x/2|/|x| - 1)$$

for $j = 1, 2$. Since $|y - x/2| \geqq \sqrt{3}|y|/2$ and $|y| \geqq |x|$ for $y \in H_0$ we obtain

$$\begin{aligned}\lambda_{R^n\setminus\{0\}}(x, y) &\leqq M(\Gamma_1) + M(\Gamma_2) \leqq 2\tau(\sqrt{3}|y|/|x| - 1)\\ &\leqq 2\tau((\sqrt{3} - 1)(|y|/|x|)) \leqq 2\tau\left(\frac{\sqrt{3} - 1}{2}\frac{|x - y|}{|x|}\right).\end{aligned}$$

By 2.9 (2) $\tau(s) \leqq 2\tau(4s)$ and hence we obtain

$$\lambda_{\mathsf{R}^n\setminus\{0\}}(x, y) \leqq 4\tau(|x-y|/|x|)$$

as desired.

4.3. REMARK. Let d be the smallest constant in Theorem 1.5 i.e.

$$d = \inf\{\alpha\colon \lambda_{\mathsf{R}^n\setminus\{0\}}(x, y) \leqq \alpha\tau(|x-y|/\min\{|x|, |y|\}) \quad \text{for all } x, y \in \mathsf{R}^n\setminus\{0\}\}.$$

Then

$$\lambda_{\mathsf{R}^n\setminus\{0\}}(x, -x) = M(\Delta([0, x], [-x, \infty))) = \tau_n(1)$$

while by the definition of d

$$\lambda_{\mathsf{R}^n\setminus\{0\}}(x, -x) \leqq d\tau_n(2).$$

In conclusion

$$d \geqq \tau_n(1)/\tau_n(2).$$

For $n = 2$ we get by (2.6) $d \geqq \mu(1/\sqrt{3})/\mu(1/\sqrt{2}) = 1.17\ldots$. It seems probable that this is the exact value of the contant for $n = 2$.

4.4. PROOF OF THEOREM 1.6. Because

$$\lambda_{\mathsf{R}^n\setminus\{0\}}(x, y) \leqq K\lambda_{\mathsf{R}^n\setminus\{0\}}(f(x), f(y))$$

holds by [Vu, 3.1] the result follows directly from Theorem 1.5.

4.5. THEOREM. *Let G be a proper subdomain of* R^n. *Then*

$$\lambda_G(x, y) \leqq \inf_{z\in\partial G} \lambda_{\mathsf{R}^n\setminus\{z\}}(x, y) \leqq 4\tau(|x-y|/m(x, y))$$

where $m(x, y) = \min\{d(x, \partial G), d(y, \partial G)\}$.

PROOF. The first inequality follows from the monotonety property of the modulus. For the second, fix $z_0 \in \partial G$ with $d(\{x, y\}, \{z_0\}) = m(x, y)$. Applying Theorem 1.5 to $\mathsf{R}^n\setminus\{z_0\}$ yields the desired result.

A closed set E in $\bar{\mathsf{R}}^n$ is said to be a *c-quasiextremal distance* or c – QED *exceptional set*, $c \in (0,1]$, if for each pair of disjoint continua $F_1, F_2 \subset \bar{\mathsf{R}}^n\setminus E$

$$M(\Delta(F_1, F_2; \bar{\mathsf{R}}^n\setminus E)) \geqq M(\Delta(F_1, F_2))c. \tag{4.6}$$

If G is a domain in $\bar{\mathsf{R}}^n$ such that $\bar{\mathsf{R}}^n\setminus G$ is a c – QED exceptional set, then we call

G a c − QED domain. Sets satisfying (4.6) have been studied by F. W. Gehring and O. Martio [GM], where also examples of sets satisfying (4.6) are given. We remark that (4.6) holds with $c = 1$, e.g. for sets of capacity zero (or more generally for sets of vanishing $(n-1)$-dimensional Hausdorff measure) and with $c = 1/(2K^2)$ if $E = \mathsf{R}^n \setminus fB^n$ and $f: \mathsf{R}^n \to \mathsf{R}^n$ is K-quasiconformal.

Next we shall prove a lower bound for $\lambda_G(x, y)$ in case G is a c − QED domain. To this end we require a variant of a well-known lemma, see [V1, 12.7], [GM, 2.6], [Vu, 2.44].

4.7. LEMMA. *Let E and F be connected disjoint sets in* R^n *with* $d(E), d(F) > 0$. *Then*

$$M(\Delta(E, F)) \geqq \tau(4m^2 + 4m) \geqq c_n \log\left(1 + \frac{1}{m}\right)$$

where c_n is a positive constant and $m = d(E, F)/\min\{d(E), d(F)\}$.

PROOF. By [V1, 10.12] we may assume that $\infty \notin \bar{E} \cap \bar{F}$. Fix $a \in \bar{E}$, $c \in \bar{F}$ with $|a - c| = d(E, F)$ and $b \in E$, $d \in F$ with $|a - b| = d(E)/2$ and $|c - d| = d(F)/2$, respectively. Applying [G2, Corollary 1, p. 226] we obtain because τ is decreasing

$$M(\Delta(E, F) \geqq \tau\left(\frac{|a-c|}{|a-b|}\cdot\frac{|b-d|}{|c-d|}\right) \geqq \tau\left(\frac{|a-c|\,(|a-b| + |a-c| + |c-d|)}{|a-b|\,|c-d|}\right)$$
$$= \tau(u).$$

Here

$$u = \frac{2d(E, F)(d(E) + 2d(E, F) + d(F))}{d(E)\,d(F)} \leqq 2m + 4m^2 + 2m$$

and the first inequality follows. The second inequality follows from [Vu, 2.14 (1)].

4.8. COROLLARY. *Let E and F be connected disjoint sets in* R^n *with* $0 < d(E) \leqq d(F)$. *Then*

$$M(\Delta(E, F)) \geqq 2^{1-n}\tau(d(E, F)/d(E)).$$

PROOF. The proof follows from 4.7 and 2.9 (1).

4.9. THEOREM. *Let G be a c* − QED *domain in* R^n. *Then*

$$\lambda_G(x, y) \geqq c\tau(s^2 + 2s) \geqq c2^{1-n}\tau(s)$$

where $s = |x - y|/\min\{d(x, \partial G), d(y, \partial G)\}$.

PROOF. Let C_x, C_y be connected sets as in the definition (1.4) with $x \in C_x$ and $y \in C_y$. Let $\Gamma_1 = \Delta(C_x, C_y; G)$ and $\Gamma_2 = \Delta(C_x, C_y)$. May assume $d(x, \partial G) \leqq$

$d(y, \partial G)$. Fix $u \in \bar{C}_x$ and $v \in \bar{C}_y$ with $|x-u| = d(x, \partial G)$ and $|y-v| = d(y, \partial G) \geqq d(x, \partial G)$. Because $|u-v| \leqq |u-x| + |x-y| + |y-v|$ we obtain by [G2, p. 226]

$$M(\Gamma_1) \geqq cM(\Gamma_2) \geqq c\tau\left(\frac{|x-y|\,|u-v|}{|x-u|\,|y-v|}\right)$$

$$\geqq c\tau\left(|x-y|\left(\frac{1}{|y-v|} + \frac{|x-y|}{|x-u|\,|y-v|} + \frac{1}{|x-u|}\right)\right) \geqq c\tau(s^2 + 2s)$$

$$\geqq c\tau(4s^2 + 4s) \geqq c2^{1-n}\tau(s),$$

where also 2.9 (1) was used in the last step. The proof follows.

If G is a proper subdomain of R^n we denote

$$r_G(x, y) = |x-y|/\min\{d(x, \partial G), d(y, \partial G)\}$$

for $x, y \in G$.

4.10. THEOREM. *Assume that $G \subsetneqq R^n$ is a $c-$ QED domain and that $f\colon G \to fG$ is K-quasiconformal with $fG \subset R^n$. Then*

$$r_{fG}(f(x), f(y)) \leqq \tau^{-1}\left(\frac{c}{2^{n+1}K}\tau(r_G(x, y))\right).$$

PROOF. The proof is similar to the proof of Theorem 1.6 except that we use now Theorems 4.5 and 4.9.

4.11. REMARKS. (1) The hypothesis that G be a $c-$ QED domain cannot be removed from 4.10. The function $f\colon B^2 \setminus [0, e_1) \to B^2 \cap \{(x, y)\colon y > 0\}$ $f(z) = \sqrt{z}$ is the desired counterexample. Indeed, let $x_j = \left(\frac{1}{2}, \frac{1}{j}\right)$, y_j $(\frac{1}{2}, -1/j)$, $j = 4, 5, \ldots$. Then $r_D(x_j, y_j) = 2$, $D = B^2 \setminus [0, e_1)$, for all $j = 4, 5, \ldots$ while $r_{fD}(f(x_j), f(y_j)) \to \infty$ as $j \to \infty$. Hence the conclusion of 4.10 cannot hold for this mapping. Note that D is not a $c-$ QED domain for any $c > 0$. Also Theorem 4.9 fails for this domain D.

(2) Theorem 4.9 and hence also Theorem 4.10 can be generalized to so-called ϕ-uniform domains (see [Vu]).

If G is a proper subdomain of R^n we set

$$j_G(x, y) = \log(1 + r_G(x, y))$$

for $x, y \in G$. Note that in [GO] a slightly different function j_G was considered. The next result follows from [GO, Theorem 4].

4.12. THEOREM. *There exist constants c and d depending only on n and K with* *the following property. If f is a K-quasiconformal mapping of* $\mathbf{R}^n$ *which maps G onto G', then*

$$j_{G'}(f(x), f(y)) \leqq c j_G(x, y) + d \tag{4.13}$$

for all $x, y \in G$.

We prove the following result.

4.14. THEOREM. *Under the hypotheses of Theorem* 4.12, *the following inequality holds for $x, y \in G$.*

$$r_{G'}(f(x), f(y)) \leqq \tau^{-1}\left(\frac{1}{4K}\tau(r_G(x, y))\right).$$

PROOF. May assume $d(f(x), \partial G') \leqq d(f(y), \partial G')$. Fix $z' \in G'$ such that $|f(x) - z'| = d(f(x), \partial G')$ and $z \in \partial G$ such that $f(z) = z'$. Then

$$\lambda_{\mathbf{R}^n \setminus \{z\}}(x, y) \leqq K \lambda_{\mathbf{R}^n \setminus \{z'\}}(f(x), f(y))$$

and hence by Theorem 1.5

$$\tau(r_G(x, y)) \leqq 4K\tau(r_{G'}(f(x), f(y)))$$

which yields the desired inequality.

Theorem 4.14 together with (4.13) yields

$$j_{G'}(f(x), f(y)) \leqq \min\{c j_G(x, y) + d, \phi(j_G(x, y))\} \tag{4.15}$$

where $\phi\colon [0, \infty) \to [0, \infty)$ is a strictly increasing function with $\phi(0) = 0$, $\phi(t) = \log\left(1 + \tau^{-1}\left(\frac{1}{4K}\tau(\exp(t) - 1)\right)\right)$, $t > 0$.

4.16. REMARK. It is clear by the proof of Theorem 4.10 (cf. 4.9) that the right side of the inequality in 4.10 can be replaced by

$$\tau^{-1}\left(\frac{c}{4K}\tau(r_D(x, y)^2 + 2r_D(x, y))\right)$$

if desired. This observation, together with the fact that the special function $t \mapsto \tau^{-1}(A\tau(t))$, $A > 0$, $t > 0$, admits dimension independent estimates, see 2.16, shows that Theorem 4.10 has a dimension-free counterpart. The same is true about Theorems 1.6 and 4.14.

5. Conformal invariance

In this section we shall give a conformally invariant version of some of the results in Sections 3 and 4. As corollaries we obtain results which are closely connected with two well-known theorems due to F. W. Gehring [G2] and S. Rickman [R], respectively. We also obtain a sharp distortion theorem for Möbius-transformations, which improves an earlier result of A. F. Beardon [B].

The spherical (chordal) metric is defined by

$$(5.1)\qquad \begin{cases} q(x,y) = |x-y|(1+|x|^2)^{-\frac{1}{2}}(1+|y|^2)^{-\frac{1}{2}}, x \neq \infty \neq y \\ q(x,\infty) = (1+|x|^2)^{-\frac{1}{2}}. \end{cases}$$

For $A \subset \bar{R}^n$, $A \neq \emptyset$, let $q(A)$ be the diameter of A and $q(A, B)$ the distance of two non-empty sets A, B in $\bar{R}^n$. The *absolute ratio* of a quadruple a, b, c, d of distinct points in R^n is defined by

$$(5.2)\qquad |a,b,c,d| = \frac{q(a,c)q(b,d)}{q(a,b)q(c,d)}$$

If all points are finite, then (5.1) yields

$$(5.3)\qquad |a,b,c,d| = \frac{|a-c|\,|b-d|}{|a-b|\,|c-d|}.$$

We also consider the maximum of two absolute ratios

$$m(a,b,c,d) = \max\{|a,b,d,c|, |a,c,d,b|\}.$$

If $G \subset \bar{R}^n$ is a domain with $\operatorname{card}(\bar{R}^n \setminus G) \geqq 2$, then let

$$m_G(b,c) = \sup\{m(a,b,c,d)\colon a, d \in \partial G\}.$$

For $D \subset \bar{R}^n$ let $GM(D)$ denote the set of all Möbius transformations f in $\bar{R}^n$ with $fD = D$. It follows that m is symmetric

$$(5.4)\qquad m(a,b,c,d) = m(a,c,b,d) = m(b,a,d,c)$$

and $GM(\bar{R}^n)$-invariant, in other words,

$$(5.5)\qquad m_f(a,b,c,d) = m(fa,fb,fc,fd) = m(a,b,c,d)$$

for all $f \in GM(\bar{R}^n)$, because the absolute ratio has this invariance property ([B, p. 32]). From (5.2) we obtain for $x, y \in R^n \setminus \{a\}, a \in R^n$,

$$(5.6)\qquad m(a,x,y,\infty) = \frac{|x-y|}{\min\{|x-a|, |y-a|\}}.$$

It follows from (5.3) and (5.5) that

$$m_G(x, y) = r_G(x, y); \; G = \mathbb{R}^n \setminus \{a\} \tag{5.7}$$

for $x, y \in G$, where r_G is as in 4.10.

Next let us consider m_G for $G \subset \bar{\mathbb{R}}^n$, $\operatorname{card}(\bar{\mathbb{R}}^n \setminus G) \geqq 2$. Clearly m_G is symmetric and $GM(\bar{\mathbb{R}}^n)$-invariant. Also the following properties are immediate

(5.8)
(1) $G_1 \subset G_2$ and $x, y \in G_1 \Rightarrow m_{G_1}(x, y) \geqq m_{G_2}(x, y)$.
(2) For a fixed $y \in G$, $m_G(x, y) \to 0$ iff $x \to y$ and $m_G(x, y) \to \infty$ iff $x \to \partial G$.
(3) $m_G(x, y) \geqq q(\partial G) q(x, y)$.
(4) $m_G(x, y) \leqq q(\partial G) q(x, y)/q(\{x, y\}, \partial G)^2$.

The Poincaré (or hyperbolic) metric ρ of B^n is defined by (see [B, p. 40])

$$\operatorname{ch} \rho(b, c) - 1 = 2 \operatorname{sh}^2 \frac{\rho(b, c)}{2} = \frac{2|b - c|^2}{(1 - |b|^2)(1 - |c|^2)}. \tag{5.9}$$

5.10. THEOREM. *$\rho(b, c) = \log(1 + m_{B^n}(b, c))$ for $b, c \in B^n$.*

PROOF. By $GM(B^n)$-invariance we may assume $b = -re_1 = -c$. Then $\rho(b, c) = 2 \log \frac{1 + r}{1 - r}$ or, equivalently, $r = \operatorname{th}(\rho(b, c)/4)$. For all $a, d \in \partial B^n$

$$m(a, b, c, d) \leqq \frac{2|b - c|}{(1 - r)^2} = \left(4 \operatorname{th} \frac{\rho}{4}\right) \Big/ \left(1 - \operatorname{th} \frac{\rho}{4}\right)^2.$$

Since $m(-e_1, -re_1, re_1, e_1) = 4r/(1 - r)^2$, it follows that

$$m_{B^n}(b, c) = \left(4 \operatorname{th} \frac{\rho}{4}\right) \Big/ \left(1 - \operatorname{th} \frac{\rho}{4}\right)^2 = e^{\rho(b, c)} - 1.$$

For f in $GM(\bar{\mathbb{R}}^n)$ let

$$\operatorname{Lip}(f) = \sup_{x \neq y} \frac{q(f(x), f(y))}{q(x, y)}.$$

Then $\operatorname{Lip}(f^{-1}) = \operatorname{Lip} f$ for all $f \in GM(\bar{\mathbb{R}}^n)$. We call f a *spherical isometry* if $\operatorname{Lip} f = 1$.

The next result was proved by A. F. Beardon [B, pp. 41–42]. It should be noted that Beardon uses $d(x, y) = 2q(x, y)$ in place of $q(x, y)$ and accordingly the constant in [B, pp. 41–42] is different from the constant 2 in 5.11.

5.11. THEOREM. *Let D be a domain in $\bar{\mathbb{R}}^n$ and let ζ and ξ be distinct points of $\bar{\mathbb{R}}^n$. If $f \in GM(\bar{\mathbb{R}}^n)$ does not assume the values ζ and ξ in D, then for all $x, y \in D$*

$$q(f(x), f(y)) \leqq \frac{2 q(x, y)}{q(\zeta, \xi) \sqrt{q(x, \partial D) q(y, \partial D)}}.$$

Moreover, the constant 2 *is best possible.*

We now prove the following sharp result.

5.12. THEOREM. *Under the assumptions of Theorem* 5.11

$$\frac{q(f(x), f(y))}{\sqrt{q(f(x), \zeta)\, q(f(y), \xi)}} \leqq \frac{\operatorname{Lip} f}{q(\zeta, \xi)} \frac{q(x, y)}{\sqrt{q(x, \partial D)\, q(y, \partial D)}}.$$

The inequality is sharp.

PROOF. Fix $a, d \in \bar{\mathsf{R}}^n \setminus D$ such that $f(a) = \zeta$ and $f(d) = \xi$. Now

$$\begin{aligned} |a, x, d, y|\,|a, y, d, x| &= \frac{q(a, d)^2\, q(x, y)^2}{q(a, x)\, q(x, d)\, q(a, y)\, q(y, d)} \\ &\leqq \frac{(\operatorname{Lip} f)^2\, q(x, y)^2}{q(a, x)\, q(y, d)\, q(fx, \xi)\, q(fy, \zeta)} \\ &\leqq \frac{(\operatorname{Lip} f)^2\, q(x, y)^2}{q(x, \partial D)\, q(y, \partial D)\, q(fx, \xi)\, q(fy, \zeta)}. \end{aligned}$$

By the $\mathrm{GM}(\bar{\mathsf{R}}^n)$-invariance of the absolute ratio, we obtain

$$|fa, fx, fd, fy|\,|fa, fy, fd, fx| = \frac{q(\zeta, \xi)^2\, q(fx, fy)^2}{q(\xi, fx)\, q(fx, \zeta)\, q(\xi, fy)\, q(fy, \xi)}.$$

These two relations together with the $\mathrm{GM}(\bar{\mathsf{R}}^n)$-invariance of the absolute ratio yield the desired conclusion.

To see the sharpness of the inequality choose $D = \mathsf{R}^n \setminus \{0\}$, $x = e_1$, $y = -e_1$, $\zeta = 0$, $\xi = \infty$, and f the identity (or the inversion in S^{n-1}).

It is clear that, in addition to being sharp, Theorem 5.12 yields a better estimate than Theorem 5.11 if $\operatorname{Lip} f < 2$.

Let a and d be distinct points in $\bar{\mathsf{R}}^n$ and $D = \bar{\mathsf{R}}^n \setminus \{a, d\}$. The next theorem is a conformally invariant version of Theorem 1.5.

5.13. THEOREM. $1 \leqq \lambda_D(b, c)/\tau(m_D(b, c)) \leqq 4$ *for* $b, c \in D$.

PROOF. The proof follows readily from (5.5) and 1.5.

5.14. THEOREM. *Let* $D \subset \bar{\mathsf{R}}^n$ *be a* c-QED *domain such that* $\operatorname{card}(\bar{\mathsf{R}}^n \setminus D) \geqq 2$. *If* $f\colon D \to fD \subset \bar{\mathsf{R}}^n$ *is* K-*quasiconformal, then for* $x, y \in D$

$$m_{fD}(f(x), f(y)) \leqq \tau^{-1}\left(\frac{c}{2^{n+1} K} \tau(m_D(x, y))\right).$$

PROOF. By the proof of Theorem 4.9

$$\lambda_D(x, y) \geqq c\tau(m_D^2 + 2m_D) \geqq c2^{1-n}\tau(m_D)$$

where $m_D = m_D(x, y)$ while

$$\lambda_{fD}(f(x), f(y)) \leqq 4\tau(m_{fD}(f(x), f(y)))$$

by 5.13 and 4.5. The proof follows now in the same way as in 4.10.

5.15. COROLLARY. *Let f: $B^n \to \bar{R}^n$ be K-quasiconformal with a, $d \in \bar{R}^n \setminus fB^n$. Then for $x, y \in B^n$*

$$q(a, d)\, q(f(x), f(y)) \leqq \tau^{-1}\left(\frac{1}{8K}\tau\left(\frac{|x-y|^2}{(1-|x|^2)(1-|y|^2}\right)\right).$$

PROOF. By [Vu, 2.23] and (5.9)

$$\lambda_{B^n}(x, y) = \frac{1}{2}\tau\left(\frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)}\right).$$

for x, $y \in B^n$. Next by (5.8) (3) and 5.13

$$\begin{aligned}\lambda_{fB^n}(f(x), f(y)) &\leqq 4\tau(m_{fB^n}(f(x), f(y)))\\ &\leqq 4\tau(q(a, d)q(f(x), f(y))).\end{aligned}$$

The proof follows from these relations as in 5.14.

5.16. COROLLARY. *Let $f: B^n \to \bar{R}^n$ be a K-quasimeromorphic mapping, let a, $d \in \bar{R}^n \setminus fB^n$ be distinct, and suppose that there exists $p \in [1, \infty)$ such that* card $\{f^{-1}(y)\} \leqq p$ *for all $y \in \bar{R}^n$. Then*

$$\frac{q(a, d)\, q(f(x), f(y))}{q(a, f(x))\, q(f(y), d)} \leqq \tau^{-1}\left(\frac{1}{8Kp}\tau\left(\frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)}\right)\right)$$

for all x, $y \in B^n$.

PROOF. Clearly

$$m_{fB^n}(f(x), f(y)) \geqq |a, f(x), d, f(y)|.$$

The proof follows now as in 5.15, except that we apply now the inequality $\lambda_{B^n}(x, y) \leqq Kp\, \lambda_{fB^n}(f(x), f(y))$ [Vu, 3.1 (2)].

In view of Theorem 5.10 we may regard Theorem 5.14 as a sort of Schwarz lemma for general domains. Corollaries 5.15 and 5.16 are close to the results in [G2, Theorem 1, p. 233] and [R, Theorem 4.4], respectively. In view of 2.15 and 2.16 these results provide some dimension-free distortion estimates. See also [V2] for some conformally invariant results.

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