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EXISTENCE THEOREMS FOR MEASURES ON CONTINUOUS POSETS, WITH APPLICATIONS TO RANDOM SET THEORY

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Abstract.

We state conditions on a partially ordered set (poset) L and a mapping λ, defined on a class $\mathscr{F}_c$ of filters on L, under which λ extends to a measure on the minimal σ-field over $\mathscr{F}_c$.

By applying this extension result to the case when L is a continuous lattice, all locally finite measures on L are identified as well as all Lévy-Khinchin measures. We then characterize these kinds of measures on continuous (inf-) semilattices and continuous posets. An interesting correspondence between Lévy-Khinchin measures and inf-infinitely divisible probability measures is presented.

The correspondence between probability measures on the line and distribution functions is a particular case of this result. So is also Choquet's characterization of the distributions of all random closed sets in a fixed locally compact second countable Hausdorff space S. Our approach to Choquet's theorem show that it holds as soon as the topology of S is continuous, second countable and sober. Our method also yields characterizations of the distributions of all random compact and all random compact convex sets in R^d for finite d. We furthermore obtain characterizations of infinite divisibility under union and sup, resp. for these kinds of random sets.

The embedding $s \to \{s\}^-$, $s \in S$, enables us to give simple proofs of existence theorems for finite and locally finite (i.e., Radon) measures on S. In the final section we give a simple proof of the Daniell-Kolmogorov existence theorem for probability measures on (countable) products of continuous lattices.

1. Introduction.

Although an extensive part of this paper is devoted to an existence theorem for measures on partially ordered sets (posets for short), we believe that the most interesting part is its application to continuous lattices and some consequences thereof. So let us begin with a description of the part.

Let L be a continuous poset and assume its Scott topology to be second countable. Write Σ, or $\Sigma(L)$, for the minimal σ-field over the sets $\uparrow x = \{y \in L; x \leqq y\}$, $x \in L$. We shall see below that Σ is the Borel-σ-field wrt. the Scott topology. Indeed Σ is generated by the collection $\mathscr{L}$ of Scott open filters on

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L. Say that a measure λ on (L, Σ) is *locally finite* if it is so wrt. the Scott topology, i.e. if

$$\lambda(\uparrow x) < \infty, \quad x \in L.$$

Assume that L is a lattice. Then the formula

$$(1.1) \qquad \Lambda(x) = \lambda(\uparrow x), \quad x \in L,$$

defines a bijection between the family of locally finite measures λ on L and the family of non-negative mappings Λ on L satisfying

$$(1.2) \qquad \Lambda(x) = \lim_n \Lambda(x_n), \quad x, x_1, x_2, \ldots \in L, x_n \uparrow x,$$

$$(1.3) \qquad \Lambda_n(x; x_1, \ldots, x_n) \geqq 0, \quad n \in N, x, x_1, \ldots, x_n \in L.$$

Here, and in the following, $x_n \uparrow x$ means that $x_1 \leqq x_2 \leqq \ldots \leqq x = \vee_n x_n$ ($x_n \downarrow x$ should be interpreted analogously), $\mathsf{N} = \{1, 2, \ldots\}$ and the left hand side of (1.3) is recursively defined by putting

$$\Lambda_1(x; x_1) = \Lambda(x) - \Lambda(x \vee x_1)$$

and letting it equal

$$\Lambda_{n-1}(x; x_1, \ldots, x_{n-1}) - \Lambda_{n-1}(x \vee x_n; x_1, \ldots, x_{n-1})$$

if $n \geqq 2$.

Clearly any function satisfying (1.3) is decreasing. It is not hard to produce a counterexample showing that decreasing functions need not satisfy (1.3). Howeer, if L is a chain, i.e. totally ordered, then all decreasing functions satisfy (1.3).

Let λ be a measure on L. Then

$$\lambda(L) = \sup_{x \in L} \lambda(\uparrow x).$$

Thus as a particular case of (1.1) we obtain a characterization of the probability measures on L in terms of *distribution functions*. These are the non-negative functions Λ on L that satisfy (1.2), (1.3) and

$$(1.4) \qquad \sup_{x \in L} \Lambda(x) = 1.$$

Note that if L has a bottom 0, which holds iff L is a continuous complete lattice, then

$$\Lambda(x) \leqq \Lambda(0), \quad x \in L.$$

In this case, (1.1) characterizes all finite measures on L.

Some of the lattices to which we shall apply (1.1) are continuous under the reverse order. In this case (1.1) takes the form

$$\Lambda(x) = \lambda\{y \in L; y \leqq x\}, \quad x \in L.$$

We talk about distribution functions in this case too, though their definition is omitted.

All existence theorems for measures on continuous posets that are proved in this paper require the Scott topology to be second countable. Below we shall give a useful characterization of this technical condition. Let us just note here that a continuous topology has a second countable Scott topology iff it is second countable in itself.

The real line R is a lattice which is not continuous. But $(-\infty, \infty]$ is one. So is also $[-\infty, \infty)$ under the reverse order. Both $(-\infty, \infty]$ and $[-\infty, \infty)$ have second countable Scott topologies. Thus, as a special case of (1.1), we obtain the well-known correspondence between measures μ on the line satisfying

$$\mu(-\infty, x] < \infty, \quad x \in \mathsf{R},$$

and increasing right-continuous functions F on R satisfying

$$\lim_{x \to -\infty} F(x) = 0,$$

which by straightforward methods extends to the correspondence between Lebesgue-Stieltjes measures on the line and increasing right-continuous functions.

The correspondending results in higher dimensions follow from the fact that L^n is a continuous lattice with a second countable Scott topology under the coordinatewise order, $(x_1, \ldots, x_n) \leqq (y_1, \ldots, y_n)$ iff $x_i \leqq y_i$, $1 \leqq i \leqq n$. Note also that this holds for the infinite product L^N if L has a bottom, i.e. if L is a continuous complete lattice. This fact enables us to give a simple derivation of the Daniell-Kolmogorov existence theorem for probability measures on products of continuous complete lattices.

We describe a special case of (1.1) which we mean is important. Let S be a locally compact second countable Hausdorff space. Write $\mathscr{F}$ for its complete lattice of closed sets, which is continuous under reverse inclusion, $F_1 \leqq F_2$ iff $F_2 \subset F_1$. Note that $\mathscr{F}$, with this order, is isomorphic to the lattice of open sets in S. Thus the Scott topology on $\mathscr{F}$ is second countable.

By a *random closed set* in S we understand an $\mathscr{F}$-valued mapping ξ, defined on some probability space $(\Omega, \mathscr{R}, P)$, which satisfies

$$\{\xi \cap K \neq \varnothing\} = \{w \in \Omega; \xi(w) \cap K \neq \varnothing\} \in \mathscr{R}, \quad K \subset S \text{ compact},$$

cf. Matheron (1975). It is easily seen that this requirement for measurability is equivalent to

$$\{\xi \subset F\} \in \mathscr{R}, \quad F \in \mathscr{F}.$$

It follows that the formula

$$P\{\xi \subset F\} = \Lambda(F), \quad F \in \mathscr{F}, \tag{1.5}$$

defines a bijection between the family of distributions $P\xi^{-1}$ of random closed sets ξ in S and the family of distribution functions Λ on $\mathscr{F}$. Choquet (1953) characterizes these distributions in terms of certain "alternating capacities" or "hitting probabilities" T defined by

$$T(K) = P\{\xi \cap K \neq \varnothing\}, \quad K \subset S \text{ compact.}$$

It is not hard to see that our result implies and is implied by this result of Choquet.

But note that (1.5) is a bijection as we asserted above as soon as the topology on S is continuous and second countable. In particular it is irrelevant whether S has a Hausdorff topology or not.

It is not possible to extend Choquet's characterization of the probability measures on $\mathscr{F}$ in terms of alternating capacities to all spaces having a continuous second countable topology. However we can come close to this generality. Below we shall see that Choquet's characterization holds on all spaces having a continuous, second countable and sober topology. But note that in the absence of the Hausdorff separation property the hitting probabilities $P\{\xi \cap K \neq \varnothing\}$ must be defined on subsets $K \subset S$ that are not only compact but also saturated (a set in a topological space is *saturated* if it coincides with the intersection of its open neighborhoods).

This extension of Choquet's theorem is a particular case of a characterization of the locally finite measures on L in terms of functions on its collection $\mathscr{L}$ of Scott open filters, which is valid as soon as L is an (inf-) semi-lattice with a top.

We shall also characterize this class of locally finite measures without further assumptions on L. This time the characterization is in terms of "additive" mappings defined on the Scott topology of L.

We mention a few more continuous semi-lattices to which our results apply. Let S be a space having a continuous, second countable and sober topology. The collection of all compact and saturated subsets of S is a continuous semi-lattice under reverse inclusion. (Note that $K_1 \wedge K_2 = K_1 \cup K_2$.) It is a lattice if S is Hausdorff. The collection of all extended real-valued lower semicontinuous functions on S is a continuous complete lattice under the pointwise order. Both examples are discussed in Giertz, Hofmann, Keimel, Lawson, Mislove & Scott (1980). Gerritse (1985) discusses an extension of the latter to lattice-valued functions.

The family $\mathscr{C}$ of compact and convex sets in R^d, equipped with reverse inclusion, is a continuous lattice for $d \in \mathsf{N}$. We regard $\varnothing$ as being convex. Cf. Giertz et al. (1980). We shall see that $\mathscr{C}$ has a second countable Scott topology. It will then follow by (1.1) that the formula

$$P\{\xi \subset C\} = \Lambda(C), \ C \in \mathscr{C}, \tag{1.6}$$

defines a bijection between the family of distributions $P\xi^{-1}$ of random variables ξ in $\mathscr{C}$ and the family of distribution functions Λ on $\mathscr{C}$. (By a random variable in a measurable space Ω we understand an Ω-valued measurable mapping of some probability space. Its distribution is the induced probability measure on Ω.)

Let ξ be a random variable in $\mathscr{C}$ with distribution function Λ. Then ξ is non-empty with probability one iff

$$\Lambda(\varnothing) = 0. \tag{1.7}$$

Many authors define a random compact convex set in R^d to be a mapping of some probability space into $\mathscr{C}' = \mathscr{C} \setminus \{\varnothing\}$, which is measurable wrt. the Borel-σ-field on $\mathscr{C}'$ generated by the Hausdorff metric. We shall see below that this σ-field is the minimal σ-field over the sets $\{D \in \mathscr{C}'; D \subset C\}, C \in \mathscr{C}'$, i.e. the relativization of $\Sigma(\mathscr{C})$ to $\mathscr{C}'$. Hence ξ is a random compact convex set iff (1.7) holds. Note also that, clearly, any random compact convex set is also a random variable (in our sense) in $\mathscr{C}$.

We thus get, as a particular case of (1.6), a bijection between the family of distributions $P\xi^{-1}$ of random compact convex sets ξ in R^d and the family of distribution functions Λ on $\mathscr{C}$ satisfying (1.7). This solves a problem which has been open since 1981, when Vitale and Trader & Eddy showed that the distribution of a random compact set in R^d is determined by its distribution function. See Vitale (1981) and Trader & Eddy (1982). Both papers use Banach space methods, while our proof of (1.6) is purely lattice theoretical. We anticipate that lattice theoretical methods in the near future will turn out to be useful in stochastic geometry.

Indeed we can already now announce a Lévy-Khinchin representation of the distributions of those random variables ξ in $\mathscr{C}$ that are infinitely divisible in the sense that, for each $n \in \mathsf{N}$, there are independent and identically distributed random variables $\xi_1, \ldots, \xi_n$ in $\mathscr{C}$ satisfying

$$\xi \stackrel{d}{=} \vee_i \ \xi_i.$$

Here $\stackrel{d}{=}$ denotes equality in distribution and $\vee_i \ \xi_i$ is the convex closure of $\cup_i \xi_i$.

This representation result is a special case of a characterization of the distributions of those random variables ξ in L which are infinitely wrt. the meet $\wedge$. We continue to describe the class of measures that appear in the characterization.

Assume that L has a top 1, nothing else (except that L is continuous and has a second countable Scott topology). By a Lévy-Khinchin measure on L we understand a measure ψ on L which concentrates its mass to $L \setminus \{1\}$ and satisfies

$$\psi(L \setminus F) < \infty, F \in \mathscr{L}.$$

If L is a semi-lattice with a top, then so is $\mathscr{L}$, and the formula

(1.8) $$\Psi(F) = \psi(L\backslash F),\ F \in \mathscr{L},$$

defines a bijection between the family of Lévy-Khinchin measures ψ on L and the family of non-negative mappings Ψ on $\mathscr{L}$ satisfying

(1.9) $$\Psi(L) = 0,$$

(1.10) $$\Psi(F) = \lim_n \Psi(F_n),\ F, F_1, F_2, \ldots \in \mathscr{L},\ F_n \uparrow F,$$

(1.11) $$\Delta_{F_n} \cdots \Delta_{F_1} \Psi(F) \leqq 0,\ n \in \mathrm{N},\ F, F_1, \ldots, F_n \in \mathscr{L}.$$

The left hand side of (1.11) is recusively defined by letting it equal

$$\Delta_{F_{n-1}} \cdots \Delta_{F_1} \Psi(F) - \Delta_{F_{n-1}} \cdots \Delta_{F_1} \Psi(F \cap F_n)$$

if $n \geqq 2$ and putting

$$\Delta_{F_1} \Psi(F) = \Psi(F) - \Psi(F \cap F_1).$$

Analogously we define $\Delta_{x_n} \ldots \Delta_{x_1} c(x)$ whenever c is a real-valued function defined on a semi-lattice.

For some L the members of $\mathscr{L}$ are hard to identify or difficult to describe. Thus there is a need for a characterization of the Lévy-Khinchin measures on L in terms of functions defined on L. Not surprisingly this can be done if L is lattice. We shall also give a characterization of this class of measures in terms of mappings defined at Scott open subsets of L. The latter characterization is valid with no further assumptions on L (except that L has a top).

Let S be a space equipped with a continuous second countable sober topology. Due to the sobriety, the mapping $s \to \{s\}^-$ maps S one-to-one onto the collection of all irreducible closed subsets of S. We shall use this embedding of S into its collection $\mathscr{F}$ of closed sets, to derive existence theorems for finite and locally finite measures on S, from existence theorems for finite and Lévy-Khinchin measures on $\mathscr{F}$.

We proceed to describe the contents of the various sections of this paper. In Section 2 we prove an existence theorem for measures on posets, from which all the other results of this paper follow more or less easy. It might be a good idea to skip this section at the first reading of this paper. Section 3 contains our characterizations of the locally finite measures on L under various presumptions on the latter, while Section 4 contains our characterizations of the Lévy-Khinchin measures on L. In Section 5 we derive the Lévy-Khinchin representation of the inf-infinitely divisible probability measures on L. Our extension of Choquet's existence theorem for random set distributions is given in Section 6. Here we also fill in those details in the discussion above on random set theory that are neither obvious nor proved elsewhere. In Section 7 we derive existence theorems for finite and locally finite measures on any space equipped with a continuous, second countable and sober topology, and in the final Section 8 we present a simple proof of the Daniell-Kolmogorov existence theorem for probability measures on countable products of continuous complete lattices.

Regarding the terminology let us just note here that $\mathsf{R}_+ = [0, \infty)$ and $\bar{\mathsf{R}}_+ = \mathsf{R}_+ \bigcup \{\infty\}$.

2. An existence theorem for measures on posets.

Let L be a poset and consider a non-negative mapping λ defined on a collection $\mathscr{F}_c$ of filters on L, which is closed under non-empty countable intersections. Assume

$$\lambda(K) = \lim_n \lambda(K_n), \quad K, K_1, K_2, \ldots \in \mathscr{F}_c,\ K_n \downarrow K, \tag{2.1}$$

$$\Delta_{K_1} \ldots \Delta_{K_n} \lambda(K) \geqq 0, \quad n \in N,\ K,\ K_1, \ldots, K_n \in \mathscr{F}_c. \tag{2.2}$$

We furthermore presumes the existence of a collection $\mathscr{F}_o$ of filters on L, which is closed under finite non-empty intersections and is "dual" to $\mathscr{F}_c$ in the following sense: Whenever $K \in \mathscr{F}_c$ and $G \in \mathscr{F}_o$, there are $K_1, K_2, \ldots \in \mathscr{F}_c$ and $G_1, G_2, \ldots \in \mathscr{F}_o$ satisfying $K_n \uparrow G$ and $G_n \downarrow K$. Moreover, if $K \subset G$ and $K_n \uparrow G$, where K, K_1, K_2, $\ldots \in \mathscr{F}_c$ and $G \in \mathscr{F}_o$, then $K \subset K_n$ for some n, and if $K_n \downarrow K \subset \bigcup_n G_n$, where K, K_1, K_2, $\ldots$ $\in \mathscr{F}_c$ and G_1, G_2, $\ldots$ $\in$ $\mathscr{F}_o$, then $K_m \subset \bigcup_{n \leqq m} G_n$ for some m. Finally, whenever $G \in \mathscr{F}_o$ we have $G \subset K$ for some $K \in \mathscr{F}_c$. Then

THEOREM 2.1. *λ extends to a measure on the minimal σ-field over $\mathscr{F}_c$. This extension is unique if $L = \bigcup_n K_n$ for some $K_1, K_2, \ldots \in \mathscr{F}_c$.*

Our proof of this theorem is given in a series of lemmata and propositions, some of which we believe have independent value.

Write

$$\mathscr{F} = \{K \bigcap G;\ K \in \mathscr{F}_c,\ G \in \mathscr{F}_o\}.$$

Then clearly, $\mathscr{F}$ is closed under finite non-empty intersections. Our first lemma depends only in this fact. Put

$$\mathscr{S} = \{F \setminus \bigcup \mathscr{A};\ F \in \mathscr{F},\ \mathscr{A} \subset \mathscr{F} \text{ finite}\}.$$

Then

LEMMA 2.2. *$\mathscr{S}$ is a semi-ring. It is a semi-algebra if $L \in \mathscr{F}$.*

We do not hesitate to leave the proof of Lemma 2.2 to the reader.

Any representation $F \setminus \bigcup \mathscr{A}$ of a non-empty member S of $\mathscr{S}$ will be called *reduced* if $\mathscr{A} = \varnothing$ or if

$$A \subset F \text{ for all } A \in \mathscr{A}, \tag{2.3}$$

$$A_1 = A_2 \text{ whenever } A_1, A_2 \in \mathscr{A},\ A_1 \subset A_2. \tag{2.4}$$

We shall show that every non-empty member of S has a unique reduced representation. We begin with two lemmata which also will be needed later in this section.

LEMMA 2.3. *Let $E, F \in \mathscr{F}$ and let $\mathscr{A} \subset \mathscr{F}$ be finite. If $\varnothing \neq F \setminus \bigcup \mathscr{A} \subset E$, then $F \subset E$.*

PROOF. Let $x \in F$. We must prove that $x \in E$. If $x \notin \bigcup \mathscr{A}$ this is obvious, so we assume $x \in A$ for some $A \in \mathscr{A}$. Choose $y \in F \setminus \bigcup \mathscr{A}$. Then there exists some $z \in F$ with $z \leqq x$, $z \leqq y$. If $z \in \bigcup \mathscr{A}$, then $y \in \bigcup \mathscr{A}$. This is not true. Hence $z \notin \bigcup \mathscr{A}$. But then $z \in E$ and $x \in E$ follows.

LEMMA 2.4. *Let $F \in \mathscr{F}$ and let $\mathscr{A} \subset \mathscr{F}$ be finite. If $F \subset \bigcup \mathscr{A}$, then $F \subset A$ for some $A \in \mathscr{A}$.*

PROOF. Clearly $\mathscr{A} \neq \varnothing$. Suppose we may choose $x_A \in F \setminus A$ for all $A \in \mathscr{A}$. We may then choose $x \in F$ such that $x \leqq x_A$, $A \in \mathscr{A}$. By assumption then $x \in A$ for some $A \in \mathscr{A}$. It follows that $x_A \in A$. This is a contradiction, from which the lemma follows.

PROPOSITION 2.5. *Every non-empty member S of $\mathscr{S}$ has a reduced representation, and if $F \setminus \bigcup \mathscr{A}$ and $E \setminus \bigcup \mathscr{B}$ are two reduced representations of S, then $F = E$ and $\mathscr{A} = \mathscr{B}$.*

PROOF. Assume $\varnothing \neq S = F \setminus \bigcup \mathscr{A}$, where $F \in \mathscr{F}$ and $\mathscr{A} \subset \mathscr{F}$ is finite. Suppose $\mathscr{A}$ is non-empty. Note that

$$S = F \setminus \bigcup (\mathscr{A} \cap F),$$

so we may assume (2.3). Fix $A \in \mathscr{A}$, and assume $A \subset A'$ for some $A' \in \mathscr{A}$, $A' \neq A$. Then

$$\bigcup \mathscr{A} = \bigcup (\mathscr{A} \setminus \{A\}).$$

Thus we may also assume (2.4). In other words, S has a reduced representation.

Let us now assume that both $F \setminus \bigcup \mathscr{A}$ and $E \setminus \bigcup \mathscr{B}$ are reduced representations of S. Then

$$F \setminus \bigcup \mathscr{A} \subset E.$$

By Lemma 2.3, $F \subset E$. By symmetry, $F = E$. But then $\bigcup \mathscr{A} = \bigcup \mathscr{B}$. Thus if $\mathscr{A}$ or $\mathscr{B}$ is empty, so is the other. Fix $A_1 \in \mathscr{A}$. By Lemma 2.4, $A_1 \subset B_1$ for some $B_1 \in \mathscr{B}$.

Similarly, $B_1 \subset A'_1$ for some $A'_1 \in \mathscr{A}$. But then $A_1 \subset A'_1$ and we conclude by (2.4) that $A_1 = A'_1$. Hence $A_1 = B_1$. This shows that $\mathscr{A} \subset \mathscr{B}$. Now $\mathscr{A} = \mathscr{B}$ follows by symmetry.

Note that the results so far do not depend on our particular choice of $\mathscr{F}$. They hold for every collection $\mathscr{F}$ of filters on L which is closed under finite non-empty intersections.

Now we extend λ to $\mathscr{F}$:

$$\lambda(F) = \sup\{\lambda(K);\ K \in \mathscr{F}_c,\ K \subset F\},\ F \in \mathscr{F}.$$

It is not hard to see that the set on the right is non-empty for every $F \in \mathscr{F}$. Moreover, λ is still real-valued. We first study the continuity properties of λ. We begin with two lemmata.

LEMMA 2.6. *Whenever $F, F_1, F_2 \in \mathscr{F}$, we have*

(i) $\lambda(F \cap F_1) \leqq \lambda(F)$,

(ii) $\lambda(F \cap F_1) + \lambda(F \cap F_2) \leqq \lambda(F) + \lambda(F \cap F_1 \cap F_2)$.

PROOF. Clearly λ is increasing. Hence (i). Choose $K_1 \subset F \cap F_1$ and $K_2 \subset F \cap F_2$. Then $K_1 \cap K_2 \subset F \cap F_1 \cap F_2$, and $K_1 \cup K_2 \subset F$. It is not hard to see that $K_1 \cup K_2 \subset K \subset F$ for some $K \in \mathscr{F}_c$. By (2.2),

$$\lambda(K_1) + \lambda(K_2) = \lambda(K \cap K_1) + \lambda(K \cap K_2)$$
$$\leqq \lambda(K) + \lambda(K \cap K_1 \cap K_2) \leqq \lambda(F) + \lambda(F \cap F_1 \cap F_2).$$

Now (ii) is obvious.

LEMMA 2.7. *Let $E_i, F_i \in \mathscr{F}$ for $1 \leqq i \leqq n$. If $E_i \subset F_i$ for all i, then $\lambda(\bigcap_{i=1}^n F_i) - \lambda(\bigcap_{i=1}^n E_i) \leqq \sum_{i=1}^n (\lambda(F_i) - \lambda(E_i))$.*

PROOF. The case $n = 1$ is trivial. Suppose $n = 2$. Put $D = F_1$, $D_1 = E_1$ and $D_2 = E_2$. By Lemma 2.6,

$$\lambda(E_1) + \lambda(F_1 \cap E_2) \leqq \lambda(F_1) + \lambda(E_1 \cap E_2).$$

Then put $D = F_2$, $D_1 = F_1$ and $D_2 = E_2$, and conclude that

$$\lambda(F_1 \cap F_2) + \lambda(E_2) \leqq \lambda(F_2) + \lambda(F_1 \cap E_2).$$

Add these expressions and cancel $\lambda(F_1 \cap E_2)$ from both sides. Conclude that the lemma is true if $n = 2$.

Now suppose we have proved the lemma for all $n \leqq m$, where $m \geqq 2$. Then

$$\begin{aligned}&\lambda(\textstyle\bigcap_{i=1}^{m+1} F_i) - \lambda(\bigcap_{i=1}^{m+1} E_i)\\ &\leqq \lambda(\textstyle\bigcap_{i=1}^{m} F_i) - \lambda(\bigcap_{i=1}^{m} E_i) + \lambda(F_{m+1}) - \lambda(E_{m+1})\\ &\leqq \textstyle\sum_{i=1}^{m+1} (\lambda(F_i) - \lambda(E_i)).\end{aligned}$$

Hence the lemma is then true for $n = m + 1$. By induction the lemma is true for all $n \in N$.

PROPOSITION 2.8. *Let $F, F_1, F_2, \ldots \in \mathscr{F}$ and suppose $F_n \downarrow F$. Then*

$$\lambda(F) = \lim_n \lambda(F_n).$$

PROOF. Clearly $\lambda(F) \leqq \lim_n \lambda(F_n) = \alpha$. Fix $\varepsilon > 0$. For every n choose $K_n \in \mathscr{F}_c$ such that $K_n \subset F_n$ and

$$\lambda(F_n) - \lambda(K_n) \leqq \varepsilon \cdot 2^{-n}.$$

By Lemma 2.7,

$$\lambda(F_n) - \lambda(\textstyle\bigcap_{i=1}^{n} K_i) \leqq \varepsilon.$$

Hence $\alpha \leqq \lambda(\bigcap_n K_n) + \varepsilon \leqq \lambda(F) + \varepsilon$.

PROPOSITION 2.9. *Let $F \in \mathscr{F}$, let $G \in \mathscr{F}_o$ and assume $K_n \uparrow G$ for some $K_1, K_2, \ldots \in \mathscr{F}_c$. Then*

$$\lambda(F \cap G) = \lim_n \lambda(F \cap K_n).$$

PROOF. Clearly $\lambda(F \cap K_n) \uparrow \lim_n \lambda(F \cap K_n) \leqq \lambda(F \cap G)$. Let $\alpha < \lambda(F \cap G)$. Then $\alpha \leqq \lambda(K')$ for some $K' \in \mathscr{F}_c$, $K' \subset F \cap G$. But then $K' \subset K_n$ for some n. Hence $K' \subset F \cap K_n$ and $\alpha \leqq \lambda(F \cap K_n)$ follows.

We write down a particular case of Proposition 2.9:

$$\lambda(K \cap G) = \lim_n \lambda(K \cap K_n), G \in \mathscr{F}_o, \ K, K_1, K_2, \ldots \in \mathscr{F}_c, \ K_n \uparrow G.$$

PROPOSITION 2.10. *Let $F \in \mathscr{F}$, $K \in \mathscr{F}_c$ and $G \in \mathscr{F}_o$. Choose $K_1, K_2, \ldots \in \mathscr{F}_c$ and $G_1, G_2, \ldots \in \mathscr{F}_o$ such that $K_n \uparrow G$ while $G_n \downarrow K$. Then*

$$\lambda(F \cap K \cap G) = \lim_n \lambda(F \cap K_n \cap G_n).$$

PROOF. Note that

$$\lambda(F \cap K \cap K_n) \leqq \lambda(F \cap K_n \cap G_n) \leqq \lambda(F \cap G \cap G_n).$$

Let $n \to \infty$. By Proposition 2.9, the left most side tends to $\lambda(F \cap K \cap G)$. But so does also the right most side by Proposition 2.8.

Fix $F, F_1, F_2 \in \mathscr{F}$. The reader easily shows that

$$\Delta_{F_2} \Delta_{F_1} \lambda(F) = \lambda(F) - \lambda(F \cap F_1) - \lambda(F \cap F_2) + \lambda(F \cap F_1 \cap F_2).$$

By symmetry,

$$\Delta_{F_2}\Delta_{F_1}\lambda(F) = \Delta_{F_1}\Delta_{F_2}\lambda(F).$$

We furthermore see that

$$\Delta_{F_1}\Delta_{F_2}\lambda(F) = \Delta_{F_2}\lambda(F)$$

if $F_1 \subset F_2$. This shows that $\Delta_{F_n}\dots\Delta_{F_1}$, for $n \in N$ and $F_1, \dots, F_n \in \mathscr{F}$, only depend on the set $\mathscr{A} = \{F_1, \dots, F_n\}$. Accordingly, we sometimes write $\Delta_{\mathscr{A}}\lambda(F)$ instead of the more cumbersome $\Delta_{F_1}\dots\Delta_{F_n}\lambda(F)$. For convenience we further put $\Delta_{\varnothing}\lambda(F) = \lambda(F)$.

We now show that the real number $\Delta_{\mathscr{A}}\lambda(F)$ only depends on the member $F\setminus\bigcup\mathscr{A}$ of $\mathscr{S}$ – not on its representation.

LEMMA 2.11. *If $F\setminus\bigcup\mathscr{A}$ and $E\setminus\bigcup\mathscr{B}$ are two representations of a member of $\mathscr{S}$, then*

$$\Delta_{\mathscr{A}}\lambda(F) = \Delta_{\mathscr{B}}\lambda(E).$$

PROOF. If $F\setminus\bigcup\mathscr{A} = \varnothing$ then $F \subset \bigcup\mathscr{A}$ and, by Lemma 2.4, $F \subset A$ for some $A \in \mathscr{A}$. Then, as the reader easily shows, $\Delta_{\mathscr{A}}\lambda(F) = 0$. Now suppose $F\setminus\bigcup\mathscr{A} \neq 0$. Note that

$$\Delta_{\mathscr{A}}\lambda(F) = \Delta_{\mathscr{A}\cap F}\lambda(F),$$

and that if A, $A' \in \mathscr{A}$, $A \subset A'$ and $A \neq A'$, then

$$\Delta_{\mathscr{A}}\lambda(F) = \Delta_{\mathscr{A}\setminus\{A\}}\lambda(F).$$

The proofs of these facts are easy. They show that we may change $F\setminus\bigcup\mathscr{A}$ into its reduced representation without changing the value of $\Delta_{\mathscr{A}}\lambda(F)$. In view of Proposition 2.5, the lemma is now obvious.

Thus we may extend λ to the semi-ring $\mathscr{S}$ by putting

$$\lambda(F\setminus\bigcup\mathscr{A}) = \Delta_{\mathscr{A}}\lambda(F),\ F \in \mathscr{F},\ \mathscr{A} \subset \mathscr{F} \text{ finite}.$$

PROPOSITION 2.12. *λ is an additive mapping on $\mathscr{S}$.*

PROOF. Fix S, $T \in \mathscr{S}$ such that $S \bigcap T = \varnothing$ while $S \bigcup T = \mathscr{S}$. Of course

$$\lambda(S \bigcup T) = \lambda(S) + \lambda(T) \tag{2.5}$$

if S or T are empty, so let us assume both to be non-empty. We further let $F\setminus\bigcup\mathscr{A}$, $E\setminus\bigcup\mathscr{B}$ and $D\setminus\bigcup\mathscr{C}$ be reduced representation of S, T and $S\bigcup T$, resp.

By Lemma 2.3, $F \bigcup E \subset D$. Clearly $F \bigcap E \subset \bigcup(\mathscr{A} \bigcup \mathscr{B})$. By Lemma 2.4, $F \bigcap E$ is included in some member of $\mathscr{A} \bigcup \mathscr{B}$. Let us assume

$$(2.6) \qquad F \bigcap E \subset A,$$

where $A \in \mathscr{A}$, and prove $F = D$. (Similarly the reader may prove that $F \bigcap E \subset \bigcup \mathscr{B}$ implies $E = D$.)

If $D \subset E$, then $F \subset E$ and $F \subset A$ follows by (2.6). This is impossible. Hence we may select a point $x \in D \setminus E$. In order to obtain a contradiction, suppose there is a point $y \in D \setminus F$. For each $C \in \mathscr{C}$, choose $x_C \in D \setminus C$. Then choose $z \in D$ such that $z \leqq x$, $z \leqq y$ and $z \leqq x_C$ for $C \in \mathscr{C}$. If $z \in C$ for some $C \in \mathscr{C}$, then $x_C \in C$. A contradiction. Thus $z \in S \bigcup T \subset F \bigcup E$. However this implies that $y \in F$ or $x \in E$. This we cannot have, and it follows that $D \subset F$. Hence $F = D$ as claimed above.

But then $E \subset F$. By (2.6), $E \subset A$. Hence $T \subset A$. This implies $(S \bigcup T) \setminus A = S$, and $(S \bigcup T) \bigcap A = T$. We thus have

$$S = D \setminus (\bigcup \mathscr{C} \bigcup A),$$

$$T = D \bigcap A \setminus \bigcup \mathscr{C}.$$

Hence

$$\lambda(S) + \lambda(T) = \Delta_{\mathscr{C} \cup \{A\}} \lambda(D) + \Delta_{\mathscr{C}} \lambda(D \bigcap A) = \Delta_{\mathscr{C}} \lambda(D) = \lambda(S \bigcup T).$$

Thus (2.5) holds if (2.6) is at hand. The remaining case is completely similar.

PROPOSITION 2.13. *λ is a non-negative mapping on $\mathscr{S}$.*

PROOF. Let $S = F \setminus \bigcup \mathscr{A} \in \mathscr{S}$. There is nothing to prove if $\mathscr{A}$ is empty, so assume not. Suppose $F = K_0 \bigcap G_0$ and $\mathscr{A} = \{K_i \bigcap G_i;\ 1 \leqq i \leqq m\}$, where $K_i \in \mathscr{F}_c$, $G_i \in \mathscr{F}_o$, $0 \leqq i \leqq m \in \mathrm{N}$. For $0 \leqq i \leqq m$ choose $K_{i1}, K_{i2}, \ldots \in \mathscr{F}_c$ such that $K_{in} \uparrow G_i$. If $0 \in I \subset \{0, 1, \ldots, m\}$ then $\bigcap_{i \in I} K_{in} \uparrow \bigcap_{i \in I} G_i$. Put $\mathscr{B}_n = \{K_i \bigcap K_{in};\ 1 \leqq i \leqq m\}$. By (2.2)

$$\Delta_{\mathscr{B}_n} \lambda(K_0 \bigcap K_{0n}) \geqq 0.$$

By Proposition 2.9, the left hand side of this inequality tends to $\Delta_{\mathscr{A}} \lambda(F)$.

Now note that λ extends to a measure on the σ-field generated by $\mathscr{F}$ if λ is countably subadditive on $\mathscr{S}$, i.e. if

$$(2.7) \qquad \lambda(\bigcup_n S_n) \leqq \sum_n \lambda(S_n), \quad S_1, S_2, \ldots, \bigcup_n S_n \in \mathscr{S}.$$

See e.g. Theorem 11.3 in Billingsley (1979).

We extend λ additively to the ring $\mathscr{R}$ generated by $\mathscr{S}$. It is not hard to see that (2.7) follows from

$$(2.8) \qquad \lim_n \lambda(R_n) = 0, \quad R_1, R_2, \ldots \in \mathscr{R}, R_n \downarrow \varnothing.$$

In order to prove (2.8) write

$$\mathscr{C} = \{K \setminus \bigcup \mathscr{A};\ K \in \mathscr{F}_c, \mathscr{A} \subset \mathscr{F}_o \text{ finite}\}.$$

It is not hard to see that $\mathscr{C}$ satisfies the following "sequential compactness property": Whenever $\bigcap_n C_n = \varnothing$ for some $C_1, C_2, \ldots \in \mathscr{C}$, we have $\bigcap_{n \leqq m} C_n = \varnothing$ for some m. Now conclude by Lemma I.6.1. in Neveu (1965), that this sequential compactness property still holds if we replace each C_n by a finite union of members of $\mathscr{C}$.

The following approximation result turns out to be handy.

PROPOSITION 2.14. *For $R \in \mathscr{R}$, we have*

$$\lambda(R) = \sup\{\lambda(\textstyle\bigcup_{i=1}^n C_i);\ n \in N,\ C_i \in \mathscr{C},\ C_i \subset R,\ 1 \leqq i \leqq n\}.$$

PROOF. It is clear that we only need to prove

$$\lambda(S) = \sup\{\lambda(C);\ C \in \mathscr{C},\ C \subset S\},\ S \in \mathscr{S}. \tag{2.9}$$

If $S = F$ for some $F \in \mathscr{F}$, (2.9) follows from Proposition 2.9. Thus we may assume $S = K_0 \bigcap G_0 \setminus \bigcup_{i=1}^m K_i \bigcap G_i$, where $m \in \mathrm{N}$ and $K_i \in \mathscr{F}_c$ while $G_i \in \mathscr{F}_o$ for $0 \leqq i \leqq m$. Choose $K_{01}, K_{02}, \ldots \in \mathscr{F}_c$ such that $K_{0n} \uparrow G_0$. Then choose, for $1 \leqq i \leqq m$, some $G_{i1}, G_{i2}, \ldots \in \mathscr{F}_o$ with $G_{in} \downarrow K_i$. Put $C_n = K_0 \bigcap K_{0n} \setminus \bigcup_{i=1}^m G_{in} \bigcap G_i$. Then $C_n \uparrow S$ and, by Proposition 2.10, $\lambda(C_n) \to \lambda(S)$. This shows formula (2.9).

Let $R_1, R_2, \ldots \in \mathscr{R}$ and assume $R_n \downarrow \varnothing$. Fix $\varepsilon > 0$ and choose for each n some finite $\mathscr{C}_n \subset \mathscr{C}$ such that $\bigcup \mathscr{C}_n \subset R_n$ and

$$\lambda(R_n) - \lambda(\textstyle\bigcup \mathscr{C}_n) \leqq \varepsilon \cdot 2^{-n}.$$

Clearly $\bigcap_n (\bigcup \mathscr{B}_n) \subset \bigcap_n R_n = \varnothing$. By the sequential compactness property (see the discussion preceeding Proposition 2.14), we must have $\bigcap_{n \leqq m} (\bigcup \mathscr{C}_n) = \varnothing$ for some m. But then

$$R_m = \textstyle\bigcup_{n \leqq m} R_m \setminus \bigcup \mathscr{C}_n \subset \bigcup_{n \leqq m} R_n \setminus \bigcup \mathscr{C}_n.$$

By finite subadditivity,

$$\lambda(R_m) \leqq \textstyle\sum_{n \leqq m} (\lambda(R_n) - \lambda(\bigcup \mathscr{C}_n)) \leqq \varepsilon.$$

This shows that $\lambda(R_n) \downarrow 0$. Thus (2.8) holds.

Hence λ extends to a measure on the σ-field generated by $\mathscr{F}$. Clearly this is the minimal σ-field over $\mathscr{F}_c$.

Let us now assume that $L = \bigcup_n K_n$, where $K_1, K_2, \ldots \in \mathscr{F}_c$. Moreover, let μ be a measure on the minimal σ-field over $\mathscr{F}_c$, satisfying

$$\mu(K) = \lambda(K),\ K \in \mathscr{F}_c.$$

Then, clearly,

$$\mu(R) = \lambda(R), \quad R \in \mathscr{R}.$$

Hence, for $m \in \mathsf{N}$,

$$\mu(\cdot \textstyle\bigcap(\bigcup_{n \leqq m} K_n)) = \lambda(\cdot \bigcap(\bigcup_{n \leqq m} K_n)).$$

We conclude that $\mu = \lambda$. This completes our proof of Theorem 2.1.

3. Locally finite measures.

Our first result characterizes, among the collection of all continuous posets, those that have a second countable Scott topology. We then discuss three characterizations of the locally finite measures on such posets. We begin by recalling some notations from Giertz et al (1980).

Let L be a poset. A set $D \subset L$ is called *directed* if it is non-empty and if $x, y \in D$ implies the existence of a $z \in D$ with $x \leqq z$ and $y \leqq z$. We assume that all directed subsets D of L have a supremum $\bigvee D$. Such posets are called *up-complete*. Let x, $y \in L$. Say that x is *way below* y, and write $x \ll y$, if $y \leqq \bigvee D$, $D \subset L$ directed imply $x \leqq z$ for some $z \in D$. The poset L is called *continuous* if the set $\{y \in L; y \ll x\}$ is directed with supremum x for all $x \in L$.

Now let L be continuous. A set $U \subset L$ is called *Scott open* if it is an *upper set* (i.e. $\uparrow x \subset U$ whenever $x \in U$) and if $x \in U$ implies the existence of some $y \in U$ with $y \ll x$. The collection of all Scott open sets is a topology on L which we denote Scott(L). It is not hard to verify that Scott(L) is a continuous topology.

We denote by CoScott (L) the collection of all Scott closed subsets of L. Note that a non-empty $F \subset L$ is Scott closed iff it is a lower set (i.e. $\downarrow x = \{y \in L; y \leqq x\} \subset F$ whenever $x \in F$) and $D \subset F$ directed implies $\bigvee D \in F$.

We denote by $\mathscr{L}$, or OFilt(L), the collection of all Scott open filters on L. Recall that a set $F \subset L$ is a *filter* if it is a non-empty upper set which is filtering in the sense that whenever $x, y \in F$ there is a $z \in F$ satisfying $z \leqq x$ and $z \leqq y$. It is easy to see that $\mathscr{L}$ is up-complete.

Let $x, y \in L$ and let $F, G \in \mathscr{L}$. Then

(3.1) $$x \ll y \Leftrightarrow y \in H \subset \uparrow x \text{ for some } H \in \mathscr{L},$$

(3.2) $$F \ll G \Leftrightarrow F \subset \uparrow z \subset G \text{ for some } z \in L.$$

See Lawson (1979). Our references to (3.1) and (3.2) are usually tacit. Note that they imply

(3.3) $$x \in F \Rightarrow x \in H \subset \uparrow z \subset F \text{ for some } z, H \in L \times \mathscr{L}.$$

Hence

$$F = \bigcup_{z \in F} \uparrow z = \bigcup \{H \in \mathscr{L};\ H \ll F\}.$$

The collection on the right is easily seen to be directed. We conclude that $\mathscr{L}$ is a continuous poset. Another useful fact which follows from (3.1) is that $\mathscr{L}$ is a base for Scott(L).

Note that the mapping

$$\mathscr{F}_x = \{F \in \mathscr{L};\ x \in F\},\quad x \in L,$$

is an isomorphism between L and OFilt$(\mathscr{L})$. This is a part of a result of Lawson (1979) which nowadays usually is referred to as the *Lawson duality.* Lawson (1979) also proves that L is a semi-lattice with a top iff $\mathscr{L}$ is so.

Let us agree to say that a subset Q of L is *separating* if $x \ll y$ implies that $x \leqq q \leqq y$ for some $q \in Q$. Assume Q is separating. If $x \ll y$ then, by (3.1) and (3.3), $x \ll q \ll y$ for some $q \in Q$. Moreover, the set $\{q \in Q;\ q \ll x\}$ is directed with supremum x for all $x \in L$. Let $F \in \mathscr{L}$. If $x \in F$ then we may choose a $q \in F \bigcap Q$ with $q \ll x$. It follows that

$$F = \bigcup_{q \in Q \cap F} \uparrow q. \tag{3.5}$$

The union to the right in (3.5) is directed.

Let $\mathscr{G}$ be a continuous topology. Of course any separating subset of $\mathscr{G}$ is a base for $\mathscr{G}$. Conversely, if $\mathscr{G}_0 \subset \mathscr{G}$ is a base for $\mathscr{G}$, then the colection of all finite unions of members of $\mathscr{G}_0 \bigcup \{\varnothing\}$ is a separating subset of $\mathscr{G}$. Hence $\mathscr{G}$ is second countable iff $\mathscr{G}$ contains a countable separating subset. By Proposition 3.1 below, this holds iff Scott$(\mathscr{G})$ is second countable.

PROPOSITION 3.1. *Let L be a continuous poset. The following four conditions are equivalent:*

(i) Scott(L) *is second countable,*
(ii) *L contains a countable separating set,*
(iii) *$\mathscr{L}$ contains a countable separating set,*
(iv) Scott$(\mathscr{L})$ *is second countable.*

PROOF. Write $U_x = \{y \in L; x \ll y\}$, $x \in L$, and let $Q \subset L$. If Q is separating, then the collection U_q, $q \in Q$, is a base for the Scott topology. To see this, let $x \in U$, where $U \in L$ is Scott open. Choose $y \in U$ such that $y \ll x$. But then $y \leqq q \ll x$ for some $q \in Q$. Hence $x \in U_q \subset U$. This shows our claim.

We see that (ii) implies (i). To see that the latter implies (iii), note that, under (i), $\mathscr{L}$ contains a countable base $\mathscr{Q}$ for the Scott topology. Let $F, G \in \mathscr{L}$, $F \ll G$. Then $F \subset \uparrow x$ for some $x \in G$. But then $x \in H \subset G$ for some $H \in \mathscr{Q}$. Clearly $F \subset H \subset G$. Hence $\mathscr{Q}$ is a separating subset of $\mathscr{L}$. This shows (iii).

Next, assume (iii) and let $\mathscr{Q} \subset \mathscr{L}$ be countable and separating. Whenever F,

$H \in \mathcal{Q}$, $F \ll H$, we choose $x_{FH} \in L$ such that $F \subset \uparrow x_{FH} \subset H$. The obtained set $\{x_{FH}\}$ is clearly countable. If $x, y \in L, x \ll y$, then $y \in F \ll H \subset \uparrow x$ for some pair F, $H \in \mathcal{Q}$. This follows by (3.1) and repeated applications of (3.3). But then $x \leqq x_{FH} \leqq y$. Hence $\{x_{FH}\}$ is separating. This shows (ii).

Thus (i), (ii) and (iii) are equivalent. But then (iii) must be equivalent to (iv) as well.

In the remaining part of this section we let L be a continuous poset with a second countable Scott topology. Then L contains a countable separating subset Q. Let $x \in L$. Then x is the supremum of the directed set $\{q \in Q; q \ll x\}$. It is now not hard to see the existence of a sequence $\{x_n\} \subset L$ satisfying $x_n \uparrow x$ and $x_n \ll x_{n+1}$ for all n. But then there are $F_1, F_2, \ldots \in \mathcal{L}$ satisfying $x_{n+1} \in F_n \subset \uparrow x_n$. Clearly $F_n \downarrow (\uparrow x)$. Similarly the reader may use (3.5) to show that, whenever $F \in \mathcal{L}$, there are $x_1, x_2, \ldots \in L$ satisfying $x_{n+1} \ll x_n$ for all n and $F = \bigcup_n \uparrow x_n$. But this follows from the Lawson duality as well.

It is obvious from the discussion above that Σ is the minimal σ-field over $\mathcal{L}$. But then Scott$(L) \subset \Sigma$. Now note that $\downarrow x$ is the closure of $\{x\}$ in Scott(L). Hence $\downarrow x \in \Sigma$ for all $x \in L$. It follows that singletons are measurable sets.

The next result needs no proof. But note that $L \ll L$ (in Scott(L)) if L has a bottom.

PROPOSITION 3.2. *Let λ be a measure on a continuous poset L with a second countable Scott topology. Then λ is locally finite iff*

$$\lambda(U) < \infty, \quad U \in \text{Scott}(L), \quad U \ll L.$$

Suppose $L \in \mathcal{L}$. Then this holds iff

$$\lambda(F) < \infty, \quad F \in \mathcal{L}, \quad F \ll L.$$

We now prove the characterization (1.1) of the locally finite measures on L, which is valid whenever L is a lattice. It is a straightforward consequence of Theorem 2.1.

THEOREM 3.3. *Let L be a continuous lattice and assume* Scott(L) *to be second countable. Then* (1.1) *defines a bijection between the family of locally finite measures λ on L and the family of mappings $\Lambda\colon L \to R_+$ satisfying* (1.2)–(1.3).

PROOF. Let λ be a locally finite measure on L and define Λ by (1.1). A routine argument yields

$$\Lambda_n(x; x_1, \ldots, x_n) = \lambda(\uparrow x \setminus \cup_{i=1}^n \uparrow x_i), \quad n \in \mathbf{N}, \; x, x_1, \ldots, x_n \in L.$$

Thus Λ satisfies (1.3). To see that Λ satisfies (1.2), it is enough to note that $x_n \uparrow x$ iff $(\uparrow x_n) \downarrow (\uparrow x)$.

Clearly different locally finite measures give rise to different functions in (1.1), since otherwise the uniqueness part of Theorem 2.1 would be violated. Thus the mapping $\lambda \to \Lambda$, defined by (1.1), is an injection.

To see that it is a bijection, let the function $\Lambda: L \to \mathsf{R}_+$ satisfy (1.2) and (1.3) and define λ by (1.1). Put $\mathscr{F}_c = \{\uparrow x;\ x \in L\}$ and let $\mathscr{F}_o = \{F \in \mathscr{L};\ F \ll L\}$. The reader easily verifies that the assumptions of Theorem 2.1 are at hand. Hence λ extends to a locally finite measure on L.

Let λ be a locally finite measure on L. Then

(3.6) $$\lambda(F) = \sup_{x \in F} \lambda(\uparrow x),\ \ F \in \mathscr{L}.$$

To see this, use the fact that whenever $F \in \mathscr{L}$ there are $x_1, x_2, \ldots \in L$ satisfying $(\uparrow x_n) \uparrow F$. A similar argument which uses Proposition 3.2 shows

(3.7) $$\lambda(\uparrow x) = \inf_{x \in F} \lambda(F),\ \ x \in L.$$

Assume, for now, that L is a semi-lattice with a top 1. Then L is an abelian semi-group under the composition $\wedge$. Its neutral element is 1. Write $\mathscr{F}$ for the space of all filters on L in the topology of pointwise convergence. Assume $M: L \to \mathsf{R}_+$ satisfies

$$\Delta_{x_1} \ldots \Delta_{x_n} M(x) \geqq 0,\ \ n \in \mathsf{N},\ \ x, x_1, \ldots, x_n \in L.$$

Berg, Christensen & Ressel (1984) show that there exists a bounded Radon measure μ on $\mathscr{F}$ satisfying (see their proposition 4.4.17)

$$\mu\{F \in \mathscr{F};\ x \in F\} = M(x),\ \ x \in L.$$

It is easily seen that the mapping c: $\mathscr{F} \to \mathscr{L}$, defined by

$$c(F) = F^{\circ},\ \ F \in \mathscr{L},$$

is continuous (we assume $\mathscr{L}$ is endowed with its Scott topology). In particular c is measurable. Hence c^{-1} transfers μ into a bounded measure $\nu = \mu c^{-1}$ on $\mathscr{L}$:

$$\nu(\mathscr{F}_x) = \mu\{F \in \mathscr{F};\ x \in F^{\circ}\},\ \ x \in L.$$

Assume M also satisfies

$$M(x) = \lim_n M(x_n),\ \ x, x_1, x_2, \ldots \in L,\ \ x_n \uparrow x.$$

Fix $x \in L$. Then there are points $x_1, x_2, \ldots \in L$ satisfying $x_n \ll x_{n+1}$ for all n and $x_n \uparrow x$. By assumption $M(x_n) \uparrow M(x)$ and it is easy to see that

$$\{F \in \mathscr{F};\ x_n \in F\} \uparrow \{F \in \mathscr{F};\ x \in F^{o}\}.$$

Hence

$$\nu(\mathscr{F}_x) = M(x),\ \ x \in L.$$

Our aim now is to extend this existence result for bounded measures on $\mathscr{L}$, to a characterization of all locally finite measures on $\mathscr{L}$. We begin with some preparations.

Obviously $\mathrm{Scott}(L)\setminus\{\emptyset\}$ is a continuous lattice with a second countable Scott topology, so Theorem 3.3 characterizes all locally finite measures on $\mathrm{Scott}(L)\setminus\{\emptyset\}$. Note that

$$\Sigma(\mathrm{Scott}(L)\setminus\{\emptyset\}) = \{B\in\Sigma(\mathrm{Scott}(L));\ \emptyset\notin B\}.$$

Hence all measures on $\mathrm{Scott}(L)\setminus\{\emptyset\}$ trivially extend to $\mathrm{Scott}(L)$. Our next result tells us when a measure on $\mathrm{Scott}(L)$ concentrates its mass to $\mathscr{L}\bigcup\{\emptyset\}$. For $x\in L$ write

$$\mathscr{U}_x = \{U\in\mathrm{Scott}(L);\ x\in U\},$$

and note that $\mathscr{U}_x$ is a Scott open filter on both $\mathrm{Scott}(L)$ and $\mathrm{Scott}(L)\setminus\{\emptyset\}$.

PROPOSITION 3.4. *Let L be a continuous semi-lattice with a top* 1 *and a second countable Scott topology. Then*

$$\Sigma(\mathscr{L}) = \{B\in\Sigma(\mathrm{Scott}(L)\setminus\{\emptyset\});\ B\subset\mathscr{L}\} \tag{3.8}$$

$$= \{B\in\Sigma(\mathrm{Scott}(L));\ B\subset\mathscr{L}\}.$$

Let μ be a measure on Scott (L). *Write $L_o = \{x\in L; x\ll 1\}$. Then μ concentrates its mass to $\mathscr{L}\bigcup\{\emptyset\}$ iff*

$$\mu(\mathscr{U}_x\textstyle\bigcap\mathscr{U}_y\setminus\mathscr{U}_{x\wedge y}) = 0,\ \ x,y\in L_o. \tag{3.9}$$

PROOF. Clearly

$$\mathscr{U}_x\in\Sigma(\mathrm{Scott}(L)\setminus\{\emptyset\})\subset\Sigma(\mathrm{Scott}(L)).$$

Fix $U\in\mathrm{Scott}(L)$, $U\neq\emptyset$. Then $U=\bigcup_n F_n$ for some $F_1,F_2,\ldots\in\mathscr{L}$. Choose $\{x_{nm}\}\subset L$ such that $F_n=\cup_m\uparrow x_{nm}$ for each n. Clearly $U=\cup_{nm}\uparrow x_{nm}$. Renumber and conclude that $U=\cup_n\uparrow x_n$ for some suitably chosen $x_1,x_2,\ldots\in L$. Hence

$$\uparrow U = \textstyle\bigcap_n\mathscr{U}_{x_n}.$$

We see that $\Sigma(\mathrm{Scott}(L)\setminus\{\emptyset\})$ and that $\Sigma(\mathrm{Scott}(L))$ are the minimal σ-fields over the sets $\mathscr{U}_x$, $x\in L$.

But $\Sigma(\mathscr{L})$ is the minimal σ-field over the sets $\mathscr{F}_x=\mathscr{U}_x\bigcap\mathscr{L}$. Hence

$$\Sigma(\mathscr{L}) = \Sigma(\mathrm{Scott}(L))\textstyle\bigcap\mathscr{L}$$

$$= \Sigma(\mathrm{Scott}(L)\setminus\{\emptyset\})\textstyle\bigcap\mathscr{L}.$$

Let $Q\subset L$ be countable and separating. We may assume that $Q\subset L_o$. Fix $U\in\mathrm{Scott}(L)$, $U\neq\emptyset$. If $U\notin\mathscr{L}$ there must be a pair $x,y\in L$ satisfying $x\in U$, $y\in U$

while $x \wedge y \notin U$. Choose $p, q \in Q \bigcap U$ such that $p \ll x$ and $q \ll y$. Of course $p \wedge q \notin U$. We see that

$$\text{Scott}(L) \setminus (\mathscr{L} \bigcup \{\varnothing\}) = \bigcup_{p,q \in Q} \mathscr{U}_p \bigcap \mathscr{U}_q \setminus \mathscr{U}_{p \wedge q}.$$

It follows that $\mathscr{L}$ is a measurable subset of both Scott (L) and $\text{Scott}(L) \setminus \{\varnothing\}$. Now (3.8) is immediate. Clearly so is also (3.9).

THEOREM 3.5. *Let L be a continuous semi-lattice with a top* 1 *and a second countable Scott topology. The formula*

$$\lambda(\mathscr{F}_x) = M(x), \ x \in L, \tag{3.10}$$

defines a bijection between the family of locally finite measures λ *on* $\mathscr{L}$ *and the family of mappings* $M: L \to \bar{\mathsf{R}}_+$ *which are finite on* $L_o = \{x \in L; x \ll 1\}$ *and satisfy*

$$M(x) = \lim_n M(x_n), \ x, x_1, x_2, \ldots \in L, x_n \uparrow x, \tag{3.11}$$

$$\Delta_{x_1} \ldots \Delta_{x_n} M(x) \geqq 0, \ n \in \mathsf{N}, \ x, x_1, \ldots, x_n \in L_o. \tag{3.12}$$

PROOF. Let λ be a locally finite measure on $\mathscr{L}$ and define M by (3.10). By Proposition 3.2, M maps L_o into R_+. Formula (3.11) is obvious, while (3.12) follows from

$$\Delta_{x_1} \ldots \Delta_{x_n} M(x) = \lambda(\mathscr{F}_x \setminus \bigcup_{i=1}^n \mathscr{F}_{x_i}), \ n \in \mathsf{N}, x, x_1, \ldots, x_n \in L_o,$$

the proof of which we leave to the reader. The uniqueness part is left to the reader too.

Next let $M: L \to \mathsf{R}_+$ map L_o into R_+ and assume (3.11)–(3.12). Whenever $U \in \text{Scott}(L)$, $U \neq \varnothing$, we write

$$\Lambda(U) = \inf\{M(\wedge_{i=1}^n x_i); \ n \in \mathsf{N}, \ x_1, \ldots, x_n \in U\}.$$

Note that $1 \in U$. Hence there is an $x \in U \bigcap L_o$. Clearly $\Lambda(U) \leqq M(x) < \infty$.

We show that Λ satisfies (1.2) and (1.3). Clearly Λ is decreasing. Thus, if $U_n \uparrow U$, $U_1 \neq \varnothing$, then $\Lambda(U_n) \downarrow \alpha \geqq \Lambda(U)$. Assume $\alpha > \Lambda(U)$. Then $\alpha > M(\wedge_{i=1}^m x_i)$ for some $x_1, \ldots, x_m \in U$. But then $x_1, \ldots, x_m \in U_n$ for n sufficiently large. This leads to a contradiction. Hence $\alpha = \Lambda(U)$ and we conclude (1.2).

To see (1.3), fix $k \in \mathsf{N}$ and $U_0, U_1, \ldots, U_k \in \text{Scott}(L) \setminus \{\varnothing\}$. For $j = 0, 1, \ldots, k$ and $n \in \mathsf{N}$, choose some finite set $\{x_{jni}\} \subset L$ such that $H_{jn} \uparrow U_j$, where $H_{jn} = \cup_i \uparrow x_{jni}$. Let $0 \in J \subset \{0, 1, \ldots, k\}$ and write $H_{Jn} = \bigcup_{j \in J} H_{jn}$. Then $H_{Jn} \uparrow U_J = \bigcup_{j \in J} U_j$. We show that

$$\Lambda(U_J) = \lim_n M(\wedge_{j \in J} \wedge_i x_{nji}).$$

Clearly $\wedge_{j \in J} \wedge_i x_{nji}$ decreases as n increases. Hence

$$\Lambda(U_J) \leqq \lim_n M(\wedge_{j \in J} \wedge_i x_{nji}).$$

Let $\Lambda(U_J) < \alpha$ and choose $y_1, \ldots, y_m \in U_J$ such that $M(\wedge_{l=1}^m y_l) < \alpha$. Clearly $\wedge_{j \in J} \wedge_i x_{nji} \leqq \wedge_{l=1}^m y_l$ for sufficiently large n. Hence

$$\alpha \geqq \lim_n M(\wedge_{j \in J} \wedge_i x_{nji}),$$

as was to be proved.

Now (1.3) is immediate. Cf the proof of Proposition 2.13. By Theorem 3.3, there is a unique locally finite measure λ on $\mathrm{Scott}(L) \setminus \{\varnothing\}$ satisfying

$$\lambda(\uparrow U) = \Lambda(U), \quad U \in \mathrm{Scott}(L) \setminus \{\varnothing\}.$$

Our aim now is to show that λ is concentrated on $\mathscr{L}$

Fix $x \in L$. By (3.6),

$$\lambda(\mathscr{U}_x) = \sup_{x \in U} \Lambda(U) \leqq M(x).$$

Assume $y \ll x$. Then $x \in U \subset \uparrow y$ for some $U \in \mathrm{Scott(L)}$. If $y_1, \ldots, y_n \in U$ then $y \leqq \wedge_{i=1}^n y_i$ and therefore $M(y) \leqq M(\wedge_{i=1}^n y_i)$. Hence

$$M(y) \leqq \Lambda(U) \leqq \lambda(\mathscr{U}_x).$$

By (3.11),

$$M(x) \leqq \lambda(\mathscr{U}_x).$$

We have shown that

$$\lambda(\mathscr{U}_x) = M(x), \quad x \in L.$$

Fix $x, y \in L$. By (3.6) and since $\mathscr{U}_{x \wedge y} \subset \mathscr{U}_x \bigcap \mathscr{U}_y$,

$$\begin{aligned} \lambda(\mathscr{U}_x \bigcap \mathscr{U}_y) &= \sup_{x, y \in U} \Lambda(U) \leqq M(x \wedge y) \\ &= \lambda(\mathscr{U}_{x \wedge y}) \leqq \lambda(\mathscr{U}_x \bigcap \mathscr{U}_y). \end{aligned}$$

Thus we have equality throughout.

By Proposition 3.4, we conclude that λ concentrates its mass to $\mathscr{L}$. Let us identify λ with its restriction to $\mathscr{L}$. Then, clearly, (3.10) holds. That λ is locally finite on $\mathscr{L}$ follows from the fact that M is finite on L_0. Cf Proposition 3.2.

We have already remarked that when L is a continuous semi-lattice with a top, so is its Lawson dual $\mathscr{L}$ and vice versa. Thus the Lawson duality allows us to replace the characterization above of the locally finite measures on $\mathscr{L}$ in terms of functions defined on L, with an equivalent characterization of the locally finite measures on L in terms of functions defined on $\mathscr{L}$.

COROLLARY 3.6. *Let L be as in Theorem* 3.5. *The formula*

$$\Phi(F) = \lambda(F), \quad F \in \mathscr{L}, \tag{3.13}$$

defines a bijection between the family of locally finite measures λ on L and the family of mappings $\Phi\colon \mathscr{L} \to \bar{\mathsf{R}}_+$ which are finite on $\mathscr{L}_o = \{F \in \mathscr{L};\ F \ll L\}$ and satisfy

$$\Phi(F) = \lim_n \Phi(F),\quad F, F_1, F_2, \ldots \in \mathscr{L},\ F_n \uparrow F, \tag{3.14}$$

$$\Delta_{F_n} \ldots \Delta_{F_1} \Phi(F) \geqq 0,\quad n \in \mathsf{N},\ F, F_1, \ldots, F_n \in \mathscr{L}_o. \tag{3.15}$$

We now prepare the proof of a characterization of the locally finite measures on L in terms of "additive" mappings on Scott(L), which is valid without any further assumptions on L.

LEMMA 3.7. *Let L be a continuous poset with a second countable Scott topology. Let $\mathscr{H} \in$ OFilt (Scott(L)), $\mathscr{H} \neq$ Scott(L). Assume*

$$U \bigcup V \in \mathscr{H} \Rightarrow U \in \mathscr{H} \ \text{ or } \ V \in \mathscr{H}.$$

Then $\mathscr{H} = \mathscr{U}_x$ for some $x \in L$.

PROOF. By assumption $\mathscr{H}^c$ is a directed and Scott closed subset of Scott(L). Hence

$$\mathscr{H}^c = \{U \in \text{Scott}(L);\ U \subset H\},$$

where $H = \bigcup \mathscr{H}^c$. Note that $H \neq L$, since $\mathscr{H} \neq \varnothing$ by assumption. Now the reader easily shows that if $H^c \subset E \bigcup F$ for some $E, F \in$ CoScott(L), then $H^c \subset E$ or $H^c \subset F$. Thus H^c is a non-empty irreducible Scott closed set. It follows by Proposition 5.2 of Lawson (1979) that $H^c = \downarrow x$ for some $x \in L$. But then $U \in \mathscr{H}$ iff $x \in U$.

Put $\mathscr{U} = \{\mathscr{U}_x; x \in L\}$. Note that, for $x, y \in L$, $x \leqq y$ iff $\mathscr{U}_x \subset \mathscr{U}_y$. Hence $\mathscr{U}$ is isomorphic to L. Note further that $\mathscr{F} \in$ OFilt$(\mathscr{U})$ iff $\mathscr{F} = \{\mathscr{U}_x;\ x \in F\}$ for some $F \in \mathscr{L}$. Hence $\Sigma(\mathscr{U})$ is the minimal σ-field over the sets $\{\mathscr{U}_x;\ x \in F\}$, $F \in \mathscr{L}$.

PROPOSITION 3.8. *Let L be a continuous poset with a second countable Scott topology. Then*

$$\Sigma(\mathscr{U}) = \{B \in \Sigma(\text{OFilt}(\text{Scott}(L)));\ B \subset \mathscr{U}\}. \tag{3.16}$$

Let μ be a measure on OFilt(Scott(L)). *Then μ concentrates its mass to $\mathscr{U} \cup \{\text{Scott}(L)\}$ iff*

$$\mu\{\mathscr{H} \in \text{OFilt}(\text{Scott}(L));\ U \bigcup V \in \mathscr{H}, U \notin \mathscr{H}, V \notin \mathscr{H}\} = 0,\quad U, V \in \text{Scott}(L). \tag{3.17}$$

PROOF. By Lemma 3.7,

$$\text{OFilt}(\text{Scott}(L)) \setminus (\mathscr{U} \bigcup (\text{Scott}(L))$$

$$= \bigcup_{U,V \in \text{Scott}(L)} \{\mathscr{H} \in \text{OFilt}(\text{Scott}(L));\ U \bigcup V \in \mathscr{H}, U \notin \mathscr{H}, V \in \mathscr{H}\}.$$

It is not hard to see that the union above may be thinned to $U, V \in \mathscr{Q}$, where $\mathscr{Q}$ is a countable separating subset of Scott(L). Moreover, $\Sigma(\text{OFilt}(\text{Scott}(L))$ is the minimal σ-field over the sets $\{\mathscr{H} \in \text{OFilt}(\text{Scott}(L)); U \in \mathscr{H}\}$, $U \in \text{Scott}(L)$, and

$$\{\mathscr{H} \in \text{OFilt}(\text{Scott}(L)); U \in \mathscr{H}\} \bigcap \mathscr{U} = \{\mathscr{U}_x; x \in U\}, \ U \in \text{Scott}(L).$$

Now both assertions of the proposition are immediate.

We can now prove the following existence theorem for locally finite measures on L.

THEOREM 3.9. *Let L be a continuous poset with a second countable Scott topology. A mapping* $\lambda: \text{Scott}(L) \to \bar{\mathsf{R}}_+$, *which is increasing and finite on* $\{U \in \text{Scott}(L); U \ll L\}$, *extends to a unique locally finite measure on L iff*

$$\lambda(\varnothing) = 0, \tag{3.18}$$

$$\lambda(U) = \lim_n \lambda(U_n), \ U, U_1, U_2, \ldots \in \text{Scott}(L), \ U_n \uparrow U, \tag{3.19}$$

$$\lambda(U \bigcup V) + \lambda(U \bigcap V) = \lambda(U) + \lambda(V), \ U, V \in \text{Scott}(L). \tag{3.20}$$

PROOF. Let $U, U_1, \ldots, U_n \in \text{Scott}(L)$ and assume $U \bigcup U_1 \bigcup \ldots \bigcup U_n \ll L$. Note that

$$\Delta_{U_1} \lambda(U) = \lambda(U) - \lambda(U \bigcap U_1) = \lambda(U \bigcup U_1) - \lambda(U_1).$$

By induction it follows readily that

$$\Delta_{U_1} \ldots \Delta_{U_n} \lambda(U) = \lambda(U \bigcup U_1 \bigcup \ldots \bigcup U_n) - \lambda(U_1 \bigcup \ldots \bigcup U_n).$$

Thus, by Theorem 3.5, there is a locally finite measure μ on OFilt(Scott(L)) satisfying

$$\mu(\mathscr{H} \in \text{OFilt}(\text{Scott}(L)); U \in \mathscr{H}\} = \lambda(U), \ U \in \text{Scott}(L).$$

Fix $U, V \in \text{Scott}(L)$. Choose $\{U_n\}, \{V_n\} \subset \text{Scott}(L)$ such that $U_n \ll U$, $V_n \ll V$ for all n and $U_n \uparrow U$ while $V_n \uparrow V$. First note that

$$\begin{aligned}&\mu\{\mathscr{H} \in \text{OFilt}(\text{Scott}(L)); U_n \bigcup V_n \in \mathscr{H}, U \notin \mathscr{H}, V \notin \mathscr{H}\} \\ &\leqq \mu\{\mathscr{H} \in \text{OFilt}(\text{Scott}(L)); U_n \bigcup V_n \in \mathscr{H}, U_n \notin \mathscr{H}, V_n \notin \mathscr{H}\} = 0.\end{aligned}$$

Then note that the sets to the left increases to

$$\{\mathscr{H} \in \text{OFilt}(\text{Scott}(L)); U \bigcup V \in \mathscr{H}, U \notin \mathscr{H}, V \notin \mathscr{H}\}.$$

Hence the latter set has μ-measure zero. By Proposition 3.8, μ concentrates its mass to $\mathscr{U} \bigcup \{\text{Scott}(L)\}$.

By (3.18), $\mu\{\text{Scott}(L)\} = 0$. Thus μ is concentrated to $\mathscr{U}$. Now note that

$$\lambda(U) = \mu(\mathscr{H} \in \mathscr{U}; U \in \mathscr{H}\} = \mu\{\mathscr{U}_x; x \in U\}, \ U \in \text{Scott}(L), \ U \neq \varnothing.$$

Hence λ extends to a measure on L. We conclude from Proposition 3.2 that λ is locally finite. Its uniqueness follows (cf. (3.7)) from

$$\lambda(\uparrow x) = \inf_{x\in U} \lambda(U), \ \ x \in L.$$

We conclude this section with an existence theorem for finite measures on L. Also this result requires some preparations.

Note that the mapping

$$x \to L \setminus \downarrow x, \ \ x \in L,$$

is a measurable injection into Scott(L). Write L^* for its range.

Proposition 3.10. *Let L be a continuous poset with a second countable Scott topology. Then*

$$\Sigma(\mathrm{Scott}(L)) \bigcap L^* = \{B \in \Sigma(\mathrm{Scott}(L));\ B \subset L^*\}. \tag{3.21}$$

Let μ be a locally finite measure on Scott(L). *Then μ concentrates its mass to $L^* \bigcup \{L\}$ iff*

$$\mu(\uparrow U \bigcap V \setminus (\uparrow U \bigcup \uparrow V) = 0, \ \ U, V \in \mathrm{Scott}(L). \tag{3.22}$$

Proof. By Proposition 5.2 of Lawson (1979),

$$\mathrm{Scott}(L) \setminus (\{L\} \bigcup L^*) = \bigcup_{U,V\in\mathrm{Scott}(L)} \uparrow U \bigcap V \setminus (\uparrow U \bigcup \uparrow V).$$

It is easy to see that the union to the right can be thinned to a countable one. Hence $L^* \in \Sigma\, \mathrm{Scott}(L))$. Now both assertions of the proposition are immediate.

Theorem 3.11. *Let L be a continuous poset and assume its Scott topology is second countable. An increasing mapping λ*: Co Scott$(L) \to \mathsf{R}_+$ *extends to a unique finite measure on L iff*

$$\lambda(\varnothing) = 0, \tag{3.23}$$

$$\lambda(F) = \lim_n \lambda(F_n), \ \ F, F_1, F_2, \ldots \in \mathrm{CoScott}(L), \ F_n \downarrow F, \tag{3.24}$$

$$\lambda(E \bigcap F) + \lambda(E \bigcup F) = \lambda(E) + \lambda(F), \ \ E, F \in \mathrm{CoScott}(L). \tag{3.25}$$

Proof. Conclude by Theorem 3.3 that there is a finite measure μ on Scott(L) satisfying

$$\mu(\uparrow U) = \lambda(U^c), \ \ U \in \mathrm{Scott}(L).$$

A straightforward calculation shows that (3.22) holds. Moreover, by (3.23), $\mu\{L\} = 0$. Hence μ concentrates its mass to L^*. Now note that

$$\lambda(F) = \mu\{L \setminus \downarrow x;\ x \in F\}, \ \ F \in \mathrm{CoScott}(L).$$

Hence λ extends to a measure on L. The uniqueless is clear.

4. Lévy-Khinchin measures.

In this section L is a fixed continuous poset with a top 1 and a second countable Scott topology. Note that, then Scott $(L)\setminus\{\varnothing\}$ is a lattice. Our first result tells us when a measure on L is a Lévy-Khinchin measure. Its proof is omitted.

PROPOSITION 4.1. *Let L be a continuous poset with a top and a second countable Scott topology. Let ψ be a measure on L which concentrates its mass to $L\setminus\{1\}$. Then ψ is a Lévy-Khinchin measure iff the following two equivalent conditions hold:*

$$\psi(L\setminus U) < \infty, \quad U \in \mathrm{Scott}(L), \quad U \neq \varnothing,$$

$$\psi(L\setminus \uparrow x) < \infty, \quad x \in L, \; x \ll 1.$$

Here is our most general existence theorem for Lévy-Khinchin measures on L.

THEOREM 4.2. *Let L be a continuous poset with a top and with a second countable Scott topology. An increasing mapping ψ: CoScott $(L)\setminus\{S\} \to \mathrm{R}_+$ extends to a unique Lévy-Khinchin measure on L iff*

$$\psi(\varnothing) = 0 \tag{4.1}$$

$$\psi(F) = \lim_n \psi(F_n), \quad F, F_1, F_2, \ldots \in \mathrm{CoScott}(L)\setminus\{S\}, \; F_n \downarrow F, \tag{4.2}$$

$$\psi(E \bigcup F) + \psi(E \bigcap F) = \psi(E) + \psi(F), \; E, F \in \mathrm{CoScott}(L)\setminus\{S\}. \tag{4.3}$$

PROOF. Conclude by Theorem 3.3 that there is a locally finite measure μ on Scott $(L)\setminus\{\varnothing\}$ satisfying

$$\mu(\uparrow U) = \psi(U^c), \quad U \in \mathrm{Scott}\,(L), U \neq \varnothing.$$

Then conclude by Proposition 3.10 that μ concentrates its mass to L^*.

Now note that

$$\psi(F) = \mu\{L\setminus \uparrow x; x \in F\}, \quad F \in \mathrm{CoScott}(L).$$

This shows that ψ extends to a measure on L satisfying

$$\psi(F) < \infty, \quad F \in \mathrm{CoScott}(L), \quad F \neq S.$$

But $L\setminus\downarrow 1 = \varnothing$ and μ concentrates its mass to Scott $(L)\setminus\{\varnothing\}$. Thus ψ is a Lévy-Khinchin measure.

It is not hard to see that ψ is unique.

Our next existence theorem presumes that L is a semi-lattice.

THEOREM 4.3. *Let L be a continuous semi-lattice with a top and a second countable Scott topology. Then formula* (1.8) *defines a bijection between the family of Lévy-Khinchin measures ψ on L and the family of mappings Ψ: $\mathscr{L} \to \mathrm{R}_+$ satisfying* (1.9)-(1.11).

PROOF. The reader easily shows (1.9)-(1.11) if Ψ, through (1.8), stems from a Lévy-Khinchin measure ψ on L. So we assume that $\Psi: L \to \mathrm{R}_+$ satisfies (1.9)-(1.11). Whenever $H \in \mathscr{L}$ we write

$$\Phi_H(F) = -\Delta_H \Psi(F) = \Psi(F \bigcap H) - \Psi(F), \quad F \in \mathscr{L}.$$

It is not hard to see that Φ_H satisfies (3.14)-(3.15). Thus there is a locally finite measure λ_H on L satisfying

$$\lambda_H(F) = \Phi_H(F) = \Psi(F \bigcap H) - \Psi(F), \quad F \in \mathscr{L}.$$

Note that $\lambda_H(L) = \Psi(H)$. Hence

$$\lambda_H(L \setminus F) = \Psi(H) - \Psi(F \bigcap H) + \Psi(F), \quad F \in \mathscr{L}.$$

Clearly λ_H concentrates its mass to $L \setminus H$. Assume $G \in \mathscr{L}$, $G \subset H$. Then the restriction of λ_G to $L \setminus H$ coincides with λ_H. This follows by the uniqueness part of Corollary 3.6.

We now put

$$\psi(B) = \sup_{H \in \mathscr{L}} \lambda_H(B), \quad B \in \Sigma.$$

Then ψ is a measure on L. Let $F \in \mathscr{L}$. Then

$$\psi(L \setminus F) = \Psi(F) + \sup_{H \in \mathscr{L}}(\Psi(H) - \Psi(F \bigcap H)) = \Psi(F).$$

This shows (1.8). Now note that, writing 1 for the top of L,

$$\psi\{1\} = \sup_{H \in \mathscr{L}} \lambda_H\{1\} = 0.$$

Thus ψ is a Lévy-Khinchin measure.

To see the uniqueness part of the theorem, consider two Lévy-Khinchin measures ψ_1 and ψ_2 on L, and assume

$$\psi_1(L \setminus F) = \psi_2(L \setminus F), \quad F \in \mathscr{L}.$$

By Corollary 3.6, ψ_1 and ψ_2 coincides on $L \setminus F$ for each $F \in \mathscr{L}$. But then we must have $\psi_1 = \psi_2$ on $L \setminus \{1\}$.

The remark preceeding Corollary 3.6 can be modified to the present study of Lévy-Khinchin measures.

COROLLARY 4.4. *Let L be as in Theorem 4.3. The formula*

$$\Phi(x) = \Phi(\mathscr{L} \setminus \mathscr{F}_x), \quad x \in L, \tag{4.4}$$

defines a bijection between the family of Lévy-Khinchin measures ψ on $\mathscr{L}$ and the family of functions $\Phi: L \to \mathrm{R}_+$ satisfying

$$\Phi(1) = 0, \tag{4.5}$$

$$\Phi(x) = \lim_n \Phi(x_n), \ x, x_1, x_2, \ldots \in x_n \uparrow x, \tag{4.6}$$

$$\Delta_{x_1} \ldots \Delta_{x_n} \Phi(x) \leqq 0, \ n \in N, x, x_1, \ldots x_n \in L. \tag{4.7}$$

Assume L to be lattice and let $\Lambda: L \to \mathsf{R}_+$. Suppose that Λ is decreasing, that $\Lambda(1) = 0$, that $\Lambda(x_n) \to \Lambda(x)$ as $x_n \uparrow x$ and that

$$\Lambda(x \vee y) + \Lambda(x \wedge y) = \Lambda(x) + \Lambda(y), \ x, y \in L.$$

By Corollary 4.4, there is a unique Lévy-Khinchin measure ψ on $\mathscr{L}$ satisfying

$$\psi(\mathscr{L} \setminus \mathscr{F}_x) = \Lambda(x), \ x \in L,$$

and, by Theorem 3.3, the formula

$$\lambda(\uparrow x) = \Lambda(x), \ x \in L,$$

defines a unique locally finite measure on L. We have here an interesting one-to-one pairing between a subclass of the Lévy-Khinchin measures on $\mathscr{L}$ and a subclass of the locally finite measures on L.

When L is a lattice we may characterize the Lévy-Khinchin measures on L in terms of functions on L. Put $L_o = \{x \in L; x \ll 1\}$, where, as usual, 1 denotes the top of L. Note that $x, y \in L_o$ iff $x \vee y \in L_o$.

THEOREM 4.5. *Let L be a continuous lattice and assume* Scott(L) *to be second countable. The formula*

$$M(x) = \psi(L \setminus \uparrow x), \ x \in L, \tag{4.8}$$

defines a bijection between the family of Lévy-Khinchin measures ψ on L and the family of mappings $M: L \to \bar{\mathsf{R}}_+$ which are finite on L_o and satisfy

$$\inf_{x \in L} M(x) = 0, \tag{4.9}$$

$$M(x) = \lim_n M(x_n), \ x, x_1, x_2, \ldots \in L, \ x_n \uparrow x, \tag{4.10}$$

$$M_n(x; x_1, \ldots x_n) \leqq 0, \ n \in \mathsf{N}, x, x_1, \ldots, x_n \in L. \tag{4.11}$$

PROOF. Assume $M: L \to \bar{\mathsf{R}}_+$ maps L_o into R_+ and that (4.9)-(4.11) hold. Write

$$\Psi(F) = \inf_{x \in F} M(x), \ F \in \mathscr{L}.$$

Note that if $F \in \mathscr{L}$, then $F \cap L_o \neq \varnothing$. Hence Ψ maps $\mathscr{L}$ into R_+. It is easy to check that (1.9)-(1.11) hold. Thus, by Theorem 4.3, there is a Lévy-Khinchin measure ψ on L satisfying

$$\psi(L \setminus F) = \Psi(F), \ F \in \mathscr{L}.$$

Let $x \in L$. Choose $F_1, F_2, \ldots \in \mathscr{L}$ such that $F_n \downarrow (\uparrow x)$. Then $\psi(L \setminus F_n) \uparrow \psi(L \setminus \uparrow x)$. Now proceed as in the proof of Theorem 3.5 and show that $\Psi(F_n) \uparrow M(x)$. This shows (4.8).

The remaining part of the proof is easy and hence omitted.

5. Infinite divisibility.

Consider a random variable ξ in a continuous semilattice L. Assume L has a top and a second countable Scott topology. Write λ for the distribution of ξ. We say that λ (or ξ) is infinitely divisible if, for every n, there are independent and identically distributed random variables $\xi_1, \ldots, \xi_n$ in L such that

$$\xi \stackrel{d}{=} \wedge_i \xi_i.$$

It is not hard to see that λ is infinitely divisible iff for all $t > 0$ there is a probability measure λ_t on L satisfying

$$\lambda_t(F)^{1/t} = \lambda(F)$$

for $F \in \mathscr{L}$. The details are routine and hence omitted.

The first result of this section characterizes infinite divisibility in an important particular case. Its proof is an immediate application of Proposition 6.10 in Berg, Christensen & Ressel (1984) and therefore omitted.

PROPOSITION 5.1. *Let L be a continuous semi-lattice with a top and a second countable Scott topology. Then the formula*

$$\lambda(F) = \exp(-\psi(L \setminus F)), \quad F \in \mathscr{L},$$

defines a bijection between the family of all Lévy-Khinchin measures ψ on L and the family of all infinitely divisible probability measures λ on L satisfying

$$\lambda(F) > 0, \quad F \in \mathscr{L}.$$

In the case when L is a lattice we may introduce the distribution function

$$\Lambda(x) = P(x \leqq \xi), \quad x \in L,$$

of ξ. Clearly ξ is infinitely divisible iff Λ^t is a distribution function on L for all $t > 0$.

COROLLARY 5.2. *Let L be a continuous lattice with a second countable Scott topology. The formula*

$$\lambda(\uparrow x) = \exp(-\psi(L \setminus \uparrow x)), \quad x \in L,$$

defines a bijection between the family of all Lévy-Khinchin measures ψ on L and the family of all infinitely divisible probability measures λ on L satisfying

$$\lambda(\uparrow x) > 0, \quad x \in L, x \ll 1.$$

PROOF. Just note that

$$\lambda(\uparrow x) = \inf_{x \in F} \lambda(F),$$
$$\psi(L \setminus \uparrow x) = \sup_{x \in F} \psi(L \setminus \mathrm{F})$$

for all $x \in L$. Then apply Proposition 5.1.

We proceed to discuss the general case. Recall that ξ is a random variable in a continuous semi-lattice L. Write

$$L_\xi = \bigcap \{F^c;\ F \in \mathscr{L},\ \xi \in F^c \text{ a.s.}\}.$$

Note that L_ξ is the (Scott) support of ξ. Hence $\xi \in L_\xi$ a.s. Moreover, L_ξ is a continuous semi-lattice. Put

$$\mathscr{L}_\xi = \mathrm{OFilt}(L_\xi).$$

It is not difficult to show that

$$\mathscr{L}_\xi = \{F \bigcap L_\xi;\ F \in \mathscr{L},\ F \bigcap L_\xi \neq \varnothing\}$$

and that, for $F \in \mathscr{L}$, $F \bigcap L_\xi \neq \varnothing$ iff $P\{\xi \in F\} > 0$. Hence $P\{\xi \in F \bigcap L_\xi\} > 0$ if $F \in \mathscr{L}$, $F \bigcap L_\xi \neq \varnothing$. Note also that $L_\xi \in \mathscr{L}_\xi$. If ξ is infinitely divisible we can say more.

PROPOSITION 5.3. *Let L be a continuous semi-lattice with a second countable Scott topology, and let ξ be an infinitely divisible random variable in L. Let*

$$x = \vee \{y \in L;\ P\{y \leqq \xi\} > 0\}. \tag{5.1}$$

Then $L_\xi = \downarrow x$.

PROOF. We first show that L_ξ has a top. By a result of Lawson (1979) (recalled in Section 3) this will follow if we can show that $\mathscr{L}_\xi$ is a semi-lattice. For this, fix $F_1, F_2 \in \mathscr{L}$ and assume $\lambda(F_1 \bigcap F_2) = 0$. where we have written $\lambda = P\xi^{-1}$. For every $t > 0$ there is a probability measure λ_t on L such that $\lambda_t(F) = \lambda(F)^t$ for $F \in \mathscr{L}$. Now note that

$$1 = \lambda_t(F_1^c \bigcup F_2^c) \leqq \lambda_t(F_1^c) + \lambda_t(F_2^c),$$

from which we conclude that

$$t^{-1} \leqq t^{-1}(1 - \lambda(F_1)^t) + t^{-1}(1 - \lambda(F_2)^t).$$

Let $t \to 0$. Then

$$\infty \leqq -\log \lambda(F_1) - \log \lambda(F_2).$$

Hence $\lambda(F_1) = 0$ or $\lambda(F_2) = 0$. This shows that $\mathscr{L}_\xi$ is a semi-lattice.

Let x be the top of L_ξ. If $y \ll x$ we may choose $F \in \mathscr{L}$ with $x \in F \subset \uparrow y$. But then

$F \cap L_\xi \neq \emptyset$. Hence $P\{y \leqq \xi\} > 0$. This shows the less than or equal to part of (5.1). To see the remaining part, assume $P\{y \in F\} > 0$. Then $P\{\xi \in F\} > 0$ for some $F \in \mathscr{L}$, $F \subset \uparrow y$. We see that $F \cap L_\xi \neq \emptyset$, i.e. $x \in F$. Hence $y \leqq x$.

We do not hesitate to omit the proof of the following Lévy-Khinohin representation of the infinitely divisible probability measures on L.

THEOREM 5.4. *Let L be a continuous semi-lattice with a top and a second countable Scott topology. The formulae*

$$x = \vee\{y \in L; \lambda(\uparrow y) > 0\},$$

$$\psi(\downarrow x \setminus F) = -\log \lambda(F), \ F \in \mathscr{L}, x \in F,$$

defines a bijection between the set of all infinitely divisible probability measures λ on L and the set of all pairs (x, ψ), where $x \in L$ and ψ is a Lévy-Khinchin measure on $\downarrow x$.

Note that it may happen that the Lévy-Khinchin measure in the representation of an infinitely divisible probability measure is identically zero.

Also note that the second formula in Theorem 5.4 may be replaced by

$$\psi(\downarrow x \setminus \uparrow y) = -\log \lambda(\uparrow y), \ y \in L, y \leqq x,$$

if L is a lattice. A similar remark applies to Propositions 5.5 and 5.6 below, the proofs of which are easy and hence omitted.

PROPOSITION 5.5. *Let λ be a probability measure on L, and put*

$$\Psi(F) = -\log \lambda(F), \ F \in \mathscr{L}, \tag{5.2}$$

$$\mathscr{L}_\Psi = \{F \in \mathscr{L}; \Psi(F) < \infty\}. \tag{5.3}$$

Then λ is infinitely divisible iff $\mathscr{L}_\Psi$ is a semi-lattice and

$$\Delta_{F_1} \cdots \Delta_{F_n} \Psi(F) \leqq 0, \ n \in \mathbf{N}, F, F_1, \ldots, F_n \in \mathscr{L}_n. \tag{5.4}$$

PROPOSITION 5.6. *Let $\Psi: \mathscr{L} \to \bar{\mathbf{R}}_+$ and define $\mathscr{L}_\Psi$ as in* (5.3). *Assume $\mathscr{L}_\Psi$ is a semi-lattice and that* (5.4) *holds together with*

$$\Psi(F) = \lim_n \Psi(F_n), \ F, F_1, F_2, \ldots \in \mathscr{L}, F_n \uparrow F,$$

$$\Psi(L) = 0.$$

Then there is a unique infinitely divisible probability measure λ on L satisfying (5.2).

6. Applications to random set theory.

Here we write down our extension of Choquet's characterization of the distributions of all random sets in a locally compact second countable Hausdorff space. We then fill in those details from the discussion in the introduction of applications to random set theory, that are neither obvious nor proved elsewhere.

Let S be a space endowed with a continuous and second countable topology $\mathscr{G}$. We write $\mathscr{F}$ for its collection of all closed sets and $\mathscr{K}$ for its collection of all compact and saturated subsets. Note that all subsets of a $T1$-space are saturated. See e.g. Giertz et al (1980).

A non-empty $F \in \mathscr{F}$ is said to be *irreducible* if $F \subset F_1 \bigcup F_2$ for some $F_1, F_2 \in \mathscr{F}$ implies that $F \subset F_1$ or $F \subset F_2$. Clearly all singleton closures are irreducible. The topological space S is called *sober* if every irreducible closed set is the singleton closure of a unique member of S. That is to say, S is sober iff the mapping $s \to \{s\}^-$ is a bijection between S and the collection of all irreducible closed subsets of S. Note that all Hausdorff spaces are sober. The verification of this fact is straightforward. Moreover, any continuous poset endowed with its Scott topology is sober. Cf Proposition 5.2 in Lawson (1979).

Assume now that S is sober. Recall that $\mathscr{F}$ is continuous and has a second countable Scott topology under reverse inclusion. It can be shown that $\mathscr{K}$ is a continuous semi-lattice under the same order. Indeed, Hofmann & Lawson (1980) proves that the mapping

$$K \to \{F \in \mathscr{F};\, K \bigcap F = \varnothing\} \in \mathrm{OFilt}(\mathscr{F}),\ \ K \in \mathscr{K}$$

is an isomorphism. In particular this shows that Scott$(\mathscr{K})$ is second countable.

Say that a real-valued mapping T on $\mathscr{K}$ is an *alternating capacity* if $K_n \downarrow K$ implies $T(K_n) \downarrow T(K)$ and if

$$T_n(K; K_1, \ldots, K_n) \leqq 0,\ \ n \in \mathsf{N},\ K, K_1, \ldots, K_n \in \mathscr{K}.$$

Note that the latter condition is equivalent to

$$T(K \bigcup K_1) \geqq T(K),\ \ K, K_1 \in \mathscr{K},$$

$$T(K \bigcup K_2) + T(K \bigcup K_1) \geqq T(K \bigcup K_1 \bigcup K_2),\ \ K, K_1, K_2 \in \mathscr{K},$$

etc, cf Matheron (1975), and Berg, Christensen & Ressel (1984).

The following extension of Choquet's theorem is, a view of the discussion above, an obvious consequence of Corollary 3.6.

THEOREM 6.1. *Let S be a space endowed with a continuous second countable sober topology. Then the formula*

$$P\{\xi \bigcap K \neq \varnothing\} = T(K),\ \ K \in \mathscr{K},$$

defines a bijection between the family of distributions $P\xi^{-1}$ of random closed sets ξ in S and the family of alternating capacities T satisfying $T(\varnothing) = 0$ and $T(K) \leqq 1$, $K \in \mathscr{K}$.

Theorem 6.1 extends to a bijection between the family of Lévy-Khinchin measures ψ on $\mathscr{F}$ and the family of all alternating capacities Ψ with $\Psi(\varnothing) = 0$, given by

$$\Psi(K) = \psi\{F \in \mathscr{F}; F \cap K \neq \varnothing\}, \quad K \in \mathscr{K}.$$

Use Theorem 4.3.

A random closed set ξ in S is called infinitely divisible if, for every $n \in \mathsf{N}$, there are independent and identically distributed random closed sets $\xi_1, \ldots, \xi_n$ in S such that $\xi \overset{d}{=} \cup_i \xi_i$ holds. Our next result gives a rather complete description of the infinitely divisible distributions. It extends the applicability of results in Matheron (1975) and Berg, Christensen & Ressel (1984). There is no need for a proof.

THEOREM 6.2. *Let S be as in Theorem 6.1. For every infinitely divisible random closed set ξ in S, the closed set*

$$H = \cap\{F \in \mathscr{F}; P\{\xi \subset F\} > 0\}$$

satisfies $H \subset \xi$ a.s., and, if $H \neq S$, the formula

$$\Psi(K) = -\log P\{\xi \cap K = \varnothing\}, \quad K \in \mathscr{K}, \quad K \cap H = \varnothing,$$

defines an alternating capacity on H^c. Conversely, for every pair (H, Ψ), where $H \subset S$, $H \neq S$, is closed and Ψ is an alternating capacity on H^c with $\Psi(\varnothing) = 0$, there exists an infinitely divisible random set ξ in S with distribution

$$P\{\xi \cap K = \varnothing\} = \begin{cases} \exp(-\Psi(K)) & \text{if } K \in \mathscr{K},\ K \cap H = \varnothing, \\ 0 & \text{if } K \in \mathscr{K},\ K \cap H \neq \varnothing. \end{cases}$$

Moreover, for $\Psi: \mathscr{K} \to \bar{\mathsf{R}}_+$, $\Psi(\varnothing) = 0$, write

$$\mathscr{K}_\Psi = \{K \in \mathscr{K}; \Psi(K) < \infty\}.$$

If $\mathscr{K}_\Psi$ is closed under finite non-empty unions and if

$$\Psi(K) = \lim_n \Psi(K_n), \quad K, K_1, K_2, \ldots \in \mathscr{K}, \ K_n \downarrow K,$$

$$\Psi_n(K; K_1, \ldots, K_n) \leqq 0, \quad n \in \mathsf{N}, \quad K, K_1, \ldots, K_n \in \mathscr{K}_\Psi,$$

then there exists an infinitely divisible random set ξ with distribution

$$P\{\xi \cap K = \varnothing\} = \exp(-\Psi(K)), \quad K \in \mathscr{K}.$$

In this case $K \in \mathscr{K}_\Psi$ iff $K \in \mathscr{K}$, $K \cap H = \varnothing$, where H is as above.

This results of Sections 3–5 may be applied to the collection $\mathscr{K}$. It is interesting to compare the conclusions with the assertions of Theorems 6.1 and 6.2, since $\mathscr{K}$ is isomorphic to the collection of all open filters on $\mathscr{F}$. However, we leave this to the reader.

Now let $\mathscr{C}$ denote the collection of compact and convex subsets of R^d for some $d \in \mathsf{N}$. We endow $\mathscr{C}$ with reverse inclusion. Note that $\mathscr{C}$ is a continuous lattice and that $\varnothing$ is the top of $\mathscr{C}$. Write $\mathscr{C}' = \mathscr{C} \setminus \{\varnothing\}$.

Let $C, D \in \mathscr{C}$. Giertz et al (1980) shows

$$C \ll D \Leftrightarrow D \subset C^\circ.$$

Let $\mathscr{G}_o$ be a countable base for the Euclidean topology on R^d consisting of convex relatively compact sets. Clearly we may choose $G_1, \ldots, G_n \in \mathscr{G}_o$ such that $D \subset \bigcup_{i=1}^n G_i \subset C^\circ$. But then $D \subset \vee_{i=1}^n G_i^- \subset C$, where we have written $\vee_{i=1}^n G_i^-$ for the convex closure of $\bigcup_{i=1}^n G_i^-$. This shows that $\mathscr{C}$ has a countable separating subset, and we conclude by Proposition 3.1 that the Scott topology on $\mathscr{C}$ is second countable.

Write $\Sigma' = \Sigma(\mathscr{C}) \bigcap \mathscr{C}'$ for the restriction of Σ to $\mathscr{C}'$. A routine argument from measure theory shows that Σ' is generated by the family $\{D \in \mathscr{C}'; D \subset C\}$, $C \in \mathscr{C}'$. Moreover,

$$\Sigma' = \{B \in \Sigma(\mathscr{C}); \varnothing \notin B\},$$

since $\mathscr{C}' \in \Sigma(\mathscr{C})$.

Consider a $\mathscr{C}'$-valued mapping ξ of some probability space $(\Omega, \mathscr{R}, P)$. Assume ξ is measurable w.r.t. Σ'. If $G \subset \mathsf{R}^d$ is open then $\{C \in \mathscr{C}; C \subset G\}$ is a Scott open set in $\mathscr{C}$, and therefore measurable w.r.t. Σ. Hence

$$\{\xi \subset G\} \in \mathscr{R}, \ G \subset \mathsf{R}^d \text{ open}.$$

Denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the Euclidean norm and inner product resp. By a closed half-space we shall understand a set of the type

$$H(x, \alpha) = \{y \in \mathsf{R}^d; \ (y, x) \leqq \alpha\}, \ x \in \mathsf{R}^d, \ \|x\| = 1, \ \alpha \in R.$$

We write $\mathscr{H}$ for the collection of all such sets. If $H \in \mathscr{H}$ then $H = \bigcap_n G_n$ for some sequence $G_1, G_2, \ldots$ of open sets. Hence

$$\{\xi \subset H\} \in \mathscr{R}, \ H \in \mathscr{H}. \tag{6.1}$$

Let d_C denote the distance function of $C \in \mathscr{C}'$, i.e.

$$d_C(x) = \inf_{y \in C} \|x - y\|, \ x \in \mathsf{R}^d.$$

Whenever $C \in \mathscr{C}'$ and $\varepsilon \geqq 0$ we furthermore introduce the parallel set

$$C_\varepsilon = \{x \in \mathsf{R}^d; \ d_C(x) \leqq \varepsilon\}.$$

Note that $C_\varepsilon \in \mathscr{C}'$. Fix $x \in \mathsf{R}^d$, $\|x\| = 1$, $\alpha \in \mathsf{R}$, $C \in \mathscr{C}'$ and $\varepsilon > 0$. The reader easily shows that $C \subset H(x, \alpha)$ iff $C_\varepsilon \subset H(x, \alpha + \varepsilon)$. Hence ξ_ε is measurable w.r.t. Σ' for all $\varepsilon > 0$.

The Hausdorff metric h on $\mathscr{C}'$ is defined by

$$h(C,D) = \inf\{\varepsilon \geqq 0; C \subset D_\varepsilon, D \subset C_\varepsilon\},\ C, D \in \mathscr{C}'.$$

It is well-known that h is complete and separable. Note that the infimum above is attained. That is to say, for $C, D \in \mathscr{C}'$ and $\varepsilon \geqq 0$, $h(C,D) \leqq \varepsilon$ iff $C \subset D_\varepsilon$ and $D \subset C_\varepsilon$. Hence

$$(6.2) \qquad \{h(\xi, C) \leqq \varepsilon\} \in \mathscr{R},\ \ C \in \mathscr{C}',\ \ \varepsilon \geqq 0.$$

We have now proved that ξ is measurable w.r.t. the Borel σ-field generated by the Hausdorff metric if ξ is measurable w.r.t. Σ'. To see the converse, assume (6.2). Fix $x \in \mathrm{R}^n$, $\|x\| = 1$ and put

$$f(C) = \sup_{y \in C}(x, y), C \in \mathscr{C}.$$

The reader easily shows

$$f(C_\varepsilon) = f(C) + \varepsilon,\ \ C \in \mathscr{C}',\ \ \varepsilon \geqq 0.$$

Now it is not hard to see that f is continuous w.r.t. the Hausdorff metric. But then we must have

$$\{\xi \subset H(x, \alpha)\} = \{f(\xi) \leqq \alpha\} \in \mathscr{R}.$$

Thus (6.1) holds.

If $C \in \mathscr{C}'$, then $C = \bigcap_n H_n$ for some $H_1, H_2, \ldots \in \mathscr{H}$. By (6.1),

$$\{\xi \subset C\} \in \mathscr{R},\ \ C \in \mathscr{C}'.$$

That is to say, ξ is measurable w.r.t. Σ'. We have thus shown that ξ is measurable w.r.t. Σ' iff ξ is measurable w.r.t. the Borel σ-field generated by the Hausdorff metric on $\mathscr{C}'$.

7. Measures on locally compact sober spaces.

The aim of this section is to prove two existence theorems for measures on a space equipped with a continuous second countable sober topology, one for finite measures and one for locally finite measures. We fix such a space S throughout this section and write $\mathscr{S}$ for its Borel σ-field. The notations $\mathscr{G}$, $\mathscr{F}$ and $\mathscr{K}$ are retained from Section 6.

It is known that S is locally compact in the sense that, whenever $s \in G \in \mathscr{G}$ we have $s \in K^\circ \subset K \subset G$ for some $K \in \mathscr{K}$. See Hofmann & Lawson (1978). On the other hand, the topology of any locally compact space is easily seen to be continuous. So, we might just as well assume S to be locally compact, second countable and sober.

Let us introduce

$$\bar{s} = \overline{\{s\}}, \quad s \in S,$$
$$\bar{S} = \{\bar{s};\ s \in S\},$$

and write $\bar{\mathscr{S}}$ for the σ-field on $\bar{S}$ generated by the sets

$$\{\bar{s};\ s \in G\}, \quad G \in \mathscr{G}.$$

Then, as the reader easily sees, the bijection $s \to \bar{s}$ is bimeasurable. Furthermore note that its range $\bar{S} \subset \mathscr{F}$ and that, for $F \in \mathscr{F}$, $F \in \bar{S}$ iff F is irreducible.

Our first result tells us that the mapping $s \to \bar{s}$ is an embedding of S into $\mathscr{F}$.

PROPOSITION 7.1. *We have*

$$\bar{S} \in \Sigma(\mathscr{F}),$$
$$\bar{\mathscr{S}} = \{B \in \Sigma(\mathscr{F});\ B \subset \bar{S}\}.$$

PROOF. Write $\mathscr{F}' = \mathscr{F} \setminus \{\varnothing\}$. It is easy to see that

$$\mathscr{F}' \setminus \bar{S} = \bigcup_{F_1, F_2 \in \mathscr{F}} \{F \in \mathscr{F};\ F \not\subset F_1, F \not\subset F_2, F \subset F_1 \bigcup F_2\}.$$

The reader easily verifies that the union on the right may be taken over all F_1, F_2 in some separating subset of $\mathscr{F}$. It follows that $\bar{S} \in \Sigma(\mathscr{F})$. Now note that $\bar{\mathscr{S}}$ is the minimal σ-field over the sets

$$\{H \in \mathscr{F}; H \subset F\} \bigcap \bar{S}, \quad F \in \mathscr{F}.$$

Hence $\bar{\mathscr{S}} = \Sigma(\mathscr{F}) \bigcap \bar{S}$. Now the final assertion is obvious.

Here is our existence theorem for finite measures on $(S, \mathscr{S})$.

THEOREM 7.2: *Let* μ: $\mathscr{F} \to \mathrm{R}_+$. *Assume* $\mu(\varnothing) = 0$, *that* μ *is increasing, that* $\mu(F_n) \to \mu(F)$ *as* $F_n \downarrow F$ *and that* μ *is additive in the following sense*

$$\mu(F_1 \bigcup F_2) + \mu(F_1 \bigcap F_2) = \mu(F_1) + \mu(F_2), \quad F_1, F_2 \in \mathscr{F}.$$

Then μ *extends to a unique finite measure on* S.

Our proof depends on the following proposition.

PROPOSITION 7.3. *Let* λ *be a finite measure on* $\mathscr{F}$, *and write*

$$\Lambda(F) = \lambda\{H \in \mathscr{F};\ H \subset F\}, \quad F \in \mathscr{F}.$$

Then λ *concentrates its mass to* $\bar{S}$ *iff* $\Lambda(\varnothing) = 0$ *and* Λ *is additive, i.e.*

$$\Lambda(F_1 \bigcup F_2) + \Lambda(F_1 \bigcap F_2) = \Lambda(F_1) + \Lambda(F_2), \quad F_1, F_2 \in \mathscr{F}.$$

PROOF. The result follows from the fact that

$$\lambda\{F \in \mathscr{F}; F \not\subset F_1, F \not\subset F_2, F \subset F_1 \bigcup F_2\} = \Lambda(F_1) - \Lambda(F_2) + \Lambda(F_1 \bigcap F_2),$$

which is valid for all F_1, $F_2 \in \mathscr{F}$. The details are left to the reader.

PROOF OF THEOREM 7.2. By Theorem 3.3, there is a finite measure λ on $\mathscr{F}$ satsfying

$$\lambda\{H \in \mathscr{F}; H \subset F\} = \mu(F), \quad F \in \mathscr{F}.$$

By Proposition 7.3, λ concentrates its mass to $\bar{S}$. The measure

$$\tilde{\mu}(B) = \lambda\{\bar{s}; s \in B\}, \quad B \in \mathscr{S},$$

is the required extension of μ. It is of course unique.

It is also possible to characterize the finite measures on S in terms of their restrictions to $\mathscr{G}$. This however is part of a characterization below of the locally finite measures on S. We say that a measure μ on $(S, \mathscr{S})$ is *locally finite* if $\mu(G) < \infty$ for all $G \in \mathscr{G}$ with $G \ll S$. Clearly this requirement holds iff $\mu(K) < \infty$ for all $K \in \mathscr{K}$. So the class of locally finite measures on S coincides with the class of Radon measures.

Whenever μ is a measure on $(S, \mathscr{S})$ we shall write $\bar{\mu}$ for the measure on $(\mathscr{F}, \Sigma(\mathscr{F}))$ given by

$$\bar{\mu}(B) = \mu\{s \in S; \bar{s} \in B\}, \quad B \in \Sigma(\mathscr{F}).$$

Note that

$$\bar{\mu}\{H \in \mathscr{F}; H \not\subset F\} = \mu(S \setminus F), \quad F \in \mathscr{F}.$$

Hence

PROPOSITION 7.4. *A measure μ on S is locally finite iff $\bar{\mu}$ is a Lévy-Khinchin measure on $\mathscr{F}$.*

The next result is similar to Proposition 7.3. We omit its proof.

PROPOSITION 7.5. *Let ψ be a Lévy-Khinchin measure on $\mathscr{F}$, and write*

$$\Psi(G) = \psi\{F \in \mathscr{F}; F \bigcap G \neq \varnothing\}, \quad G \in \mathscr{G}.$$

Put further $\mathscr{G}_o = \{G \in \mathscr{G}; G \ll S\}$. Then ψ is concentrated on $\bar{S}$ iff

$$\Psi(G_1 \bigcup G_2) + \Psi(G_1 \bigcap G_2) = \Psi(G_1) + \Psi(G_2), \quad G_1, G_2 \in \mathscr{G}_o.$$

It is now not hard to prove the following existence theorem for locally finite measures on S.

THEOREM 7.6. *Let $\mu: \mathscr{G} \to \bar{\mathsf{R}}_+$ be finite on $\mathscr{G}_o = \{G \in \mathscr{G}; G \ll S\}$. Assume further* *that $\mu(\varnothing) = 0$, that μ is increasing, that $\mu(G_n) \uparrow \mu(G)$ as $G_n \uparrow G$ and that μ is additive on $\mathscr{G}_o$, that is*

$$\mu(G_1 \bigcup G_2) + \mu(G_1 \bigcap G_2) = \mu(G_1) + \mu(G_2),\ G_1, G_2 \in \mathscr{G}_o.$$

Then μ is the restriction to $\mathscr{G}$ of a unique locally finite measure on S.

PROOF. By Theorem 4.5, there is a Lévy-Khinchin measure ψ on $\mathscr{F}$ satisfying

$$\psi\{H \in \mathscr{F};\ H \not\subseteq F\} = \mu(F^c),\ F \in \mathscr{F}.$$

Use Proposition 7.5 to conclude that ψ concentrates its mass to $\bar{S}$.

8. The Daniell-Kolmogorov theorem.

Let $L_1, L_2, \ldots$ be a sequence of continuous complete lattices and assume Scott(L_i) to be second countable for each i. Let us provide each finite product $\prod_{i=1}^n L_i$, and also the infinite product $L = \prod_i L_i$, with the coordinatewise order. Note that all these product posets are continuous complete lattices with second countable Scott topologies. By Theorem 3.3, probability measures on these product lattices may be characterized in terms of their distribution functions.

Assume now that we are given a projective sequence of distribution functions Λ_n on $\prod_{i=1}^n L_i$, $n \in \mathbf{N}$. That is to say,

$$\Lambda_{n+1}(x_1, \ldots, x_n, 1) = \Lambda_n(x_1, \ldots, x_n),\ n \in \mathbf{N},\ x_i \in L_i,\ 1 \leqq i \leqq n.$$

Here, and below, we write 0 for the bottom of L_n, $n \in \mathbf{N}$. Let us now write

$$\Lambda(x_1, x_2, \ldots) = \lim_n \Lambda_n(x_1, \ldots, x_n),\ x_i \in L_i,\ i \in N.$$

Then, clearly, Λ satisfies the requirement (1.3). Moreover, $\Lambda(0, 0, \ldots) = 1$. Thus it will follow that Λ is a distribution function on L if we can prove (1.2).

This is however easy: Fix $x_n = (x_{n1}, x_{n2}, \ldots) \in L$, $n \in \mathbf{N}$, such that $x_n \uparrow x = (x_1, x_2, \ldots) \in L$. Clearly $\Lambda(x) \leqq \Lambda(x_n) \downarrow \inf_n \Lambda(x_n)$. But

$$\begin{aligned}\inf_n \Lambda(x_n) &= \inf_n \inf_m \Lambda_m(x_{n1}, \ldots, x_{nm}) \\ &= \inf_m \inf_n \Lambda_m(x_{n1}, \ldots, x_{nm}) = \inf_m \Lambda_m(x_1, \ldots, x_m) = \Lambda(x).\end{aligned}$$

This shows (1.2). By Theorem 3.3, there is a unique probability measure λ on L with distribution function Λ.

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