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ORDERS, IN NON-EICHLER (R)-ALGEBRAS OVER GLOBAL FUNCTION FIELDS, HAVING THE CANCELLATION PROPERTY

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Abstract.

We study R-orders θ in central simple algebras A over global function fields K. In studying the locally free left θ-ideals, one has to deal with the following difficulty: the set of isomorphism classes of locally free left θ-ideals, $LF_1(\theta)$ is in general not a group; but one can consider the group $\mathscr{C}\ell(\theta)$ of stable isomorphism classes of these ideals. We say that θ has the cancellation property if $LF_1(\theta) = \mathscr{C}\ell(\theta)$, i.e. if all stably free θ-ideals are free.

A theorem of Jacobinski and Swan states that if A satisfies the Eichler condition with respect to R, then all R-orders in A do have the cancellation property.

In [DVG 1] we proved that there only exist R-orders with the cancellation property in a *non Eichler* (R)-*algebra* A if the center K of A is a rational function field.

In this paper we obtain a complete characterization of the non Eichler (R)-algebras which contain R-orders with the cancellation property. This characterization, although obtained by analytic-numbertheoretic methods, is of geometrical nature. We also determine all hereditary R-orders in these algebras which have the cancellation property.

0. Preliminaries.

For a complete discussion of the definitions and methods we refer to [D 1], [DVG 1] and [DVG 2].

Let A be a central simple K-algebra of index n (i.e. $n^2 = (A:K)$) and $K = \mathsf{F}_q(\mathscr{C})$ the function field of a complete regular curve $\mathscr{C}$, defined over F_q. F_q is the finite field with q elements which we suppose to be algebraically closed in K.

Recall that for $\mathfrak{p} \in \mathscr{C}$, the completion $A_{\mathfrak{p}} \cong M_{\kappa_{\mathfrak{p}}}(D_{\mathfrak{p}})$ with $D_{\mathfrak{p}}$ a skewfield of index $e_{\mathfrak{p}}$ over $K_{\mathfrak{p}}$, $\kappa_{\mathfrak{p}}$ is called the capacity and $e_{\mathfrak{p}}$ the ramificationindex of A at $\mathfrak{p}$ and $e_{\mathfrak{p}}\kappa_{\mathfrak{p}} = n$. We denote $S_{\mathrm{ram}} = \{\mathfrak{p} \in \mathscr{C} \mid 1 \neq e_{\mathfrak{p}}\}$, $S^t_{\mathrm{ram}} = \{\mathfrak{p} \in \mathscr{C} \mid e_{\mathfrak{p}} = n\}$ and $r_t = \# S^t_{\mathrm{ram}}$.

We fix a Dedekind ring R in K by choosing a finite non-empty subset $\mathscr{T}$ of

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$\mathscr{C}$ and $R = \left(\bigcap_{\mathfrak{p}\notin\mathscr{T}} R_{\mathfrak{p}}\right)\bigcap K$ where $R_{\mathfrak{p}}$ is the completion of the valuation ring corresponding to $\mathfrak{p}$. We denote $t = \#\mathscr{T}$ and for $\mathfrak{p}\in\mathscr{C}$ the absolute norm of $\mathfrak{p}$ is $\mathscr{N}\mathfrak{p} = q^{\varphi_{\mathfrak{p}}} = \#(R_{\mathfrak{p}}/\mathfrak{p})$.
In this paper we always restrict to *non Eichler (R)-algebras*, which means that $\mathscr{T} \subset S^{t}_{\text{ram}}$ (so $t \leqq t_t$), cf. [DVG 1].

Fix a genus $\mathscr{G}$ of R-orders in A ($\theta_1, \theta_2 \in \mathscr{G}$ then $\theta_{1,\mathfrak{p}} \cong \theta_{2,\mathfrak{p}}$ for every $\mathfrak{p}\notin\mathscr{T}$), then both $\mathscr{C}\ell(\theta)$ and $LF_1(\theta)$ do not depend on the choice of $\theta\in\mathscr{G}$, so the cancellation property (C.P.) is a property of the genus $\mathscr{G}$, cf. [V].
For $\theta\in\mathscr{G}$ we choose a set $\{\theta x_i \mid i \leqq i \leqq r_\theta\}$ of representatives for the isomorphism-classes of stably free left θ-ideals (x_i are idèles of A). Remark that θ has the C.P. iff $r_{\theta_1} = 1$ for some $\theta_1\in\mathscr{G}$.

Denote $\theta_i = x_i^{-1}\theta x_i \in\mathscr{G}$, the right order of θx_i, and $\hat{w}_i = \#\theta'_i$, where $X' =$ kernel $(nr/_X)$ is the kernel of the reducec norm map nr.

The *measure of stably free θ-ideals* is given by (cf. [V]):

$$\mathscr{M}_\theta(\mathscr{S}) = \#\mathscr{C}\ell(\theta) \sum_{1\leqq i\leqq r_\theta} \frac{(R^\circ : nr\,\theta_i^*)}{\hat{w}_i}$$

where $R^\circ = R^* \bigcap \left(\bigcap_{\mathfrak{p}\notin\mathscr{T}} nr(\theta_{\mathfrak{p}}^*)\right)$, $\theta\in\mathscr{G}$.

For hereditary orders θ one has $R^\circ = R^*$, cf. [R], [D 1]. By using analytic methods, M.F. Vigneras was able to calculate $\mathscr{M}_\theta(\mathscr{S})$ for Z-orders θ in totally definite quaternion algebras over number fields, cf. [V]. These methods can be extended to calculate $\mathscr{M}_\theta(\mathscr{S})$ for R-orders in non Eichler (R)-algebras over global function fields of any index $n \geqq 2$:

(F1) $$\mathscr{M}_\theta(\mathscr{S}) = \#\mathscr{C}\ell(\theta)\, q^{(n^2-1)(g_K-1)}\zeta_K(2)\ldots\zeta_K(n)\prod_{\mathfrak{p}\in\mathscr{C}} T_{\mathfrak{p}}$$

where g_K is the genus of K, $\zeta_K(s) = \prod_{\mathfrak{p}\in\mathscr{C}} (1 - \mathscr{N}\mathfrak{p}^{-s})^{-1}$ is the zetafunction of K and the factors $T_{\mathfrak{p}}$ are given by:

$$T_{\mathfrak{p}} = \prod_{\substack{1\leqq i\leqq n-1\\ i\not\equiv 0 \bmod e_{\mathfrak{p}}}} (\mathscr{N}\mathfrak{p}^i - 1)\cdot\frac{v_{\mathfrak{p}}(\Lambda'_{\mathfrak{p}})}{v_{\mathfrak{p}}(\theta'_{\mathfrak{p}})}$$

with $\Lambda_{\mathfrak{p}}$ a maximal $R_{\mathfrak{p}}$-order containing $\theta_{\mathfrak{p}}$ and $v_{\mathfrak{p}}$ a bi-invariant Haarmeasure on $\Lambda'_{\mathfrak{p}}$.

Remark that $T_{\mathfrak{p}} = 1$ for almost every $\mathfrak{p}\in\mathscr{C}$.

In the function field case (F1) can also be deduced using the relation between 'θ-divisors' and the genus zetafunction, cf. [D 1].
From all this we obtain the following '*cancellation formula*': All R-orders in

$\mathscr{G}$ have the C.P. iff there exists $\theta \in \mathscr{G}$ such that:

$$\text{(F2)} \qquad q^{(n^2-1)(g_K-1)}\zeta_K(2)\dots\zeta_K(n)\prod_{\not{p}\in\mathscr{G}} T_{\not{p}} \overset{(\leqq)}{=} \frac{(R^\circ : nr\,\theta^*)}{\hat{w}}$$

Remark that if (F2) holds with the inequality sign '$\leqq$' then the equality follows.

To calculate the right-hand side of (F2) we introduce: $\overline{\mathsf{F}_q(\theta)} = \theta \cap \left(\bigcap_{\not{p}\in\mathscr{T}} \Lambda_{\not{p}} \right)$ where $\Lambda_{\not{p}}$ is the unique maximal $R_{\not{p}}$-order in the division algebra $A_{\not{p}}$. One can think of $\overline{\mathsf{F}_q(\theta)}$ as being the algebraic closure of F_q in θ, cf. [DVG 2]; so $\overline{\mathsf{F}_q(\theta)} \cong \mathsf{F}_{q^\ell}$ with $\ell \mid n$. For hereditary R-orders θ we obtain:

$$\frac{(R^*: nr\,\theta^*)}{\hat{w}} = \frac{(\mathsf{F}_q^*: nr(\mathsf{F}_{q^\ell}^*))}{\hat{w}} \cdot (R^*/\mathsf{F}_q^*: nr(\theta^*/\mathsf{F}_{q^\ell}^*)) = \frac{q-1}{q^\ell - 1} x_\theta$$

with $x_\theta = (R^*/\mathsf{F}_q^*: nr(\theta^*/\mathsf{F}_{q^\ell}^*)) \mid n^{t-1}$.

A *n*on *E*ichler (R)-algebra which contains R-orders with the *c*ancellation property is called an NEC(R)-*algebra.*

1. Characterization of the NEC(R)-algebras.

From [DVG 1] theorem 2.1, the center K of NEC (R)-algebras is a rational function field, so $K = \mathsf{F}_q(t)$, $g_K = 0$ and the zetafunction of K is

$$\zeta_K(s) = \frac{1}{(1-q^{-s})(1-q^{1-s})}.$$

If we put $\mathscr{T} = \{\not{p}_1, \dots, \not{p}_t\}$ and denote φ_i the degree of $\not{p}_i$, (F2) yields:

$$\text{(C.F.)} \qquad \frac{\prod_{1\leqq i\leqq t} (q^{\varphi_i}-1)\dots(q^{(n-1)\varphi_i}-1)}{(q-1)^2\dots(q^{n-1}-1)^2} \prod_{\not{p}\notin\mathscr{T}} T_{\not{p}} \overset{(\leqq)}{=} x_\theta \frac{q^n-1}{q^\ell-1}$$

with $\overline{\mathsf{F}_q(\theta)} \cong \mathsf{F}_{q^\ell}$ and $x_\theta \mid n^{t-1}$.

By a result of Fröhlich, cf. [F, p. 117], all maximal R-orders in NEC (R)-algebras have the C.P., so A is an NEC (R)-algebra iff (C.F.) holds for some $\Lambda \in \mathscr{G}_{\max}$ ($\mathscr{G}_{\max}$ is the genus of maximal R-orders in A).

Before passing to the main theorem we prove two lemmas about $\overline{\mathsf{F}_q(\Lambda)}$ and x_Λ, $\Lambda \in \mathscr{G}_{\max}$.

LEMMA 1. *Let A be a central simple algebra of index n over $\mathsf{F}_q(t)$ such that $S_{\text{ram}} = S_{\text{ram}}^\ell =$ then $\overline{\mathsf{F}_q(\Lambda)} \simeq \mathsf{F}_{q^n}$ for some $\Lambda \in \mathscr{G}_{\max}$ if and only if $\varphi_{\not{p}} \not\equiv 0 \bmod n$ for every $\not{p} \in S_{\text{ram}}$.*

PROOF. The proof in [DVG 2], where we assumed n to be prime extends directly since $S_{\text{ram}} = S_{\text{ram}}^\ell$.

LEMMA 2: *Let A be a non Eichler (R)-algebra of index n over $\mathsf{F}_q(t)$ with $q = 2$, $n = 3$ and $S_{\mathrm{ram}} = \{t^{-1}, t, t+1\}$ such that $\imath = \# \mathcal{T} \geqq 2$ then $x_\theta \geqq 3$ for every R-order θ in A.*

PROOF. Remark that $\mathcal{T} \subset S_{\mathrm{ram}}$, so $\imath \leqq 3$. Let $\imath = 3$ ($\imath = 2$ can be treated similarly) then $R^*/\mathsf{F}_q^* = \{t^k(t+1)^l \mid k, l \in \mathsf{Z}\}$.

It is sufficient to show that for $c = t, t^2, (t+1)^2$ we have $c \notin nr\,\theta^*$.

Assume that $nr(\alpha) = c$ for some $\alpha \in \theta$ then $L = \mathsf{F}_q(t)(\alpha)$ defines a splitting field in A and $f_\alpha(Y)$, the minimal polynomial of α, has coefficients in R. Moreover t^{-1}, $t, t+1$ may not decompose in L, cf. [R, p. 238–239], from which we easily obtain a contradiction.

In central simple algebras over function fields of curves there exists a notion of "*genus of the algebra*" g_A, and one also has a Riemann-Roch Theorem (due to E. Witt), cf. [Wi].

Using this "geometric" invariant we can characterize NEC(R)-algebras as follows:

THEOREM 3. *A non Eichler (R)-algebra A over a global function field K is an NEC(R)-algebra if and only if $g_A \leqq \varphi_b$ with $\varphi_b \in \mathsf{N}$ such that all primes in K of degree $\leqq \varphi_b$ are totally ramified in A. Moreover if $\varphi_b \neq 0$ then the choice of R is restricted by $\imath \geqq q - \varphi_b$*

It turns out that, at least partially, our results can be extended to central simple algebras over function fields of curves, defined over some infinite field ($\mathsf{Q}, \mathsf{Q}_{p}, \ldots$). This is done by P. Salberger cf. [V – Sa, appendix]. Salberger's proof is geometrical in the sense that it describes the problem in terms of the Brauer-Severi scheme of an order; it also relies on Witt's Riemann-Roch.

Before proving theorem 3 we determine the non Eichler(R)-algebras with $g_A \leqq \varphi_b$, using the Hasse-invariants of the algebra. Recall that the *Hasse-invariants* of a central simple algebra A determine (A), the class of A in $\mathrm{Br}(K)$, completely. Since non Eichler (R)-algebras are division algebras, we can characterize them by giving their Hasse-invariants, cf. [R, p. 266–277].

For the Hasse-invariants $\left\{\frac{s_p}{e_p} \,\middle|\, p \in \mathscr{C}\right\}$ one has $(s_p, e_p) = 1$ and $\sum_{p \in S_{\mathrm{ram}}} \frac{s_p}{e_p} \equiv 0 \bmod \mathsf{Z}$.

We will see that if n, q, $\mathcal{T} \subset S_{\mathrm{ram}}$ and $\{e_p \mid p \in S_{\mathrm{ram}}\}$ satisfy certain conditions A will be an NEC(R)-algebra. This means that for every choice of $\{s_p \mid p \in S_{\mathrm{ram}}\}$ for which $(s_p, e_p) = 1$ and $\sum_{p \in S_{\mathrm{ram}}} \frac{s_p}{e_p} \equiv 0 \bmod \mathsf{Z}$; the corresponding class (A) in $\mathrm{Br}(K)$ contains an NEC(R)-algebra.

We reformulate the condition $g_A \leqq \varphi_b$ in terms of the Hasse-invariants, using the generalization of the Hurwitz formula, cf. [V – V, Theorem 0.3], [D 1, lemma III.5]:

(H.F.) $$g_A = 1 + n^2(g_K - 1) + \tfrac{1}{2} \sum_{\not{p} \in \mathscr{C}} n \kappa_{\not{p}} \varphi_{\not{p}} (e_{\not{p}} - 1)$$

If $\varphi_b \neq 0$ then all primes of degree $\leqq \varphi_b$ are contained in S^{ℓ}_{ram}, so $g_A \geqq 1 + n^2(g_K - 1) + \frac{n(n-1)}{2}(q^{\varphi_b} + 1)$. For $\varphi_b \geqq 3$ this yields $g_A > \varphi_b$. For $\varphi_b \leqq 2 (\# S_{\text{ram}} \geqq 2, \imath_\ell \geqq 1)$ $g_A \leqq 2$ yields $g_K = 0$. We deduce further conditions on the Hasse-invariants for the particular values of $\varphi_b = 0, 1, 2$, (for $\imath_\ell = 1$ we also use that $\sum_{\not{p} \in S_{\text{ram}}} \frac{s_{\not{p}}}{e_{\not{p}}} \equiv 0 \bmod \mathbf{Z}$ with $(s_{\not{p}}, e_{\not{p}}) = 1$).

We can reformulate Theorem 3 as follows:

THEOREM 3 (*bis*): *A non Eichler (R)-algebra A over a global function field K is an NEC(R)-algebra if and only if K is a rational function field* $\mathsf{F}_q(t)$ *and one of the following conditions is satisfied.*

i) $S_{\text{ram}} = \{\not{p}_1, \not{p}_2\}$ *with* $\varphi_1 = \varphi_2 = 1$
ii) $n = 2$ *and* $S_{\text{ram}} = \{\not{p}_1, \not{p}_2\}$ *with* $\varphi_1 \cdot \varphi_2 = 2$
iii) $n = 2, q = 2$ *and* $S_{\text{ram}} = \{t^{-1}, t, t+1, t^2+t+1\}$
iv) $n = 2, q = 3, S_{\text{ram}} = \{t^{-1}, t, t+1, t-1\}$ *and* $\# \mathscr{T} \geqq 3$
v) $n = 3, q = 2, S_{\text{ram}} = \{t^{-1}, t, t+1\}$ *and* $\# \mathscr{T} \geqq 2$

PROOF. If $g_K \neq 0$ the statement follows from [DVG 1, theorem 2.1.]. So we can assume that $K = \mathsf{F}_q(t)$.

We only prove the theorem for $n > 2$. For $n = 2$ the proof is similar and is done in [DVG 3], remark that for $n = 2$ the result is slightly different.

We must show that (C.F.) is only satisfied for maximal orders in non Eichler (R)-algebras corresponding to i) or v).

We introduce the following notation:
L_{hs} (resp. R_{hs}) is the left-hand (resp. right-hand) side of (C.F.)

$$T_{\not{p}} = \prod_{\substack{1 \leqq i \leqq n-1 \\ i \not\equiv 0 \bmod e_{\not{p}}}} (\mathscr{N}\not{p}^i - 1) = (q^{\varphi_{\not{p}}} - 1)./\dots/.(q^{\varphi_{\not{p}}(n-1)} - 1)$$

First we check that if A corresponds to i) or v) then $L_{\text{hs}} \leqq R_{\text{hs}}$ for a good choice of $\Lambda \in \mathscr{G}_{\max}$:

i) $L_{\text{hs}} = 1 \leqq x_\Lambda \dfrac{q^n - 1}{q^\ell - 1} = R_{\text{hs}}$

v) $L_{\text{hs}} = 3$ and the lemmas 1 and 2 provide that $\ell = n$ and $x_\Lambda \geqq 3$ for some $\Lambda \in \mathscr{G}_{\max}$; $L_{\text{hs}} \leqq R_{\text{hs}}$ follows.

Now we prove the converse: for all the other non Eichler (R)-algebras A over $\mathsf{F}_q(t)$ (of index $n > 2$) $L_{hs} > R_{hs}$ for some $\Lambda \in \mathscr{G}_{max}$.

If A corresponds to v) and $\imath = 1$ then $L_{hs} = 3 > 1 = R_{hs}$. The remaining non Eichler (R)-algebras are split up, using the following scheme ($n > 2$):

A.1) $q \neq 2$ or $n \neq 3$ and $\imath_\ell \geqq 3$.
A.2) $q \neq 2$ or $n \neq 3$, $\imath = r_\ell = 2$ and at least one prime in S_{ram} has degree > 1.
A.3) $q \neq 2$ or $n \neq 3$, $\imath = 1 \leqq \imath_\ell \leqq 2$ and at least one prime in S_{ram} has degree > 1.
A.4) $q \neq 2$ or $n \neq 3$, $\imath_\ell \leqq 2$, $\# S_{ram} > 2$ and all primes in S_{ram} have degree 1.
A.5) $q = 2$, $n = 3$ and S_{ram} is not as in i) or v).

A.1) $q \neq 2$ or $n \neq 3$ and $\imath_\ell \geqq 3$:
$L_{hs} \geqq \{(q-1)\dots(q^{n-1}-1)\}^{\imath_\ell - 2}$ and $R_{hs} \leqq n^{\imath - 1}(q^{n-1} + .. + 1)$;
For $n \geqq 6$ or $n \geqq 5, q \neq 2$ or $n \geqq 4, q \geqq 4$ or $n \geqq 3, q \geqq 11$ one can easily show that $(q-1)\dots(q^{n-1}-1) > n^2(q^{n-1} + \dots + 1)$; since $\imath_\ell \geqq \imath$ this implies $L_{hs} > R_{hs}$ in these cases.
For the remaining "small values" of n and q, remark first that if $\varphi_{\not p} > 1$ for some $\not p \in S_{ram}^\ell$ then the corresponding factor $T_{\not p}$ in L_{hs} satisfies $T_{\not p} > n^2(q^{n-1} + \dots + 1)$ and $L_{hs} > R_{hs}$ follows.
If $\varphi_{\not p} = 1$ for every $\not p \in S_{ram}^\ell$, we rewrite (C.F.) as follows:

$$(q^\ell - 1)\{(q-1)\dots(q^{n-1}-1)\}^{\imath_\ell - 2} \prod_{1 < e_{\not p} < n} T_{\not p} = x_\Lambda(q^n - 1)$$

Since $x_\Lambda \mid n^{\imath - 1}$ and $\imath_\ell \geqq 3$ we find in all the remaining cases, a prime which divides the left-hand side, but does not divide the right-hand side, so (C.F.) cannot hold.

A.2) $q \neq 2$ or $n \neq 3$, $\imath_\ell = \imath = 2$ and at least one prime $\not p$ in S_{ram} has degree $\varphi_{\not p} > 1$: Let $\mathscr{T} = \{\not p_1, \not p_2\}$ with respective degrees φ_1, φ_2.

* $\varphi_i > 1$ *for* $i = 1$ *or* 2: $L_{hs} \geqq (q+1)\dots(q^{n-1}+1)$; we calculate this product and deduce $(q+1)\dots(q^{n-1}+1) > (n-2)\dfrac{q^n - 1}{q - 1} + q^{\frac{n(n-1)}{2}}$. But $q \neq 2$ or $n \neq 3$ so $q^{\frac{n(n-1)}{2}} \geqq 2\dfrac{q^n-1}{q-1}$ and $L_{hs} > R_{hs}$ follows.

** $\varphi_1 = \varphi_2 = 1$: $L_{hs} \geqq (q^{\varphi_{\not p}} - 1)(q^{(n-1)\varphi_{\not p}} - 1) > n\dfrac{q^n - 1}{q - 1} \geqq R_{hs}$

A.3) $q \neq 2$ or $n \neq 3$, $\imath = 1 \leqq \imath_\ell \leqq 2$ and $\varphi_{\not p} > 1$ for some $\not p \in S_{ram}$:
* $\varphi_{\not p} > 1$ for some $\not p \in S_{ram}^\ell$:

$$L_{hs} \geqq \frac{(q+1)\dots(q^{n-1}+1)}{(q-1)\dots(q^{n-1}-1)}(q-1)(q^n-1) > \frac{q^n-1}{q-1} \geqq R_{hs}.$$

** $\mathcal{T} = \{\not{p}_1\}$ with $\varphi_1 = 1$ and $\varphi_2 > 1$ with $1 < e_2 < n$:
If $\imath_\ell = 2$ then $L_{\text{hs}} > R_{\text{hs}}$ follows as in A.2**. So assume that $\imath_\ell = 1$ and denote $S_{\text{ram}} = \{\not{p}_1, \not{p}_2, \ldots \not{p}_k\}$. The conditions on the Hasse-invariants provide that $k \geqq 3$ and l.c.m $\{e_2, \ldots, e_k\} > n$. We rewrite (C.F.) as follows:

$$(q^\ell - 1)(q^{\varphi_2} - 1)./../.(q^{(n-1)\varphi_2} - 1)\ldots(q^{\varphi_k} - 1)./.. /. (q^{(n-1)\varphi_k} - 1) \overset{(\leq)}{\geqq} (q-1)(q^2-1)\ldots(q^{n-1}-1)(q^n-1)$$

For $1 \leqq m \leqq n-2$ we find $e_i, i \neq 1$ such that $e_i \nmid m$ and the factor $(q^{m\varphi_i} - 1)$ is not barred is $T_{\not{p}_i}$. Moreover (since $k \geqq 3$ and $\varphi_2 > 1$):

$$(q^{(n-1)\varphi_2} - 1)(q^{(n-1)\varphi_3} - 1) > (q^n - 1)(q^{n-1} - 1)$$

so (C.F.) cannot hold.

A.4) $q \neq 2$ or $n \neq 3$, $\imath_\ell \leqq 2$, $\# S_{\text{ram}} > 2$ and $\varphi_{\not{p}} = 1$ for every $\not{p} \in S_{\text{ram}}$:
By lemma 1, we can find $\Lambda \in \mathcal{G}_{\max}$ such that $\ell = n$, so $R_{\text{hs}} \leqq n$. If $\imath_\ell = 2$ then $L_{\text{hs}} \geqq (q-1)./../.(q^{n-1} - 1) > n \geqq R_{\text{hs}}$ and if $\imath_\ell = 1$ a similar reduction as in A.3)** is possible.

A.5) $q = 2, n = 3$ and $\varphi_{\not{p}} \geqq 2$ for some $\not{p} \in S_{\text{ram}} = S^\ell_{\text{ram}}$:
If $\varphi_{\not{p}} \geqq 3$ then $L_{\text{hs}} \geqq \frac{1}{9}(2^3 - 1)(2^6 - 1)\, 3^{\imath_\ell - 1} > 7.3^{\imath - 1} \geqq R_{\text{hs}}$
If $\varphi_{\not{p}} \leqq 2$ for every $\not{p} \in S_{\text{ram}}$ then $\ell = n$ and $L_{\text{hs}} > R_{\text{hs}}$ follows.
This settles theorem 3.

2. The cancellation property for non-maximal orders.

For R-orders θ in an arbitrary genus $\mathcal{G}$ it is not possible to calculate $v_{\not{p}}(\theta'_{\not{p}})$ explicitly, therefore using the methods of the preceding paragraphs we are unable to decide whether a given R-order θ has the cancellation property.
However, all hereditary R-orders with the cancellation property can be determined (since $v_{\not{p}}(\theta'_{\not{p}})$ can be calculated in this case) and we prove a finiteness theorem for general R-orders with the cancellation property in non Eichler (R)-algebras.

THEOREM 4 (*Finiteness theorem*). *Let A be a non Eichler (R)-algebra over a global function field then there are only finitely many genera of R-orders in A having the cancellation property. So up to isomorphism there are only finitely many R-orders in A with the cancellation property*

PROOF. In [V] this was proved for totally definite quaternion algebras over number fields. To extend that proof to non Eichler (R)-algebras of any index over global function fields, we only remark that $v_{\not{p}}(\Lambda'_{\not{p}})$ is known for maximal orders, cf. lemma 5, and if an R-order θ has the C.P. then $L_{\text{hs}} \leqq x_\theta n^{\imath - 1} \leqq n^3(q^n - 1)$ (since A is an NEC(R)-algebra).

For hereditary R-orders θ the structure theorem of Harada-Brumer, cf. [R, p. 358], enables us to calculate $v_{\mathfrak{p}}(\theta'_{\mathfrak{p}})$ explicitly.

Recall that a genus $\mathscr{G}$ of hereditary R-orders is determined by giving the local type $r_{\mathfrak{p}}$ and local invariants $(n_j) = (n_1, \ldots, n_r)$ for every $\mathfrak{p} \notin \mathscr{T}$; furthermore $r_{\mathfrak{p}} = 1$ for almost every $\mathfrak{p} \notin \mathscr{T}$, the local invariants are determined up to a cyclic permutation and $\sum_{1 \leq j \leq r_{\mathfrak{p}}} n_j = \kappa_{\mathfrak{p}}$ for every $p \notin \mathscr{T}$.

LEMMA 5. *If $\theta_{\mathfrak{p}}$ is an hereditary $R_{\mathfrak{p}}$-order in $A_{\mathfrak{p}} \simeq M_{\kappa_{\mathfrak{p}}}(D_{\mathfrak{p}})$ of local type $r_{\mathfrak{p}}$ and local invariants* (n_j), *and $\Lambda_{\mathfrak{p}} \simeq M_{\kappa_{\mathfrak{p}}}(\Delta_{\mathfrak{p}})$ (where $\Delta_{\mathfrak{p}}$ is the unique maximal $R_{\mathfrak{p}}$-order in $D_{\mathfrak{p}}$) then*

$$\frac{v_{\mathfrak{p}}(\Lambda'_{\mathfrak{p}})}{v_{\mathfrak{p}}(\theta'_{\mathfrak{p}})} = \frac{\prod_{1 \leq j \leq \kappa_{\mathfrak{p}}} (q_{\Delta}^{j} - 1)}{\prod_{1 \leq j \leq r_{\mathfrak{p}}} \prod_{1 \leq j \leq n_j} (q_{\Delta}^{i} - 1)}$$

with $q_{\Delta} = \sharp(\Delta_{\mathfrak{p}}/_{\bar{\pi}\Delta_{\mathfrak{p}}}) = q^{\varphi_{\mathfrak{p}} e_{\mathfrak{p}}}$ where $\bar{\pi}$ is the generator of rad $\Delta_{\mathfrak{p}}$.

PROOF. Since $v_{\mathfrak{p}}$ is bi-invariant we can assume that $\Lambda_{\mathfrak{p}} = M_{\kappa_{\mathfrak{p}}}(\Delta_{\mathfrak{p}})$ and

$$\theta_{\mathfrak{p}} = \begin{bmatrix} (\Delta_{\mathfrak{p}})(\bar{\pi}\Delta_{\mathfrak{p}}) \ldots (\bar{\pi}\Delta_{\mathfrak{p}}) \\ \vdots \quad \ddots \quad \vdots \\ (\Delta_{\mathfrak{p}}) \quad \ldots \quad (\Delta_{\mathfrak{p}}) \end{bmatrix}^{(n_j)}$$

Note that $nr(\theta^*_{\mathfrak{p}}) = nr(\Lambda^*_{\mathfrak{p}}) = R^*_{\mathfrak{p}}$ so $\frac{v_{\mathfrak{p}}(\Lambda'_{\mathfrak{p}})}{v_{\mathfrak{p}}(\theta'_{\mathfrak{p}})} = \frac{\omega_{\mathfrak{p}}(\Lambda^*_{\mathfrak{p}})}{\omega_{\mathfrak{p}}(\theta^*_{\mathfrak{p}})}$ with $\omega_{\mathfrak{p}}$ a bi-invariant Haarmeasure on $A^*_{\mathfrak{p}}$, cf. [V], [We].

We normalize $\omega_{\mathfrak{p}}$ by stating that $\omega_{\mathfrak{p}}$ is defined with respect to an $R_{\mathfrak{p}}$-basis $\{u_i\}$ of $\Lambda_{\mathfrak{p}}$, that is

$$\omega_{\mathfrak{p}}(X) = \int_X \|x\|^{-1}\, dx \text{ with } dx = \prod dx_i \text{ for } x = \sum x_i u_i \text{ and } \int_{R_{\mathfrak{p}}} dx_i = 1.$$

Then $\omega_{\mathfrak{p}}(\Lambda^*_{\mathfrak{p}}) = \omega_{\mathfrak{p}}(\mathrm{GL}_{\kappa_{\mathfrak{p}}}(\Delta_{\mathfrak{p}})) = \prod_{1 \leq j \leq \kappa_{\mathfrak{p}}} (1 - q_{\Delta}^{-j})$, cf. [We].

Furthermore $\theta^*_{\mathfrak{p}} = \begin{bmatrix} (\Delta_{\mathfrak{p}})^*(\bar{\pi}\Delta_{\mathfrak{p}}) \ldots (\bar{\pi}\Delta_{\mathfrak{p}}) \\ \vdots \quad \ddots \quad \vdots \\ (\Delta_{\mathfrak{p}}) \quad \ldots \quad (\Delta_{\mathfrak{p}})^* \end{bmatrix}^{(n_j)}$

with $(\Delta_{\mathfrak{p}})^*_{j,j} = \mathrm{GL}_{n_j}(\Delta_{\mathfrak{p}})$; analogue calculations yield:

$$\omega_{\mathfrak{p}}(\theta^*_{\mathfrak{p}}) = q_{\Delta}^{\frac{-\kappa_{\mathfrak{p}}^2 + \sum n_j^2}{2}} \prod_{1 \leq j \leq r_{\mathfrak{p}}} \prod_{1 \leq i \leq n_j} (1 - q_{\Delta}^{-i})$$

and the assertion follows.

We fix a genus $\mathscr{G} = \mathscr{G}_{D_1 D_2}$ of hereditary R-orders in A with $D_2 = \prod_{r_{\not p} \neq 1} \not p^{(n_j)}$ is the contribution of the non-maximal part of θ to the discriminant $D(\theta/R) = D_1 D_2$, cf. [DVG 2].

As for maximal orders, we first prove some lemmas concerning x_θ and $\overline{\mathsf{F}_q(\theta)}$, $\theta \in \mathscr{G}_{D_1 D_2}$.

LEMMA 6. *Let A be a non Eichler (R)-algebra over $\mathsf{F}_q(t)$ such that $S_{\mathrm{ram}} = S_{\mathrm{ram}}^{\ell}$ then $\overline{\mathsf{F}_q(\theta)} \simeq \mathsf{F}_{q^n}$ for some $\theta \in \mathscr{G}_{D_1 D_2}$ if and only if $\varphi_{\not p} \not\equiv 0 \bmod n$ for every $\not p \in S_{\mathrm{ram}}$ and $\varphi_{\not p} \equiv 0 \bmod n$ for every $\not p \mid D_2$.*

PROOF. The proof in [DVG 2] extends since $S_{\mathrm{ram}} = S_{\mathrm{ram}}^{\ell}$.

LEMMA 7. *Let A be a non Eichler (R)-algebra over $\mathsf{F}_q(t)$ with $\mathscr{T} = S_{\mathrm{ram}} = \{t, t^{-1}\}$. If $q = 2$, $n = 3$ and $D_2 = (t^2 + t + 1)^{(2,1)}$ or $D_2 = (t + 1)^{(1,1,1)}$ then $x_\theta \geqq 3$ for every $\theta \in \mathscr{G}_{D_1 D_2}$; if $q = 2, n = 5$ and $D_2 = \not p^{(2,3)}$ with $\varphi_{\not p} = 1$ then $x_\theta \geqq 5$ for every $\theta \in \mathscr{G}_{D_1 D_2}$.*

PROOF. As in lemma 2 it is sufficient to show that for $c = t$ or t^2: $c \notin nr\,\theta^*$. Assume that $\alpha \in \theta$ and $nr(\alpha) = c$ then similar arguments as in lemma 2 yield that $f_\alpha(Y) = Y^n - c$.

If $n = 3$ and $D_2 = (t^2 + t + 1)^{(2,1)}$ then $\alpha \in \theta_{\not p}^{(1,2)}$ ($\not p = t^2 + t + 1$) implies that $f_\alpha(Y)$ reduces modulo $\not p$, a contradiction.

If $n = 3$ and $D_2 = (t + 1)^{(1,1,1)}$ then $\alpha \in \theta_{\not p}^{(1,1,1)}$ ($\not p = t + 1$) implies that $f_\alpha(Y) \equiv (Y - a_1)(Y - a_2)(Y - a_3) \bmod \not p$, a contradiction.

If $n = 5$ and $D_2 = \not p^{(2,3)}$ with $\varphi_{\not p} = 1$ then $\alpha \in \theta_{\not p}^{(2,3)}$ implies that $f_\alpha(Y) \equiv g_1(Y) \cdot g_2(Y) \bmod \not p$ with $\deg g_1 = 2$, $\deg g_2 = 3$, a contradiction.

Now we determine all non-maximal hereditary R-orders in non Eichler (R)-algebras, having the cancellation property:

THEOREM 8: *There are non-maximal hereditary R-orders having the cancallation property in the non Eichler (R)-algebra A if and only if $g_A < 0$ and the non-maximal hereditary R-orders in $\mathscr{G}_{D_1 D_2}$ have the C.P. iff D_2 (and $\mathscr{T}$) satisfy one of the following conditions.*

i) $D_2 = \not p^{(n-1,1)}$ *with* $\varphi_{\not p} = 1$
ii) $n = 3, q = 2, \mathscr{T} = \mathscr{S}_{\mathrm{ram}}$ *and* $D_2 = \not p^{(1,2)}$ *with* $\varphi_{\not p} = 2$
or $D_2 = \not p^{(1,1,1)}$ *with* $\varphi_{\not p} = 1$.
iii) $n = 5, q = 2, \mathscr{T} = S_{\mathrm{ram}}$ *and* $D_2 = \not p^{(2,3)}$ *with* $\varphi_{\not p} = 1$.

PROOF. We only give the proof for $n > 2$ and refer to [DVG 3] for $n = 2$. We can restrict to NEC(R)-algebras which are given by theorem 3. (Remark that $S_{\mathrm{ram}} = S_{\mathrm{ram}}^{\ell}$ and $(D_1, D_2) = 1$ for all NEC(R)-algebras.) For $\not p \mid D_2$ of local type $r_{\not p}$

and local invariants (n_j) we find that

$$T_{\not p} = \frac{\prod_{1 \leqq j \leqq n} (q^{\varphi_{\not p} j} - 1)}{\prod_{1 \leqq j \leqq r_{\not p}} \prod_{1 \leqq i \leqq n_j} (q^{\varphi_{\not p} i} - 1)}, \text{ so } T_{\not p} \geqq \frac{q^{\varphi_{\not p} n} - 1}{q^{\varphi_{\not p}} - 1}.$$

I. *A is an* NEC (R)-*algebra with* $g_A < 0$:

Remark that A corresponds to i) of theorem 3(bis) and thus $L_{\text{hs}} = \prod_{\not p | D_2} T_{\not p}$. Using lemmas 6 & 7 it follows directly that $L_{\text{hs}} \leqq R_{\text{hs}}$ for the hereditary R-orders in $\mathscr{G}_{D_1 D_2}$ corresponding to i) ii) or iii).

And if the order corresponds to ii) or iii) except for the restriction on $\mathscr{T}$ then clearly $L_{\text{hs}} > R_{\text{hs}}$.

We now show that, also for all the other non-maximal hereditary R-orders in A, (C.F.) cannot hold.

If two or more primes divide D_2: $L_{\text{hs}} \geqq \left(\frac{q^n - 1}{q - 1}\right)^2 > n\frac{q^n - 1}{q - 1} \geqq R_{\text{hs}}$; so we assume now that $D_2 = \not p^{(n_j)}$. The proof depends on $\varphi_{\not p}$:

* $\varphi_{\not p} > 1$: $L_{\text{hs}} \geqq \frac{\mathscr{N}\not p^n - 1}{\mathscr{N}\not p - 1} > n\frac{q^n - 1}{q - 1} \geqq R_{\text{hs}}$ if $q \neq 2$ or $n \neq 3$ or $\varphi_{\not p} > 2$ ($\mathscr{N}\not p \geqq q^3$);

For $q = 2$, $n = 3$, $\varphi_{\not p} = 2$ we can assume that $(n_j) = (1, 1, 1)$ (in view of ii) and $L_{\text{hs}} > R_{\text{hs}}$ follows.

** $\varphi_{\not p} = 1$ and $(n_j) \neq (n - 1, 1)$ up to a cyclic permutation:
The proof depends on the local type $r_{\not p}$:

* $r_{\not p} > 2$: We can assume $q \neq 2$ or $n \neq 3$ in view of ii) and

$$L_{\text{hs}} \geqq \frac{(q^n - 1)(q^{n-1} - 1)}{(q - 1)^2} > n\frac{q^n - 1}{q - 1} \geqq R_{\text{hs}} \text{ follows.}$$

** $r_{\not p} = 2 (n \geqq 4)$: $L_{\text{hs}} \geqq \frac{(q^n - 1)(q^{n-1} - 1)}{(q^2 - 1)(q - 1)}$

If $t = 1$, $L_{\text{hs}} > \frac{q^n - 1}{q - 1} \geqq R_{\text{hs}}$ follows and if $t = 2$ we can assume $q \neq 2$ or $n \neq 5$ in view of iii); $L_{\text{hs}} > n\frac{q^n - 1}{q - 1} \geqq R_{\text{hs}}$ follows if $n \neq 4$ or $q \geqq 4$.

For the remaining cases $n = 4$, $q = 2$ or 3, $(n_j) = (2, 2)$ we argue on the prime decomposition of both sides of (C.F.) to conclude that (C.F.) cannot hold.

II. *A is an NEC(R)-algebra with* $n = 3$, $q = 2$, $S_{\mathrm{ram}} = \{t^{-1}, t, t+1\}$:

In this algebra $L_{\mathrm{hs}} = 3 \prod_{\not{p} \mid D_2} T_{\not{p}}$ and $R_{\mathrm{hs}} \leqq 9(2^3 - 1)$.

If $\not{p} \mid D_2$ and $\varphi_{\not{p}} \geqq 3$ then $L_{\mathrm{hs}} \geqq 3 \dfrac{2^9 - 1}{2^3 - 1} > R_{\mathrm{hs}}$, so we assume that $\varphi_{\not{p}} \leqq 2$ for every $\not{p} \mid D_2$. Then there is only one prime $\not{p} = t^2 + t + 1$ dividing D_2 and $\ell = 1$. Let us calculate x_θ in this case: $f_\alpha(Y) = Y^3 + t(t+1)$ has a root α in $\Lambda \in \mathscr{G}_{\max}$; and since $f_\alpha(Y) \equiv (Y+1)(Y+t)(Y+t+1) \bmod t^2 + t + 1$, we can choose $\alpha \in \theta$, $\theta \in \mathscr{G}_{D_1 D_2}$ for both $(n_j) = (1,1,1)$ or $(2,1)$ and thus $x_\theta \leqq 3$.

We conclude that $L_{\mathrm{hs}} \geqq 3 \cdot 21 > x_\theta(2^3 - 1) = R_{\mathrm{hs}}$.

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