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CORONA C*-ALGEBRAS AND THEIR APPLICATIONS TO LIFTING PROBLEMS

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1. Introduction.

In the good old days every C*-algebra had a unit. And if it didn't, you immediately adjoined it. Nowadays – when crossed products abound, and K-theory is the order of the day – the situation is reversed: If A is a C*-algebra we do not expect that it contains 1; and if it does, we quickly replace A by the stable algebra $A \otimes \mathscr{K}$ to ensure that $1 \notin A$. This change of attitude has kindled interest in the structure of the algebra $M(A)$ of multipliers of a non-unital (but usually σ-unital) C*-algebra A. From relatively modest beginnings in [7] and [3], the concepts of multipliers/centralizers now pervade vital areas of C*-algebra theory. For technical reasons, some of the relations among elements in $M(A)$, that are invariant under perturbations by elements from A, are best expressed in the quotient algebra $C(A) = M(A)/A$ – the corona of A. The motivating examples are firstly commutative: if $A = C_0(X)$ for some locally compact (σ-compact) Hausdorff space X, then $M(A) = C_b(X) = C(\beta X)$, so that $C(A) = C(\beta X \backslash X)$ – the continuous functions on the corona space of X, cf. [8]. Even more illustrative is the non-commutative paragon $A = \mathscr{K}$ – the compact operators on the separable Hilbert space $\mathscr{H}$. Here $M(A) = \mathrm{B}(\mathscr{H})$, so that $C(A)$ is the Calkin algebra. In all cases the elements in A represent local or (better) quasi-local properties, whereas relations in $C(A)$ describe truly global phenomena.

The present paper grew out of a desire to understand, and thereby generalize, some of the results in the Smith-Williams papers, [22] and [23]. We soon realized that our work had implications for the so-called technical theorems of Kasparov, which play a key part in establishing his KK-theory. These theorems were originally proved by bare hands methods, see [11], but can now, thanks to N. Higson, be established in a more civilized (but still non-trivial) way, cf. [5, 12.4.2]. We show in Corollary 3.4 that every corona C*-algebra enjoys what might be called the asymptotically abelian countable Riesz separation property. From this property it is easy to deduce Kasparov's results. The converse is unlikely to be the

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case: The Kasparov theorems are concerned with orthogonality relations, and use the AW*- (or rather the SAW*-) character of $C(A)$. Our, AA-CRISP, result deals with the order structure in $C(A)_{sa}$, and establishes a monotone sequential property of the corona algebras.

As a generalization of a theorem of Handelman ([10]), we show that every element x in $C(A)$ admits a weak polar decomposition: $x = v|x|$. The elements v in $C(A)$ is not a partial isometry in general (although $\|v\| \leqq 1$), and therefore not necessarily unique. Using the asymptotically commutative techniques from the previous section, we show that v can be chosen normal and commuting with x whenever x is normal. Moreover, v can be chosen unitary with $\operatorname{Re} v \geqq 0$, whenever $\operatorname{Re} x \geqq 0$. The method of proof is again based on the principle that if a property is asymptotically valid in $M(A)$, it holds exactly in the quotient $C(A)$. The same principle applies to show that derivations of $C(A)$ are "locally inner", in the sense that they are inner on each prescribed separable C*-subalgebra of $C(A)$. Finally we investigate the special morphism $\tilde{\rho}$ between corona C*-algebras $C(A)$ and $C(B)$ that obtains from a surjective morphism ρ: $A \to B$. From [19] we already know that $\tilde{\rho}(D)^{\perp} = \tilde{\rho}(D^{\perp})$ for every σ-unital, hereditary C*-subalgebra D of $C(A)$; a condition that in the commutative case expresses openness (more precisely σ-openness) of the restriction map between the underlying topological spaces. We now show that $\tilde{\rho}(D)' = \tilde{\rho}(D')$ for every separable C*-subalgebra D of $C(A)$.

In the last section we apply an equivalent version of Kasparov's theorem to solve a lifting problem for general C*-algebras: If I is a closed ideal in a C*-algebra A, and $x \in A$ such that $x^n \in I$ for some n, there is an element a in I with $(x + a)^n = 0$. The case $A = \mathrm{B}(\mathscr{H})$, $I = \mathscr{K}$ was solved in [14], and the case $n = 2$ (A and I arbitrary) was solved in [2]. The general solution is an instance surely recurring) of K-theoretic machinery being useful for something completely different.

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2. Notation and Preliminary Results.

Throughout this article, A will denote a C*-algebra represented as operators on its universal Hilbert space (unless otherwise specified), and A'' will denote the enveloping von Neumann algebra for A, cf. [18, 3.7]. We assume that A is σ-unital, but non-unital. Thus A has a strictly positive element and a countable approximate identity (e_n). Indeed, we may take $e_n = f_n(h)$, if h is strictly positive and if the functions f_n increase pointwise to 1 on $\mathrm{sp}(h)\setminus\{0\}$, [18, 3.10.5]. Moreover, the approximate unit (e_n) may be chosen quasi-central with respect to any fixed separable subset of A'' that derives A, [18, 3.12.14] and [5, 12.4.1]. As in [18, 3.12] we let $\mathrm{QM}(A)$ and $M(A)$ denote the sets of quasi-multipliers and two-sided) multipliers of A in A''. We shall be almost exclusively concerned with the unital C*-algebra $M(A)$ of multipliers, and we recall from [18, 3.12.8], that if B is any C*-algebra containing A as an essential ideal, there is a natural embedding $A \subseteq B \subseteq M(A)$. On $M(A)$ we have the strict topology, generated by the semi-norms $x \to \|xa\|$ and $x \to \|ax\|$, $a \in A$; and $M(A)$ is the strict completion of A in A'', cf. [7]. Our first result uses none of the notions above, however, but deals with estimates of commutators $[x, y] = xy - yx$.

2.1. LEMMA. *If x and y are elements in a C*-algebra and $x \geqq 0$, then for every exponent β, $0 < \beta < 1$, we have*

$$\|[x^\beta, y]\| \leqq (1-\beta)^{\beta-1}\|[x,y]\|^\beta \|y\|^{1-\beta}.$$

PROOF. We may assume that the algebra is unital and that $\|x\| = \|y\| = 1$. For $|t| < 1$ we have the power series expansions

$$1-(1-t)^\beta = \sum_{n=1}^{\infty} \alpha_n t^n, \quad \beta(1-t)^{\beta-1} = \sum_{n=1}^{\infty} n\alpha_n t^{n-1},$$

where $\alpha_n > 0$ for all n. (Actually $\alpha_n = (-1)^{n-1}\binom{\beta}{n}$). This implies that for any selfadjoint element z with $\|z\| < 1$ we can estimate

$$\begin{aligned}\|[(1-z)^\beta, y]\| &= \Big\|\sum_{n=1}^{\infty} \alpha_n [z^n, y]\Big\| \\ &= \Big\|\sum_{n=1}^{\infty}\sum_{k=0}^{n-1} \alpha_n z^k [z,y] z^{n-1-k}\Big\| \leqq \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} \alpha_n \|z\|^{n-1}\|[z,y]\| \\ &= \sum_{k=1}^{\infty} n\alpha_n \|z\|^{n-1}\|[z,y]\| = \beta(1-\|z\|)^{\beta-1}\|[z,y]\|.\end{aligned}$$

Applied with $z = 1 - x - \delta$, so that $-\delta \leqq z \leqq 1-\delta$, and for $0 < \delta < 1$, we get

$$\begin{aligned}\|[(\delta+x)^\beta, y]\| &\leqq \beta(1-\|1-x-\delta\|)^{\beta-1}\|[x,y]\| \\ &\leqq \beta\delta^{\beta-1}\|[x,y]\|.\end{aligned}$$

Note now that since the root functions are (operator) monotone, cf. [18, 13.8],

$$0 \leqq (\delta+x)^\beta - x^\beta \leqq (\delta^\beta + x^\beta) - x^\beta = \delta^\beta.$$

Thus with $d = (\delta + x)^\beta - x^\beta$ we have

$$\begin{aligned}\|[x^\beta, y]\| &\leqq \|[(\delta + x)^\beta, y]\| + \|[d, y]\| \\ &\leqq \beta\delta^{\beta-1}\|[x, y]\| + \|[d - \tfrac{1}{2}\|d\|, y]\| \\ &\leqq \beta\delta^{\beta-1}\|[x, y]\| + 2\|(d - \tfrac{1}{2}\|d\|)\|\,\|y\| \leqq \beta\delta^{\beta-1}\|[x, y]\| + \delta^\beta.\end{aligned}$$

Taking $\delta = (1 - \beta)\|[x, y]\|$ we obtain the desired result.

2.2. REMARK. Arveson proved in [4, Lemma to Theorem 2] that for x and y in the unit ball, and for each continuous function on $\mathrm{sp}(x)$ there is a function δ, with $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that $\|[f(x), y]\| \leqq \delta(\|[x, y]\|)$. The argument in Lemma 2.1 shows that for $f(t) = t^\beta$ one may take $\delta(\varepsilon) = (1 - \beta)^{\beta-1}\varepsilon^\beta$.

For $\beta = \frac{1}{2}$ (the only case we shall need) we get $\|[x^{\frac{1}{2}}, y]\| \leqq (2\|[x, y]\|)^{\frac{1}{2}}$. This result with a slightly larger constant) was obtained independently by K. R. Davidson, and he raised the question whether $\sqrt{2}$ is best possible. The answer is no! An argument by U. Haagerup, based on the integral representation of t^β found e.g. in the proof of [18, 1.3.8], shows that the constant (for any β) may be reduced to $(\pi\beta(1 - \beta))^{-1}\sin\beta\pi$. Thus for $\beta = \frac{1}{2}$ to $4\pi^{-1}$. Experiments with matrices indicate that the best constant should be 1. And another argument by Haagerup shows that indeed $\|[x^\beta, u]\| \leqq \|[x, u]\|^\beta$ if u is unitary. The quest for the best constant will be pursued elsewhere.

2.3. LEMMA. *Let (e_n) be a countable approximate unit in a C*-algebra A. Then for every bounded sequence (b_n) in $\mathrm{QM}(A)$, the element*

$$b = \sum (e_n - e_{n-1})^{\frac{1}{2}}\, b_n (e_n - e_{n-1})^{\frac{1}{2}}$$

(computed as a strongly convergent sum in A'', and taking $e_0 = 0$) belongs to $M(A)$.

PROOF. Since $\mathrm{QM}(A)$ is a *-subspace of A'', containing 1, it suffices to consider the case where $0 \leqq b_n \leqq 1$ for all n. Then with $h_n = (e_n - e_{n-1})^{\frac{1}{2}}$ we have $h_n b_n h_n \in A_+$ and $h_n(1 - b_n)h_n \in A_+$ for every n. Thus in the notation of [18, 3.11.4]

$$b \in (A_+)^m \text{ and } 1 - b \in (A_+)^m,$$

since evidently $\sum h_n^2 = 1$. Consequently

$$b \in (\tilde{A}_{sa})^m \cap (\tilde{A}_{sa})_m = M(A)_{sa}$$

by [18, 3.12.9], as desired.

2.4. REMARK. The result above is not new. A special case of it appears in [13, 6.3], and it has certainly been known to L. G. Brown for a long time. A slightly stronger version can be found in [12, 2.2]. In this paper, Lemma 2.3 will be used repeatedly to convert an approximate result in $M(A)$ (represented by a sequence (b_n)) into an exact formula in the corona C*-algebra $C(A) = M(A)/A$.

2.5. LEMMA. *Let* (e_n) *and* (b_n) *be as Lemma* 2.3, *and assume furthermore that* $\|b_n\| \leqq 1$ *for all* n. *Then with* $b = \sum h_n b_n h_n$ *as before*, $\|b\| \leqq 1$.

PROOF. Consider the operators h and d in $A \otimes \mathrm{B}(\mathscr{H})$ given by

$$(h)_{ij} = \begin{cases} h_j & \text{if } i = 1 \\ 0 & \text{if } i \neq 1, \end{cases} \qquad (d)_{ij} = \begin{cases} b_j & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Note that d is a diagonal operator with $\|d\| \leq 1$, and that h is a partial isometry with $hh^* = 1 \otimes e_{11}$. As $hdh^* = b \otimes e_{11}$, it follows that

$$b^*b \otimes e_{11} = (hd^*h^*)(hdh^*) \leqq hd^*dh^* \leqq hh^* = 1 \otimes e_{11};$$

whence $\|b^*b\| \leqq 1$, as claimed.

3. Technical Theorems.

As in §2, A denotes a σ-unital C*-algebra with multiplier algebra $M(A)$. The *corona C*-algebra* for A is the quotient $C(A) = M(A)/A$. An account of the elementary properties of $C(A)$ and its analogues in general topology can be found in [19].

Following a terminology introduced by R. R. Smith and D. P. Williams [22, 4.5] we say that a C*-algebra B has the *countable Riesz separation property* (CRISP), if for any two sequences (x_n) and (y_n) in B_{sa} satisfying

$$(*) \qquad x_n \leqq x_{n+1} \leqq \cdots \leqq y_{n+1} \leqq y_n,$$

for all n, there is an element z in B_{sa} such that $x_n \leqq z \leqq y_n$ for all n.

3.1. THEOREM. *Every corona* C*-*algebra* $C(A)$ *has the countable Riesz separation property*.

PROOF. Consider sequences (x_n) and (y_n) in $C(A)_{sa}$ satisfying $(*)$ above. If h is a strictly positive element in A and $\pi\colon M(A) \to C(A)$ denotes the quotient map, assume that we have constructed elements a_k and b_k in $M(A)_{sa}$, for $1 \leqq k \leqq n$, such that

$$\text{(i)} \qquad a_{k-1} \leqq a_k \leqq b_k \leqq b_{k-1},$$

$$\text{(ii)} \qquad \pi(a_k) = x_k, \quad \pi(b_k) = y_k,$$

$$\text{(iii)} \qquad \|h(b_k - a_k)h\| \leqq k^{-1},$$

for all $k \leqq n$. By [18, 1.5.10] there is an element b_{n+1} in $M(A)_{sa}$ satisfying $a_n \leqq b_{n+1} \leqq b_n$ and $\pi(b_{n+1}) = y_{n+1}$. By the same result there is also an element c in $M(A)_{sa}$ with $a_n \leqq c \leqq b_{n+1}$ and $\pi(c) = x_{n+1}$. Now let $(e_\lambda)_{\lambda \in \Lambda}$ be an approximate unit for A, and put

$$a_\lambda = c + (b_{n+1} - c)^{\frac{1}{2}} e_\lambda (b_{n+1} - c)^{\frac{1}{2}}.$$

Then $c \leqq a_\lambda \leqq b_{n+1}$ and $\pi(a_\lambda) = x_{n+1}$ for all λ. As $(b_{n+1} - c)^{\frac{1}{2}} h \in A$ we have

$$h(b_{n+1} - a_\lambda)h = h(b_{n+1} - c)^{\frac{1}{2}}(1 - e_\lambda)(b_{n+1} - c)^{\frac{1}{2}} h \to 0;$$

so we may take $a_{n+1} = a_\lambda$ for λ sufficiently large, to obtain $\|h(b_{n+1} - a_{n+1})h\| \leqq (n+1)^{-1}$. By induction we can therefore find sequences (a_k) and (b_k) in $M(A)_{sa}$ satisfying (i), (ii) and (iii) for all k.

Working in the enveloping von Neumann algebra A'' of A, we let a and b denote the strong limits of the bounded monotone sequences (a_n) and (b_n) in A''_{sa}. We have $a \leqq b$ by (i), and since the norm is strongly lower semi-continuous, $h(b - a)h = 0$ by (iii). Since hA is norm dense in A, it is weakly dense in A'', and thus $b - a = 0$. Therefore, in the language of [18, 3.12.9],

$$a = b \in (M(A)_{sa})^m \bigcap (M(A)_{sa})_m \subset (\tilde{A}_{sa})^m \bigcap (\tilde{A}_{sa})_m = M(A)_{sa}.$$

Alternatively, one may argue that the sequence (a_k) converges strictly to a (whence $a \in M(A)$), by observing that

$$\begin{aligned} \|(a - a_n)h\| &= \|h(a - a_n)\| = \|h(a - a_n)^2 h\|^{\frac{1}{2}} \\ &\leqq (\|a - a_n\| \, \|h(a - a_n)h\|)^{\frac{1}{2}} \leqq (\|a - a_n\| \, \|h(b_n - a_n)h\|)^{\frac{1}{2}} \\ &\leqq \|a - a_n\|^{\frac{1}{2}} n^{-\frac{1}{2}} \to 0. \end{aligned}$$

Let $z = \pi(a) = \pi(b)$ in $C(A)_{sa}$. Then by (ii),

$$x_n = \pi(a_n) \leqq \pi(a) = z = \pi(b) \leqq \pi(b_n) = y_n$$

for all n, as desired.

3.2. Remarks. The CRISP condition gives a very easy proof that $C(A)$ is an SAW*-algebra. Indeed, if x and y are orthogonal, positive elements in $C(A)$, with $x, y \leqq 1$ for convenience, then taking $x_n = x^{1/n}$ and $y_n = (1 - y)^n$ we have monotone sequences satisfying the condition (*). If $z \in C(A)_{sa}$ with $x_n \leqq z \leqq y_n$ for all n, then $0 \leqq z \leqq 1$; and as

$$\|x(1 - z)\| \leqq \lim \|x(1 - x^{\frac{1}{n}})\| = 0,$$

$$\|yz\| \leqq \lim \|y(1 - y)^n\| = 0,$$

we see that $x(1 - z) = zy = 0$, which is precisely the SAW*-condition, cf. [19].

In [22, 4.6] it is shown that a compact Hausdorff space X is sub-Stonean if and only if $C(X)$ is CRISP, and again if and only if $C(X)$ is separably injective. This is extended in [23, 3.4] to the case of an n-homogeneous C*-algebra A, which is shown to be separably injective, hence CRISP, if and only if $\hat{A}$ is sub-Stonean or, equivalently, by a short argument, A is an SAW*-algebra. R. R. Smith has informed us that the method of proof extends to cover also the case where A is (finitely) sub-homogeneous.

There is no reason to believe that all SAW*-algebras are CRISP, although we see from above that counterexamples are not readily accessible. By contrast it is obvious that separable injectivity is in general a much stronger condition than either the CRISP or the SAW*-condition. In fact, the Calkin algebra $\mathbf{B}(\mathscr{H})/\mathscr{K}$ is not even finitely injective.

3.3. THEOREM. *Let (x_n) be a monotone increasing sequence in $C(A)_{sa}$ and D a separable subset of $C(A)$, such that $[d, x_n] \to 0$ for every d in D. If $x_n \leqq y$ for some y in $C(A)_{sa}$ and all n, there is a z in $C(A)_{sa}$, commuting with D, such that $x_n \leqq z \leqq y$ for all n.*

PROOF. Choose a separable subset B of $M(A)$ such that $\pi(B) = D$, and let (b_n) be a dense sequence in B. Taking $d_n = \pi(b_n)$, and passing if necessary to a subsequence of (x_n), we may assume that

(i) $$\|[d_k, x_n]\| < 2^{-n}$$

for all k and n with $k \leqq n$. From the proof of Theorem 3.1 we obtain an increasing sequence (a_n) in $M(A)_{sa}$, converging strictly to an element a in $M(A)$, such that

(ii) $$\pi(a_n) = x_n, \quad x_n \leqq \pi(a) \leqq y$$

for all n.

Now choose a countable approximate unit (e_n) for A, that is quasi-central with respect to the elements b_n, a_n and a, cf. [18, 3.12.14]. Specifically we may assume, passing if necessary to a subsequence of (e_n) and using Lemma 2.1, that with $h_n = (e_n - e_{n-1})^{\frac{1}{2}}$ (and $e_0 = 0$) we have

(iii) $$\|[a_k, h_n]\| \leqq 2^{-n},$$

(iv) $$\|[a, h_n]\| \leqq 2^{-n},$$

(v) $$\|[b_k, h_n]\| \leqq 2^{-n},$$

for all k and n with $k \leqq n$. Furthermore, since by [18, 1.5.4] we have

$$\|[a_n, b_k](1 - e_\lambda)\| \to \|\pi([a_n, b_k])\| = \|[x_n, d_k]\| < 2^{-n}$$

for $k \leqq n$ by (i), for every approximate unit (e_λ) of A; we may assume also, passing if necessary to a further subsequence of (e_n), that

(vi) $$\|h_n[a_n, b_k]h_n\| \leqq 2^{-n+2}$$

for $k \leqq n$. This follows from the estimate

$$\begin{aligned} &\|h_n[a_n, b_k]h_n\| \leqq \|[a_n, b_k]h_n^2\| + 2\cdot 2^{-n} \\ &\leqq \|[a_n, b_k](1 - e_n)\| + \|[a_n, b_k](1 - e_{n-1})\| + 2^{-n+1} < 2\cdot 2^{-n} + 2^{-n+1} = 2^{-n+2}. \end{aligned}$$

By Lemma 2.3 we have $b = \sum h_n a_n h_n \in M(A)$. For a fixed m, let

$$c_1 = \sum_{n=1}^{m-1} h_n a_n h_n, \quad c_2 = \sum_{n=m}^{\infty} h_n [a_m, h_n],$$

and note from (iii) that $c_2 \in A$, since the sum is uniformly convergent in A. Of course, $c_1 \in A$ as well. Thus

$$b \geqq \sum_{n=1}^{m-1} h_n a_n h_n + \sum_{n=m}^{\infty} h_n a_m h_n$$

$$= c_1 + c_2 + \sum_{n=m}^{\infty} h_n^2 a_m = c_1 + c_2 + (1 - e_{m-1}) a_m.$$

This shows that if $z = \pi(b)$, then

$$z = \pi(b) \geqq \pi(a_m) = x_m$$

for all m. Similarly, if we set

$$c_3 = \sum_{n=1}^{\infty} h_n [a, h_n],$$

then $c_3 \in A$ by (iv), and we have

$$b \leqq \sum h_n a h_n = c_3 + \sum h_n^2 a = c_3 + a;$$

which shows that

$$z = \pi(b) \leqq \pi(a) \leqq y.$$

Finally, for each k,

$$[b, b_k] = \sum [h_n a_n h_n, b_k]$$
$$= \sum h_n a_n [h_n, b_k] + \sum h_n [a_n, b_k] h_n + \sum [h_n, b_k] a_n h_n.$$

Each of these three sums converge in A by (v) and (vi), and thus $[b, b_k] \in A$; whence

$$[z, d_k] = \pi([b, b_k]) = 0.$$

Thus z commutes with D, and the proof is complete.

3.4. Corollary. *If (x_n) is a monotone increasing sequence in $C(A)_{sa}$ and (y_n) is a monotone decreasing sequence such that $x_n \leqq y_n$ for all n, and if furthermore D is a separable subset of $C(A)$ such that $[d, x_n] \to 0$ for every d in D, then there is an element z in $C(A)_{sa}$, commuting with D, such that $x_n \leqq z \leqq y_n$ for all n.*

Proof. Combine Theorems 3.1 and 3.3 to get AA-CRISP.

3.5. THEOREM. (*Kasparov*). *Let B_1 and B_2 be orthogonal, σ-unital* C*-*subalgebras of $C(A)$. Assume further that D is a separable subset of $C(A)$ deriving B_1 (i.e. $[d, b_1] \in B_1, \forall\, d \in D, b_1 \in B_1$). There is then an element z in $C(A)$, commuting with D, with $0 \leqq z \leqq 1$, such that $B_1(1 - z) = zB_2 = 0$.*

PROOF. Let x and y be strictly positive elements in the unit balls of B_1 and B_2, respectively. Then with $e_n = x^{1/n}$ we have an approximate unite for B_1. For each b_1 in B_1 and d in D we therefore have

$$\begin{aligned}\lim\, [e_n, d] b_1 &= \lim e_n[d, b_1] + [e_n b_1, d] \\ &= [d, b_1] + [b_1, d] = 0.\end{aligned}$$

This means that $[e_n, d] \to 0$, strictly ([7]), hence weakly in B_1. By the Hahn-Banach theorem we can therefore find a sequence (x_n) such that $\|[x_n, d]\| \to 0$, and each x_n is an convex combination of e_n's. With (d_k) a dense sequence in D we may thus by induction choose an increasing sequence (x_n) in $C^*(x)$, such that $x_n \geqq x^{1/n}$ and $\|[x_n, d_k]\| < n^{-1}$ for all $k \leqq n$.

Let $y_n = (1 - y)^n$ and observe that (y_n) is decreasing, with $x_n \leqq y_n$ for every n. By Corollary 3.4 there is an element z in $C(A)$, commuting with D, with $0 \leqq z \leqq 1$, such that $x_n \leqq z \leqq y_n$ for all n. This means that

$$\begin{aligned}\|x(1 - z)\| &\leqq \lim \|x(1 - x_n)\| \leqq \lim \|x(1 - x^{\frac{1}{n}})\| = 0, \\ \|yz\| &\leqq \lim \|y(1 - y)^n\| = 0;\end{aligned}$$

so that $x(1 - z) = yz = 0$. Since $B_1 x$ and $y B_2$ are dense in B_1 and B_2, respectively, it follows that $B_1(1 - z) = zB_2 = 0$.

3.6. REMARK. The result above is known as Kasparov's Technical Theorem [11, §3 Theorem 3]. Our quotient formulation (with the elements in $C(A)$ and not in $M(A)$) can be found in [5, p. 123]. In his excellent book, B. Blackadar reproduces N. Higson's proof of the KTT [5, 12.4.2], and the present authors certainly used it as an inspiration for the arguments leading to Theorem 3.3 and its corollary.

The following equivalent formulation of KTT treats the subalgebras symmetrically, and is phrased entirely in terms of orthogonality relations. It could therefore conceivably be valid in any SAW*-algebra.

3.7. THEOREM. *Let D be a separable, unital* C*-*subalgebra of $C(A)$ such that $xDy = 0$ for some elements x, y in $C(A)$. There is then an element z in $C(A)$, commuting with D, with $0 \leqq z \leqq 1$, such that $x(1 - z) = zy = 0$.*

PROOF. Let (d_k) be a dense sequence in D, and with $d_0 = 1$ put

$$x_0 = \sum_{n=0}^{\infty} 2^{-n} d_n^* x^* x d_n \|d_n\|^{-2}.$$

We still have $x_0 D y = 0$, but now if $d = d^*$,

$$2i[d, x_0] = (1 + id)x_0(1 - id) - (1 - id)x_0(1 + id)$$

$$\leqq (1 + id)x_0(1 - id) \leqq 2(x_0 + dx_0 d).$$

We claim that for every $\varepsilon > 0$ there is a constant γ such that $dx_0 d \leqq \gamma x_0 + \varepsilon$. Indeed, we have

$$dx_0 d \leqq \sum_{n=0}^{m} 2^{-n} dd_n^* x^* x d_n d \|d_n\|^{-2} + \tfrac{1}{2}\varepsilon$$

for m sufficiently large; and choosing now $d_{k(n)}$ in (d_k) for $0 \leqq n \leqq m$, such that $\|d_n\|^{-1} d_n d = d_{k(n)} + c_n$, where $\|c_n\| \leqq \frac{1}{2}\|x\|^{-1}\varepsilon^{\frac{1}{2}}$, we get

$$dx_0 d \leqq \sum_{n=0}^{m} 2^{-n}(d_{k(n)} + c_n)^* x^* x(d_{k(n)} + c_n) + \tfrac{1}{2}\varepsilon$$

$$\leqq \sum_{n=0}^{m} 2^{-n} \cdot 2(d_{k(n)}^* x^* x d_{k(n)} + c_n^* x^* x c_n) + \tfrac{1}{2}\varepsilon$$

$$\leqq \sum_{n=0}^{m} 2^{-n+1} d_{k(n)}^* x^* x d_{k(n)}$$

$$+ \sum_{n=0}^{m} 2^{-n+1} \|c_n\|^2 \|x\|^2 + \tfrac{1}{2}\varepsilon \leqq \gamma x_0 + \varepsilon$$

for $\gamma \geqq \max\{2^{k(n)+1-n}\|d_{k(n)}\|^2 \mid 0 \leqq n \leqq m\}$. If therefore B_1 denotes the hereditary C*-subalgebra of $C(A)$ generated by x_0, i.e. $B_1 = (x_0 C(A) x_0)^=$, then $dx_0 d \in B_1$. By the first argument, both $i[d, x_0]$ and $-i[d, x_0]$ are dominated by $x_0 + dx_0 d$, whence $[d, x_0] \in B_1$. Consequently D derives B_1. If we now set $B_2 = (yC(A)y^*)^=$ we have exactly the assumptions in Theorem 3.5, hence the conclusion.

4. Polar Decompositions.

4.1. LEMMA. *If x and y are elements in $C(A)$ with $x^*x \leqq y^*y$, then $x = vy$ for some v in $C(A)$ with $\|v\| \leqq 1$.*

PROOF. Let $\pi: M(A) \to C(A)$ denote the quotient map, and take b in $M(A)$ such that $\pi(b) = y$. By a result of F. Combes, cf. [18, 1.5.10], there is then an element

a in $M(A)$ such that $\pi(a) = x$ and $a^*a \leqq b^*b$. Put $\varepsilon_n = 4^{-n+1}$ and define

$$(*) \qquad w_n = a(\varepsilon_n + b^*b)^{-1}b^*.$$

Note that

$$\begin{aligned}\|a - w_nb\| &= \|a(1 - (\varepsilon_n + b^*b)^{-1}b^*b)\| = \|a\varepsilon_n(\varepsilon_n + b^*b)^{-1}\| \\ &= \|(\varepsilon_n + b^*b)^{-1}\varepsilon_n a^*a\varepsilon_n(\varepsilon_n + b^*b)^{-1}\|^{\frac{1}{2}} \\ &\leqq \|(\varepsilon_n + b^*b)^{-1}\varepsilon_n b^*b\varepsilon_n(\varepsilon_n + b^*b)^{-1}\|^{\frac{1}{2}} \\ &= \|\varepsilon_n(b^*b)^{\frac{1}{2}}(\varepsilon_n + b^*b)^{-1}\|.\end{aligned}$$

With $c = |b|$ $(= (b^*b)^{\frac{1}{2}})$ we see from spectral theory that

$$\text{(i)} \qquad \|a - w_nb\| \leqq \|\varepsilon_n c(\varepsilon_n + c^2)^{-1}\| \leqq \tfrac{1}{2}\varepsilon_n^{\frac{1}{2}} = 2^{-n}.$$

Choose now a countable approximate unit (e_n) for A which is quasi-central with respect to the elements a and b. Specifically we may assume, invoking Lemma 2.1, that with $h_n = (e_n - e_{n-1})^{\frac{1}{2}}$ we have

$$\text{(ii)} \qquad \|[h_n, a]\| \leqq 2^{-n} \quad\text{and}\quad \|[h_n, b]\| \leqq 2^{-n}.$$

Define $w = \sum h_nw_nh_n$ and note that $w \in M(A)$ by Lemma 2.3. Moreover, $\|w\| \leqq 1$ by Lemma 2.5, since $\|w_n\| \leqq 1$ for every n by an argument similar to the one above, cf. [18, 1.4.4]. Finally,

$$\begin{aligned}a - wb &= \sum ah_n^2 - h_nw_nh_nb \\ &= \sum [a, h_n]h_n + \sum h_n(a - w_nb)h_n + \sum h_nw_n[b, h_n].\end{aligned}$$

It follows immediately from (i) and (ii) that $a - wb \in A$. Take now $v = \pi(w)$. Then $\|v\| \leqq 1$ and

$$vy = \pi(wb) = \pi(a) = x.$$

4.2. Remark. The result above was suggested by D. Handelman. He proves in [10, 2.1] that if A and M are, respectively, the direct sum and the direct product of a sequence of unital C*-algebras, the quotient M/A satisfies the conclusions in Lemma 4.1 (called $\aleph_0$-injectivity by him). Since in this case $M = M(A)$, our result is a natural generalization.

Appealing to the asymptotically abelian techniques from §3, we obtain a considerably stronger version of the lemma.

4.3. Proposition. *If x and y are elements in $C(A)$ with $x^*x \leqq y^*y$, and if D is a separable subset of $C(A)$ commuting with x, y^* and y^*y, then $x = vy$ for some v in $C(A)$, commuting with D, with $\|v\| \leqq 1$.*

PROOF. Let (b_n) be a countable subset of $M(A)$, chosen such that $(\pi(b_n))$ is dense in D. Then, with notations as in the proof of Lemma 4.1, we have $[b_k, w_n] \in A$ for all n and k, since

$$\pi([b_k, w_n]) = [\pi(b_k), x(\varepsilon_n + y^*y)^{-1}y^*] = 0.$$

Thus $[b_k, w_n](1 - e_\lambda) \to 0$ for every approximate unit (e_λ) of A. Choosing the countable approximate unit (e_n) to be quasi-central for the sequence (b_k) as well, and noting that

$$[b_k, w_n]h_m^2 = [b_k, w_n](1 - e_{m-1}) - [b_k, w_n](1 - e_m),$$

we may therefore assume (in addition to (i) and (ii)) that (e_n) also satisfies

(iii) $$\|h_n[b_k, w_n]h_n\| \leqq 2^{-n}$$

(iv) $$\|[h_n, b_k]\| \leqq 2^{-n},$$

for all $k \leqq n$. This implies that when we set $w = \sum h_n w_n h_n$, then

$$[b_k, w] = \sum [b_k, h_n]w_n h_n + \sum h_n[b_k, w_n]h_n + \sum h_n w_n[b_k, h_n] \in A.$$

Consequently $v = \pi(w)$ commutes with D, as required.

4.4. THEOREM. *For every x in $C(A)$ there is an element u in $C(A)$, with $\|u\| \leqq 1$, commuting with every separable subset D that commutes with x and $|x|$, such that $x = u|x|$. Thus $u^*u|x| = |x|$, $uu^*x = x$ and $u^*x = |x|$. If x is normal we can moreover choose u to be a normal element commuting with x and x^*.*

PROOF. Since $x^*x \leqq |x|^2$, we have $x = u|x|$ for some element u as specified above, by Proposition 4.3. As $|x|^2 = x^*x = |x|u^*u|x|$, it follows that $|x|(1 - u^*u)|x| = 0$. Since $1 - u^*u \geqq 0$ this means that $u^*u|x| = |x|$. Consequently $u^*x = u^*u|x| = |x|$, and $uu^*x = uu^*u|x| = u|x| = x$, as claimed.

If x is normal, both x and x^* commute with x and $|x|$. They can therefore be included in D to produce a decomposition $x = u|x|$, where u commutes with x and x^* (as well as with the rest of D). Consequently

$$u^*u|x| = u^*x = xu^* = u|x|u^* = uu^*|x|,$$

so that $(u^*u - uu^*)|x| = 0$. Applying Theorem 3.7 with x and y replaced by $u^*u - uu^*$ and $|x|$, and with the C*-algebra generated by D, u and u^*, we obtain an element z in $C(A)$, with $0 \leqq z \leqq 1$, such that

$$(u^*u - uu^*)z = 0, \quad (1 - z)|x| = 0;$$

and such that z commutes with D, u and u^*. Put $v = uz$. Then

$$v^*v = u^*uz^2 = uu^*z^2 = vv^*,$$

so that v is normal. Moreover, v commutes with x, x^* and D, because both u and z do so. Finally,

$$v|x| = uz|x| = u|x| = x.$$

Replacing u with v we obtain the desired conclusion.

4.5. REMARK. It is not in general possible to write $x = u|x|$ with u unitary, even if x is normal. Take for example a locally compact, σ-compact Hausdorff space Y such that $\dim(Y \setminus C) \geqq 2$ for every compact subset C of Y. Then the corona set $X = \beta Y \setminus Y$ has $\dim X \geqq 2$ by [8, 3.6]. By definition there is therefore a unitary function $f_0: E \to S^1$, defined on a closed subset E of X, that has no continuous extension as a unitary function on all of X. However, by Tietze's extension theorem there is an extension f in $C(X)$, with $\|f\| = 1$ (identifying S^1 with the unit circle in C). In the corona C*-algebra $C(A) = C(X)$ (where $A = C_0(Y)$ and $M(A) = C_b(Y) = C(\beta Y)$) the normal element f has no unitary polar decomposition $f = u|f|$, because $u \mid E = f \mid E = f_0$, so that u would be a unitary extension of f_0.

As an ilustration of this topological obstruction, let Y be the disjoint union of a countable number of compact spaces Δ_n, each homeomorphic to the unit disk in C; and define f in $C_b(Y)$ as $f = \sum f_n$, $f_n \in C(\Delta_n)$, where $f_n(z) = z$, $\quad z \in \Delta_n$, for every n, cf. [25, Example 6].

4.6. REMARK. If $x = x^*$ in $C(A)$, then $x = u|x|$ for some normal element u in $C(A)$ by Theorem 4.4. Thus $(u - u^*)|x| = 0$, whence $(u - u^*)z = 0, (1 - z)|x| = 0$ for some z in $C(A)_+$ commuting with u, u^* and $|x|$, by Theorem 3.7. Replacing u by uz it follows that we have $x = u|x|$ with $u^* = u$. In this case we can dilate u to a unitary by writing $w = u + i(1 - u^2)^{\frac{1}{2}}$. Since $(1 - u^2)|x| = 0$ it follows that we have a unitary polar decomposition $x = w|x|$.

Even so, we can not in general hope to write $x = u|x|$ with a self-adjoint unitary u when $x = x^*$. The obstruction to such a decomposition is the possible scarcity of projections (hence of symmetries = self-adjoint unitaries) in $C(A)$. Indeed, if Y is a locally compact, σ-compact space which is connected at infinity (e.g. $Y = \mathrm{R}^d$, $d > 1$), then with $A = C_0(Y)$ we have $C(A) = C(\beta Y \setminus Y$, and $\beta Y \setminus Y$ is connected by [8, 3.5]. Thus $C(A)$ has no non-trivial projections.

4.7. LEMMA. *If* $x \in C(A)$ *such that* $x + x^* \geqq 0$, *then* $x = v|x|$ *for some* v *in* $C(A)$ *with* $\|v\| \leqq 1$ *and* $v + v^* \geqq 0$.

PROOF. Choose y in $M(A)$ with $\pi(y) = x$ and $y + y^* \geqq 0$. Then $y + \varepsilon_n$ is invertible for every $\varepsilon_n > 0$, so that $y + \varepsilon_n = w_n|y + \varepsilon_n|$ with w_n unitary in $M(A)$. Since

$$w_n|y + \varepsilon_n| + |y + \varepsilon_n|w_n^* = y + y^* + 2\varepsilon_n,$$

we see from [13, 6.1] or [26] that $w_n + w_n^* \geqq 2\varepsilon_n\|y + \varepsilon_n\|^{-1}$. In particular,

$w_n + w_n^* \geqq 0$. Now

$$\|y - w_n|y|\| \leqq \|y - w_n|y + \varepsilon_n|\| + \|w_n(|y + \varepsilon_n| - |y|)\|$$
$$= \|y - (y + \varepsilon_n)\| + \||y + \varepsilon_n| - |y|\| \leqq 2\varepsilon_n.$$

Thus with $\varepsilon_n = 2^{-n}$ we may use the w_n's constructed above in place of those in (*) in the proof of Lemma 4.1, to obtain $w = \sum h_n w_n h_n$ in $M(A)$ such that $y - w|y| \in A$. Evidently $w + w^* \geqq 0$, and thus with $v = \pi(w)$ we have the desired conclusion.

4.8. PROPOSITION. *If $x \in C(A)$ such that $x + x^* \geqq 0$, then $x = u|x|$ for some unitary u in $C(A)$ with $u + u^* \geqq 0$.*

PROOF. Assuming, as we may, that $C(A) \subset \mathrm{B}(\mathscr{H})$ for some (huge!) Hilbert space $\mathscr{H}$, we take ξ in $\ker x$. Writing $x = h + ik$ we compute $(h\xi \mid \xi) + i(k\xi \mid \xi) = 0$, whence $(h\xi \mid \xi) = (k\xi \mid \xi) = 0$. Since $h \geqq 0$ by assumption, this implies that $h\xi = 0$, whence also $k\xi = 0$. We conclude that $x^*\xi = (h - ik)\xi = 0$, so that, in fact, $\ker x = \ker x^*$. Taking complements, $(x^*\mathscr{H})^= = (xH)^=$, and we let p denote the projection on this subspace. Note that p is the smallest unit for $|x|$ (and for $|x^*|$) in $\mathrm{B}(\mathscr{H})$.

By Lemma 4.7 we have $x = v|x|$ for some v in $C(A)$ with $\|v\| \leqq 1$ and $v + v^* \geqq 0$. Thus $x = vp|x|$, and vp is the canonical partial isometry in the polar decomposition of x in $\mathrm{B}(\mathscr{H})$, cf. [18, 2.2.9]. In our case it means that vp is a unitary on $p\mathscr{H}$ and vanishes on $(1 - p)\mathscr{H}$. Consequently $vpv^* = pv^*vp = p$, whence $vp = vpv^*v = pv$. Write $v = a + ib$ and note that $[v, p] = 0$ implies $[a, p] = [b, p] = 0$. As $ap + ibp$ is unitary (on $p\mathscr{H}$) and $ap \geqq 0$ we conclude that

$$(*) \qquad ap = (p - (bp)^2)^{\frac{1}{2}} = (1 - b^2)^{\frac{1}{2}} p.$$

Put $u = (1 - b^2)^{\frac{1}{2}} + ib$, and note that u is unitary in $C(A)$ with $u + u^* \geqq 0$. Furthermore, by (*) we have

$$u|x| = up|x| = (a + ib)p|x| = v|x| = x.$$

4.9. REMARK. The authors suspect that every element in the closure of the group of invertible elements in $C(A)$ has a unitary polar decomposition in $C(A)$. As shown in [20] and [21], the condition $\operatorname{dist}(x, \mathrm{GL}(B)) = 0$ is necessary to have $x = u|x|$, $u \in \mathscr{U}(B)$, for any element x in a unital C*-algebra B. In general the condition is not sufficient; one must ask also that x has vanishing index, cf. [15]. For corona C*-algebras, however, we have the feeling that all notions of index should be trivial in the closure of the invertibles. Certainly this happens in our guiding example $A = \mathscr{K}$, where $C(A)$ is the Calkin algebra. In this case our conjecture about unitary polar decomposition is easily proved. For more general algebras, even commutative, we do not know the answer. We can show, however, that another obstacle is not present in corona algebras: any element in the unit ball with unitary rank 2 admits a unitary polar decomposition.

4.10. PROPOSITION. *If $x = \frac{1}{2}(v + w)$ for some unitaries v and w in $C(A)$, then $x = u|x|$ for some unitary u in $C(A)$.*

PROOF. Set $y = 2v^*x = 1 + v^*w$. Then y is normal and $y + y^* \geqq 0$. By Proposition 4.8 we therefore have $y = u_0|y|$ for some unitary u_0 in $C(A)$. With $u = vu_0$ this means that

$$x = \tfrac{1}{2}vy = vu_0\tfrac{1}{2}|y| = u|v^*x| = u|x|.$$

5. Derivations and Morphisms.

5.1. PROPOSITION. *If δ is a (bounded) derivation of $C(A)$, there is for each separable C*-subalgebra B of $C(A)$ an element b in $C(A)$, with $0 \leqq b \leqq \|\delta\|$ if $\delta^* = -\delta$, such that $\delta(x) = [b, x]$ for every x in B.*

PROOF. Without loss of generality we may assume that $\delta = -\delta^*$ and that B is δ-invariant. Choose a separable C*-subalgebra D of $M(A)$ such that $\pi(D) = B$, where $\pi\colon M(A) \to C(A)$ denotes the quotient map. Let (y_k) be a dense sequence in D and let (x_k) be its image in B. Now recall from [18, 8.6.12] that there is an increasing sequence (b_n) in B_+, bounded by $\|\delta\|$, such that

(i) $$\|\delta(x_k) - [b_n, x_k]\| < 2^{-n},$$

for all n and k with $k \leqq n$. We claim that there is a corresponding increasing sequence (d_n) in D_+ with $0 \leqq d_n \leqq \|\delta\|$ and $\pi(d_n) = b_n$, such that

(ii) $$\|[d_n - d_{n-1}, y_k]\| < 2^{-n+2}$$

for $1 \leqq k \leqq n - 1$. Suppose that we have already chosen $d_1, d_2, \ldots, d_{n-1}$ subject to the conditions in (ii). By [18, 1.5.10] there is an element d in D with $\pi(d) = b_n$, such that $d_{n-1} \leqq d \leqq \|\delta\|$. In order also to satisfy (ii), let (e_λ) be an approximate unit for A which is quasi-central for (y_k) and for $(d - d_{n-1})^{\frac{1}{2}}$, cf. [18, 3.12.14]. Consider the element

$$d_\lambda = d - (d - d_{n-1})^{\frac{1}{2}} e_\lambda (d - d_{n-1})^{\frac{1}{2}},$$

and note that $d_{n-1} \leqq d_\lambda \leqq d$ and $\pi(d_\lambda) = b_n$. Moreover, for $1 \leqq k \leqq n - 1$ we have

$$\begin{aligned} &\limsup \|[d_\lambda - d_{n-1}, y_k]\| \\ &= \limsup \|[(d - d_{n-1})^{\frac{1}{2}}(1 - e_\lambda)(d - d_{n-1})^{\frac{1}{2}}, y_k]\| \\ &= \limsup \|[d - d_{n-1}, y_k](1 - e_\lambda)\| \\ &= \|[b_n - b_{n-1}, x_k]\| < 2^{-n} + 2^{-n+1} < 2^{-n+2} \end{aligned}$$

by (i) and [18, 1.5.4]. Taking therefore $d_n = d_\lambda$ for λ sufficiently large we have $\|[d_n - d_{n-1}, y_k]\| < 2^{-n+2}$ for $k \leqq n-1$. By induction the sequence (d_n) can therefore be constructed as claimed.

Define $\delta_0(y_k) = \lim\, [d_n, y_k]$, cf. (ii). Then δ_0 extends uniquely to a derivation of D with $\delta_0^* = -\delta_0$ and $\|\delta_0\| \leqq \|\delta\|$. Moreover,

$$\begin{aligned}(\pi\delta_0)(x_k) &= \pi(\delta_0(y_k)) = \lim \pi([d_n, y_k]) \\ &= \lim\, [b_n, x_k] = \delta(x_k),\end{aligned}$$

which shows that δ_0 is a lifting of $\delta \mid B$, cf. [18, 8.6.15].

Choose now a countable approximate unit (e_n) for A which is quasi-central for (y_k) and for $(\delta_0(y_k))$. Specifically we may assume, applying Lemma 2.1, that with $h_n = (e_n - e_{n-1})^{\frac{1}{2}}$ (and $e_0 = 0$) we have

(iii) $$\|[h_n, y_k]\| < 2^{-n}, \quad \|[h_n, \delta_0(y_k)]\| < 2^{-n},$$

for $1 \leqq k \leqq n$. Define $d = \sum h_n d_n h_n$, and note from Lemmas 2.3 and 2.4 that $d \in M(A)_+$ with $\|d\| \leqq \|\delta\|$. We have

$$[d, y_k] = \sum h_n d_n [h_n, y_k] + \sum h_n [d_n, y_k] h_n + \sum [h_n, y_k] d_n h_n,$$

and it follows from (iii) that the first and the third sum converges in A. For the middle terms we use the fact that $\sum h_n^2 = 1$ to compute

$$\begin{aligned}&\sum h_n [d_n, y_k] h_n - \delta_0(y_k) \\ &= \sum h_n([d_n, y_k] - \delta_0(y_k)) h_n + \sum [h_n, \delta_0(y_k)] h_n \in A,\end{aligned}$$

because $\|[d_n, y_k] - \delta_0(y_k)\| < 2^{-n+2}$ for $k \leqq n$. Taken together it means that $[d, y_k] - \delta_0(y_k) \in A$ for every k. Now let $b = \pi(d)$. It follows immediately that $[b, x] = \delta(x)$ for every x in B.

The argument above – to the effect that derivations of $C(A)$ are "locally inner" – uses only the existence of the derivation on the subalgebra. It applies therefore immediately to give the following

5.2. COROLLARY. *Let I be an essential ideal in a separable* C*-*algebra A (so that we have a canonical embedding $I \subset A \subset M(I)$). There is then for each derivation δ of A an element d in $M(I)$ such that $\delta(x) - [d, x] \in I$ for every x in A.*

5.3. REMARK. It is instructive to compare Corollary 5.2 with the main result in [17]: Given a derivation δ of A there is for each $\varepsilon > 0$ an essential ideal I of A and an element d in $M(I)$ such that $\|\delta(x) - [d, x]\| \leqq \varepsilon \|x\|$ for every x in I. Despite the similar vocabulary the two statements are quite different.

Let $\rho: A \to B$ be a morphism between σ-unital C*-algebras A and B, and assume that $\rho(A)$ contains an approximate unit for B. As proved in [19, 23] there is then a natural extension $\rho'': M(A) \to M(B)$, with $\operatorname{Ker} \rho'' = M(A, \operatorname{Ker} \rho)$, and an induced morphism $\tilde{\rho}: C(A) \to C(B)$, with

$$\operatorname{Ker} \tilde{\rho} = M(A, \operatorname{Ker} \rho)/\operatorname{Ker} \rho.$$

The morphism $\tilde{\rho}$ is an analogue of a σ-open map (a σ-morphismo, cf. [19, 17]), which means that $\tilde{\rho}(D)^{\perp} = \operatorname{her} \tilde{\rho}(D^{\perp})$ for every σ-unital, hereditary C*-subalgebra D of A. If ρ is surjective, so is ρ'' by [19, 10]; whence also $\tilde{\rho}$ is surjective. The condition above then takes the simple form: $\tilde{\rho}(D)^{\perp} = \tilde{\rho}(D^{\perp})$. We shall investigate a similar condition, where annihilators are replaced by commutants.

5.4. PROPOSITION. *Let $\rho: A \to B$ be a surjective morphism between σ-unital C*-algebras, and let ρ also denote its extension to a surjective morphism $\rho: M(A) \to M(B)$. If Δ is a separable subset of derivations of $M(A)$, and $y \in M(B)_{sa}$ such that $(\rho\delta)(y) \in B$ for every δ in Δ (where $(\rho\delta)(b) = \rho(\delta(\rho^{-1}(b)))$, $b \in M(B)$, by definition), then there is an element x in $M(A)_{sa}$, with $\rho(x) = y$, such that $\delta(x) \in A$ for every δ in Δ.*

PROOF. Let h be a strictly positive element for A, so that $k = \rho(h)$ is strictly positive for B. Reasoning exactly as in the proof of [19, 10] we find sequences (x_n) and (y_n) in $\tilde{B}_{sa}$, such that (x_n) is increasing, (y_n) is decreasing, $x_n \leqq y \leqq y_n$, and $\|k(y_n - x_n)k\| < n^{-1}$ for all n.

For each δ in Δ we know that $\rho\delta$ is a derivation of B, hence σ-weakly continuous on B'' (cf. [18, 8.6.6]). As $y_n \to y$ strictly, hence σ-weakly, it follows that $\rho\delta(y_n)) \to \rho\delta(y)$, σ-weakly. Since, however, $\rho\delta(y) \in B$ by assumption, it follows from the Hahn-Banach theorem that $\rho\delta(y) \in (\operatorname{Conv} \rho\delta(y_n))^{=}$. If (δ_k) is a dense sequence in Δ we may therefore, working by induction, replace (y_n) with another decreasing sequence (z_n), such that each z_n belongs to $\operatorname{Conv}(y_m)$ and satisfies the conditions

$$(*) \qquad \|k(z_n - x_n)k\| < n^{-1}, \quad \|\rho\delta_k(y) - \rho\delta_k(z_n)\| < 2^{-n}$$

for every $k \leqq n$.

Assume that we have found sequences (a_m) and (b_m), $1 \leqq m \leqq n-1$, in $\tilde{A}_{sa}$, the first increasing and the second decreasing, but with $a_m \leqq b_m$ for all m, satisfying the conditions

$$\text{(i)} \qquad \rho(a_m) = x_m, \quad \rho(b_m) = z_m, \quad 1 \leqq m \leqq n-1,$$

$$\text{(ii)} \qquad \|h(b_m - a_m)h\| < m^{-1}, \quad 1 \leqq m \leqq n-1,$$

$$\text{(iii)} \qquad \|\delta_k(b_{m-1} - b_m)\| < 2^{-m+2}, \quad 1 \leqq m \leqq n-1.$$

By [18, 15.10] there is an element b_n in $\tilde{A}_{sa}$ with $a_{n-1} \leqq b_n \leqq b_{n-1}$, such that $\rho(b_n) = z_n$. If (e_λ) is an approximate unit for $\ker \rho$ in A, then $e_\lambda \uparrow p$ for some central projection p in A'', whence $\delta(e_\lambda) \to 0$, σ-weakly, for every δ in Δ. Passing if necessary to another approximate unit in Conv (e_λ) we may therefore assume that $\|\delta(e_\lambda)\| \to 0$ for every δ in Δ. We may furthermore assume that (e_λ) is quasi-central with respect to $(b_n - b_{n-1})^{\frac{1}{2}}$. This means that

$$\begin{aligned}
&\limsup \|\delta((b_{n-1} - b_n)^{\frac{1}{2}}(1 - e_\lambda)(b_{n-1} - b_n)^{\frac{1}{2}})\| \\
&= \limsup \|\delta((b_{n-1} - b_n)(1 - e_\lambda))\| \\
&\leqq \limsup \|(\delta(b_{n-1} - b_n))(1 - e_\lambda)\| + \|(b_{n-1} - b_n)\delta(e_\lambda)\| \\
&= \|\rho\delta(z_{n-1} - z_n)\| + 0,
\end{aligned}$$

by [18, 1.5.4]. Replacing therefore if necessary b_n by

$$b_n' = b_n + (b_{n-1} - b_n)^{\frac{1}{2}} e_\lambda (b_{n-1} - b_n)^{\frac{1}{2}}$$

(so that we still have $a_n \leqq b_n' \leqq b_{n-1}$ and $\rho(b_n') = z_n$), we may assume that

$$\|\delta_k(b_{n-1} - b_n)\| < 2^{-n+1} + 2^{-n} < 2^{-n+2}$$

for $1 \leqq k \leqq n-1$, by (*). Again by [18, 1.5.10] there is an element a_n in $\tilde{A}_{sa}$, with $a_{n-1} \leqq a_n \leqq b_n$, such that $\rho(a_n) = x_n$. As above we see that

$$\limsup \|h(b_n - a_n)^{\frac{1}{2}}(1 - e_\lambda)(b_n - a_n)^{\frac{1}{2}} h\| = \|k(z_n - x_n)k\|$$

for a suitably quasi-central approximate unit (e_λ). Replacing therefore if necessary a_n by

$$a_n' = a_n + (b_n - a_n)^{\frac{1}{2}} e_\lambda (b_n - a_n)^{\frac{1}{2}}$$

(so that we still have $a_{n-1} \leqq a_n' \leqq b_n$ and $\rho(a') = x_n$), we may assume that $\|h(b_n - a_n)h\| < n^{-1}$ by (*). By induction we can therefore find sequences (a_n) and (b_n) satisfying (i), (ii) and (iii). These sequences converge strongly in A'' to elements a and x, with $a \leqq x$. However, $h(x-a)h = 0$ by (ii), whence $a = x$. Consequently,

$$x = a \in (\tilde{A}_{sa})^m \bigcap (\tilde{A}_{sa})_m = M(A)_{sa}$$

by [18, 3.12.9]. Since ρ is normal from A'' onto B'' it follows that

$$\rho(x) = \lim \rho(b_n) = \lim y_n = y.$$

Finally, by (iii), $(\delta_k(b_n))$ is norm convergent for every k. Since $b_n \to x$, strongly, we have $\delta_k(b_n) \to \delta_k(x)$, strongly, whence $\|\delta_k(b_n) - \delta_k(x)\| \to 0$. As $b_n = c_n + \lambda_n 1$ for some c_n in A and λ_n in C, and $\delta_k(1) = 0$, we conclude that $\delta_k(b_n) \in A$, and therefore $\delta_k(x) \in A$, as desired.

5.5. REMARK. It should be evident to the reader that, working along the lines of the previous proof, we can prove versions of Theorems 3.3, 3.4, 3.5 and 4.3, where the subset D is replaced by a separable subset Δ of derivations of the corona algebra. On the other hand we see from Proposition 5.1 that nothing much would be gained; because the derivations in these embellished theorems only operate on countable families of elements, and thus can be replaced by inner derivations.

5.6. THEOREM. *Let $\rho: A \to B$ be a surjective morphism between σ-unital C*-algebras, and let $\tilde{\rho}: C(A) \to C(B)$ denote the induced surjective morphism between the corona algebras. Then (with ′ meaning relative commutant) we have*

$$\tilde{\rho}(D)' = \tilde{\rho}(D')$$

for every separable C-subalgebra D of $C(A)$.*

PROOF. Clearly $\tilde{\rho}(D') \subset \tilde{\rho}(D)'$. Take now y_0 in $\tilde{\rho}(D)'$, and note that since $\tilde{\rho}(D)$ is self-adjoint we may assume that $y_0 = y_0^*$.

Let y be a counter-image of y_0 in $M(B)_{sa}$, and let C be a separable C*-subalgebra of $M(A)$ such that $C + A/A = D$. Then $[\rho(c), y] \in B$ for every c in C (cf. the commutative diagram in the proof of [19, 23]). By Proposition 5.4 there is therefore an x in $M(A)_{sa}$ with $\rho(x) = y$, such that $[c, x] \in A$ for every c in C. Taking x_0 as the image of x in $C(A)$ we see that $\tilde{\rho}(x_0) = y_0$ and that $x_0 \in D'$.

6. Applications to Lifting Problems.

We shall use the technical results for corona C*-algebras established in [19] to solve some extension or lifting problems for quotient maps between C*-algebras.

6.1. LEMMA. *If $\{x_1, \ldots, x_n\} \subset M(A)$ such that $\Pi x_k \in A$ (where A is σ-unital), there are elements $a_1, \ldots, a_n$ in A such that $\Pi(x_k - a_k) = 0$.*

PROOF. The Lemma is valid for $n = 2$ by [2, 2.3] (for any closed ideal in every C*-algebra). Assume now that it has been established for n factors ($n \geqq 2$), and take $x_1, \ldots, x_{n+1}$ in $M(A)$ with Πx_k in A. If $y = \Pi_{k=1}^n x_k$, and $\pi: M(A) \to C(A)$ denotes the quotient map, then $\pi(y)\pi(x_{n+1}) = 0$. Since $C(A)$ is an SAW*-algebra by [19, 13] or Remark 3.2, there are elements d and e in $M(A)_+$, with $\pi(d)\pi(e) = 0$, such that

$$(*) \qquad \pi(y)(1 - \pi(d)) = 0, \quad (1 - \pi(e))\pi(x_{n+1}) = 0.$$

By [2, 2.4], we may assume that $de = 0$. The induction hypothesis, applied to the elements $x_1, \ldots, x_{n-1}$, $x_n(1 - d)$ (whose product lies in A by (*)) assures the existence of elements $a_1, \ldots, a_n$ in A, such that with $z = \Pi_{k=1}^{n-1}(x_k - a_k)$ we have $z(x_n(1 - d) - a_n) = 0$. Set $a_{n+1} = (1 - e)x_{n+1}$, and note from (*) that $a_{n+1} \in A$.

Finally,

$$\begin{aligned}\Pi_{k=1}^{n+1}(x_k - a_k) &= z(x_n - a_n)(x_{n+1} - a_{n+1})\\ &= z(x_n(1-d) - a_n + x_n d)ex_{n+1}\\ &= z(x_n(1-d) - a_n)ex_{n+1} + zx_n dex_{n+1}\\ &= 0 + 0 = 0.\end{aligned}$$

The proof is completed by induction.

6.2. THEOREM. *If I is a closed ideal in a* C*-*algebra A, and if $\{x_1,\dots,x_n\} \subset A$ such that $\Pi x_k \in I$, there are elements $a_1,\dots,a_n$ in I with $\Pi(x_k - a_k) = 0$.*

PROOF. Let B denote the C*-subalgebra of A generated by $\{x_1,\dots,x_n\}$. Replacing A and I by B and $B \bigcap I$, we may assume that A is separable. Let $I^\perp$ denote the annihilator of I in A. Then $I^\perp$ is a closed ideal in A, orthogonal to I, and $I + I^\perp$ is an essential ideal in A. By [18, 3.12.8] there is therefore a natural embedding

$$I + I^\perp \subset A \subset M(I + I^\perp).$$

Applying Lemma 6.1 to the separable, hence σ-unital C*-algebra $I + I^\perp$, we find elements $c_1,\dots,c_n$ in $I + I^\perp$ such that $\Pi(x_k - c_k) = 0$. Since $I \bigcap I^\perp = 0$, each element has a unique decomposition $c_k = a_k + b_k$ in $I + I^\perp$. Thus

$$0 = \Pi(x_k - a_k - b_k) = \Pi(x_k - a_k) + b,$$

where the element b is the sum of products, each of which contains at least one factor b_k. Thus $b \in I^\perp$, whereas $\Pi(x_k - a_k) \in I$ by assumption. We conclude that $b = 0$ and, more importantly, $\Pi(x_k - a_k) = 0$.

6.3. LEMMA. *Let $C(A)$ be the corona of a σ-unital* C*-*algebra A. If $x \in C(A)$ and $x^n = 0$ for some n, there are elements $e_0, e_1,\dots,e_n$ in $C(A)$, $0 \leqq e_k \leqq 1$, such that*

(i) $$(1 - e_k)x^{n-k} = 0, \quad 0 \leqq k \leqq n,$$

(ii) $$(1 - e_{k-1})xe_k = 0, \quad 1 \leqq k \leqq n,$$

(iii) $$e_k e_{k-1} = e_{k-1} \quad 1 \leqq k \leqq n,$$

with $e_0 = 0$ and $e_n = 1$ by definition.

PROOF. Suppose that we have already constructed $e_0, e_1,\dots,e_m$ in $C(A)$ satisfying the conditions in (i)–(iii) for all $k \leqq m$, where $0 \leqq m < n$. Then

$$(1 - e_m)xx^{n-m-1} = 0$$

by (i). Furthermore, by (ii) and (iii)

$$(1 - e_m)xe_m = (1 - e_m)e_{m-1}xe_m = 0,$$

so that

$$((1 - e_m)x)(x^{n-m-1}(x^{n-m-1})^* + e_m) = 0.$$

Since $C(A)$ is an SAW*-algebra by 3.2, there is an element e_{m+1}, with $0 \leqq e_{m+1} \leqq 1$, such that

$$(1 - e_m)xe_{m+1} = 0,$$
$$(1 - e_{m-1})x^{n-m-1} = (1 - e_{m+1})e_m = 0.$$

Thus $e_0, \ldots, e_{m+1}$, satisfy (i)–(iii) and the proof is completed by induction.

6.4. REMARK. In a von Neumann algebra, Lemma 6.3 is quite easy to prove: If $x^n = 0$, let e_k be the range projection of x^{n-k}, and check that $e_0, e_1, \ldots, e_n$ satisfy (i), (ii) and (iii). In a corona algebra the elements e_k, $0 \leqq k \leqq n$, need not be projections, and there is no canonical choice for them. Nevertheless they serve the same function of establishing a triangular form for x relative to a commutative algebra, and thus prepare the way for a lifting.

6.5. LEMMA. *If $\rho: A \to B$ is a surjective morphism between* C**-algebras A and B, and (e_n) is a sequence in B such that $0 \leqq e_n \leqq 1$ and $e_n e_{n+1} = e_n$ for all n, then there is a sequence (d_n) in A with*

$$0 \leqq d_n \leqq 1, \quad d_n d_{n+1} = d_n, \quad \rho(d_n) = e_n$$

for all n.

PROOF. We may assume that A and B are unital and that $\rho(1) = 1$. Since $e_1(1 - e_2) = 0$ there are by [2, 2.4] orthogonal positive elements d_1 and x_2 of norm one in A such that $\rho(d_1) = e_1$ and $\rho(x_2) = 1 - e_2$. Take $A_1 = A$ and let A_2 be the hereditary C*-subalgebra of A generated by x_2. Then $d_1 A_2 = 0$ and $1 - e_2 = \rho(x_2) \in \rho(A_2)$, whence $1 - e_n \in \rho(A_2)$ for $n \geqq 2$. Set $\tilde{A}_2 = \mathbf{C}1 + A_2$. Now $e_2(1 - e_3) = 0$ in $\rho(\tilde{A}_2)$, so as above we can find orthogonal elements d_2 and x_3 in $\tilde{A}_2$ with $\rho(d_2) = e_2$ and $\rho(x_3) = 1 - e_3$. Note that $d_2 = \lambda 1 + a$ with a in A_2 and λ in $\mathbf{C}$. But

$$\rho(1 - d_2) = 1 - \lambda - \rho(a) \in \rho(A_2),$$

which is a proper hereditary subalgebra of B. Thus $\lambda = 1$, so that $1 - d_2 \in A_2$. Continuing with $A_3 = (x_3 A x_3)^=$ and $\tilde{A}_3 = \mathbf{C}1 + A_3$, and so on by induction, we find a sequence (d_n) in A with $0 \leqq d_n \leqq 1$ and $\rho(d_n) = e_n$; and a sequence (A_n) of hereditary C*-subalgebras of A such that $d_n A_{n+1} = 0$, $1 - d_n \in A_n$ and $1 - e_m \in$

$\rho(A_n)$ for all $m \geqq n$. From this we immediately see that $d_n(1 - d_{n+1}) = 0$ for all n, as desired.

6.6. LEMMA. *Let A be a σ-unital* C*-*algebra, and let $\pi: M(A) \to C(A)$ denote the quotient map onto the corona* C*-*algebra of A. If $x \in C(A)$ such that $x^n = 0$ for some n, then $x = \pi(y)$ for some y in $M(A)$ with $y^n = 0$.*

PROOF. Choose by Lemma 6.3 elements $e_0, e_1, \dots, e_n$ in $C(A)$ satisfying (i), (ii) and (iii) of that Lemma. Then use Lemma 6.5 to lift the e's to a set $\{d_0, d_1, \dots, d_n\}$ in $M(A)$, such that $0 \leqq d_k \leqq 1$ and $d_k d_{k-1} = d_{k-1}$ for $1 \leqq k \leqq n$, with $d_0 = 0$, $d_n = 1$.

Take continuous functions f and g on $[0, 1]$ with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$, such that $fg = f$. Then let z be an element in $M(A)$ with $\pi(z) = x$, and define

$$y_k = f(d_{k-1})z(g(d_k) - g(d_{k-1}))$$

for $1 \leqq k \leqq n$. Note that $y_1 = 0$ and that

$$\pi(y_k) = f(e_{k-1})x(g(e_k) - g(e_{k-1})) = x(g(e_k) - g(e_{k-1})),$$

since $(1 - f(e_{k-1}))xg(e_k) = 0$ by (ii). Therefore, with $y = \sum y_k$ we have

$$\pi(y) = \sum_{k=1}^{n} x(g(e_k) - g(e_{k-1})) = x(g(e_n) - g(e_0)) = x.$$

Since $(g(d_k) - g(d_{k-1}))f(d_{j-1}) = 0$ if $j \leqq k$, we see from the construction of the y_k's that $y_k y_j = 0$ if $j \leqq k$. Now y^n is a sum of products, each of which must contain n factors from the set $\{y_2, \dots, y_n\}$. It must therefore have the form $y_{\alpha(1)}y_{\alpha(2)} \cdots y_{\alpha(n)}$, where $\alpha(k-1) \geqq \alpha(k)$ for some k, which means that the product is zero. It follows that $y^n = 0$, as desired.

6.7. THEOREM. *If I is a closed ideal in a* C*-*algebra A, and $x \in A$ such that $x^n \in I$ for some n, then $(x + a)^n = 0$ for some a in I.*

PROOF. As in Theorem 6.2 we may assume that A is separable, passing if necessary to the C*-algebra generated by x. Also, we may take $I^\perp$ as the annihilator of I in A, to obtain an embedding

$$I + I^\perp \subset A \subset M(I + I^\perp).$$

Applying Lemma 6.6 to the σ-unital algebra $I + I^\perp$ we find an element $c = a + b$ in $I + I^\perp$ such that

$$0 = (x + c)^n = (x + a)^n + b_0,$$

where b_0 is a sum of products, each of which contains a factor b. Thus $b_0 \in I^\perp$, whereas $(x+a)^n \in I$ by assumption. Since $I \bigcap I^{\perp} = 0$ we conclude that $(x+a)^n = 0$, as desired.

6.8. REMARK. If $A = \mathrm{B}(\mathscr{H})$ and $I = \mathscr{K}$, the first named author proved in [13, 2.4], that if $x \in A$ such that $f(x) \in I$ for some complex polynomial f, then $f(x-a) = 0$ for some a in I. The method of proof (existence of range projections in the algebra) immediately generalizes to the case where I is a (norm) closed ideal in a von Neumann algebra A, cf. [2, 4.3].

When A is only a C*-algebra there are topological obstructions to the lifting of polynomially ideal elements. Already in the case $A = C([0,1])$, $I = C_0(]0,1[)$ and $f(t) = t^2 - t$ we have a counterexample, cf. [2, 2.9]. For the monomials $f(t) = t^n$ no such obstructions exist, and the question was raised in [2, 2.7], whether $x^n \in I$ would imply $(x-a)^n = 0$ for some a in I. Although the case $n = 2$ was solved in [2, 2.8], we had to wait for the $\mathscr{K}$-theoretical machinery, viz. Theorem 3.7, in order to prove the general case.

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