MATH SCAND. 64 (1989), 87–108

COMMUTATORS AND GENERATORS II

DEREK W. ROBINSON

Abstract.

Let $\mathscr{B}$ be a Banach space and σ a C_0-group of isometries of $\mathscr{B}$ with generator H. Set $\mathscr{B}_n = D(H^n)$, $\mathscr{B}_\infty = \bigcap_{n=1}^{\infty} B_n$, and $\|a\|_n = \sup_{m \leqq n} \|H^m a\|$ for all $a \in \mathscr{B}_n$. If K is a dissipative operator from $\mathscr{B}_\infty$ into $\mathscr{B}$ satisfying

$$\|(\operatorname{ad} \sigma_t)(K)a\| \leqq k \|a\|_1 |t|$$

for some $k \geqq 0$ and all $|t| \leqq 1$, $a \in \mathscr{B}_\infty$, then we prove that the closure $\bar{K}$ of K generates a C_0-semigroup of contractions S. Moreover, if $K\mathscr{B}_\infty \subseteqq \mathscr{B}_1$, then $S\mathscr{B}_1 \subseteqq \mathscr{B}_1$ and S defines a $\|\cdot\|_1$-continuous semigroup on $\mathscr{B}_1$. Analogous results are proved for non-isometric groups σ, or for holomorphic semigroups, but at the cost of imposing additional multi-commutator conditions.

1. Introduction

Let σ be a C_0-group of isometries of a Banach space $\mathscr{B}$ with infinitesimal generator H and set $\mathscr{B}_n = D(H^n)$ and $\mathscr{B}_\infty = \bigcap_{n=1}^{\infty} B_n$. Then $\mathscr{B}_n$ is a Banach space with respect to the norm

$$\|a\|_n = \sup_{0 \leqq m \leqq n} \|H^m a\|$$

and $\mathscr{B}_\infty$ is a Fréchet space with respect to the topology induced by the family of norms $\{\|\cdot\|_n; n \geqq 1\}$.

The primary aim of this paper is to prove the following commutator theorem.

THEOREM 1.1. *Let K be a dissipative operator from $\mathscr{B}_\infty$ into $\mathscr{B}$ satisfying*

$$\|(\operatorname{ad} \sigma_t)(K)a\| \leqq k|t| \cdot \|a\|_1 \tag{1.1}$$

for some $k \geqq 0$ and all $a \in \mathscr{B}_\infty$, $|t| \leqq 1$.

Then the closure $\bar{K}$ of K is the generator of a C_0-semigroups S of contractions. Moreover, if $K\mathscr{B}_\infty \subseteqq \mathscr{B}_1$ then $S\mathscr{B}_1 \subseteqq \mathscr{B}_1$ and $S|_{\mathscr{B}_1}$ is a $\|\cdot\|_1$-continuous semigroup, generated by the $\|\cdot\|_1$-closure of K, satisfying

$$\|S_t a\|_1 \leqq e^{kt} \|a\|_1, \ t \geqq 0, \quad a \in \mathscr{B}_1.$$

Received December 1, 1987.

Our secondary aim is to prepare the ground for a subsequent extension of the above theorem [Rob3] in which σ is replaced by an isometric representation of a general Lie group, and (1.1) is modified accordingly. The proof of this generalization requires a number of new techniques together with a version of Theorem 1.1 in which σ is a holomorphic semigroup. We prove the appropriate result in Section 4 but as an intermediate step we establish an analogous result for a non-isometric group σ in Section 3.

A result similar to Theorem 1.1 was proved in [BaR] for dissipative operators from $\mathscr{B}_1$ into $\mathscr{B}$ and in fact the proof of the second statement of the theorem is identical to the proof in [BaR]. The proof of the generator result is based on arguments of "singular perturbation theory" similar to those used in [BaR] but now we use perturbation theory of holomorphic semigroups instead of contraction semigroups. This requires new growth estimates for perturbations of holomorphic semigroups which are based on the approximate commutation of K and H.

If $\mathscr{B}$ is a Hilbert space, and σ a unitary group, then Theorem 1.1 is an elaboration of a commutator theorem of Glimm and Jaffe. The Hilbert space theory, which has been studied extensively, is described in [Rob1] and this paper contains references to earlier work.

In the case $K\mathscr{B}_\infty \subseteq \mathscr{B}_1$ general theory gives another characterization of the generator of the $\|\cdot\|_1$-continuous semigroup obtained by restricting S to $\mathscr{B}_1$. Since $\mathscr{B}_1$ is continuously embedded in $\mathscr{B}$ it follows that this generator is the restriction of $\bar{K}$ to the domain $\{a \in D(\bar{K}) \bigcap \mathscr{B}_1; \bar{K}a \in \mathscr{B}_1\}$ (see, for example, [Paz] Chapter 4, Section 4.5).

The starting point of the proof of the theorem is the observation that dissipativity of K implies closability and hence K is a closable from the Fréchet space $\mathscr{B}_\infty$ into the Banach space $\mathscr{B}$. Hence K is continuous in the sense that

$$\|Ka\| \leqq c\|a\|_p \tag{1.2}$$

for some $c \geqq 0$, $p \geqq 0$, and all $a \in \mathscr{B}_\infty$. Since by standard reasoning $\mathscr{B}_\infty$ is $\|\cdot\|_p$-dense in $\mathscr{B}_p$ it follows that $\mathscr{B}_p \subseteq D(\bar{K})$ and (1.2) is valid for the closure $\bar{K}$ on $\mathscr{B}_p$. Another version of Theorem 1.1 follows if one assumes directly that K is a dissipative operator from $\mathscr{B}_p$ into $\mathscr{B}$, for some $p > 0$, which satisfies (1.2) on $\mathscr{B}_p$. Alternatively it suffices to assume (1.2) is satisfied on some norm-dense σ-invariant subspace $\mathscr{D}$ which is contained in $\mathscr{B}_p$. Similarly for the second statement of the theorem it suffices that there is a norm-dense σ-invariant subspace $\mathscr{D}_1$ such that $\mathscr{D}_1 \subseteq D(\bar{K}) \bigcap \mathscr{B}_p$ and $\bar{K}\mathscr{D}_1 \subseteq \mathscr{B}_1$.

If one relaxes the requirement that the group σ is isometric then the problem becomes more complicated. Nevertheless one can prove versions of Theorem 1.1 if one has additional higher-order estimates

$$\|(\operatorname{ad}\sigma_t)^m(K)a\| \leqq k_m \|a\|_{n_m} |t|^m$$

for all $a \in \mathscr{B}_\infty$, $|t| \leqq 1$, and $m = 1, 2, \ldots, p$. This will be discussed in further detail in Sections 3 and 4 but we note in passing that higher-order estimates lead in general to stronger smoothing properties, e.g. $S\mathscr{B}_m \subseteqq \mathscr{B}_m$.

To conclude this introduction we give a brief sketch of the proof of the generator property of Theorem 1.1. First, by the previous discussion (1.2) is satisfied for $\bar{K}$ on $\mathscr{B}_p$ for some $p \geqq 0$. Hence if $2n > p$ it follows that $\bar{K}$ is relatively bounded by $(-H^2)^n$ with relative bound zero. Second, $(-H^2)^n$ generates a holomorphic semigroup and therefore by perturbation theory $\beta(-H^2)^n + \bar{K}$ also generates a holomorphic semigroup for each $\beta > 0$. In particular, for each $\beta > 0$ there is an $\varepsilon_\beta > 0$ such that the resolvents $r_\beta(\varepsilon) = (I + \varepsilon(\beta(-H^2)^n + \bar{K}))^{-1}$ exist as bounded operators for all $\varepsilon \in [0, \varepsilon_\beta]$. Third, by the Hille-Yosida theorem $\bar{K}$ is a generator if, and only if, $R(I + \varepsilon\bar{K}) = \mathscr{B}$ for all small $\varepsilon > 0$. But if there is a $\phi \in \mathscr{B}^*$ such that

$$\phi((I + \varepsilon\bar{K})a) = 0$$

for all $a \in \mathscr{B}_p$ then

$$\phi((I + \varepsilon\bar{K})r_\beta(\varepsilon)b) = 0$$

for all $b \in \mathscr{B}$. Therefore one has the estimate

$$\begin{aligned} |\phi(b)| &= \varepsilon\beta|\phi((-H^2)^n r_\beta(\varepsilon)b)| \\ &\leqq \varepsilon\beta \, \|\phi\| \cdot \|r_\beta(\varepsilon)b\|_{2n} \end{aligned}$$

for all $b \in \mathscr{B}$ and ε in the interval $[0, \varepsilon_\beta]$. Now if one can prove that ε_β is bounded away from zero as $\beta \to 0$ and that $\|r_\beta(\varepsilon)b\|_{2n}$ is bounded for $b \in \mathscr{B}_{2n}$ uniformly in $\beta \in [0, 1]$ then in the limit $\beta \to 0$ one concludes that $\phi(b) = 0$. Since $\mathscr{B}_{2n}$ is norm-dense this implies $\phi = 0$ and $\bar{K}$ is a generator. Thus the proof is reduced to establishing the boundedness properties of ε_β and $r_\beta(\varepsilon)$. Unfortunately the bounds on $r_\beta(\varepsilon)$ are not generally true under the hypothesis of Theorem 1.1 and in order to establish them it is necessary to replace $\bar{K}$ by a regularized operator. This complicates the proof somewhat but it nevertheless follows the above lines.

2. Resolvent Bounds

In this section we derive resolvent bounds of the type indicated above and then we apply them to the proof of Theorem 1.1.

If H generates a C_0-group σ then the operators $(-H^2)^n$, $n = 1, 2, \ldots$ generate holomorphic C_0-semigroups $S^{(n)}$ defined by

$$S_t^{(n)} = \int ds \, \mu_t^{(n)}(s) \, \sigma_s$$

where the $\mu^{(n)}$ are convolution semigroups given by

$$\mu_t^{(n)}(s) = (2\pi)^{-1}\int dq\, e^{-tq^{2n}} e^{iqs}.$$

Continuity and holomorphy of the $S^{(n)}$ follow directly from the explicit form of the $\mu^{(n)}$ and in fact one has the following basic estimates.

LEMMA 2.1. *There is an $M_n \geqq 1$ and $\omega_n \geqq 0$ such that*

$$\|H^m S_t^{(n)}\| \leqq M_n e^{\omega_n t} t^{-m/2n}$$

for all $t > 0$ and $m = 0, 1, 2, \ldots, 2n$. Moreover if σ is isometric then one can choose $\omega_n = 0$.

PROOF. Set $D = d/ds$ then

$$H^m S_t^{(n)} = \int ds\,(D^m \mu_t^{(n)})(s)\,\sigma_s.$$

But it follows from the explicit form of $\mu^{(n)}$ by scaling that

$$(D^m \mu_t^{(n)})(s) = t^{-(m+1)/2n}(D^m \mu_1^{(n)})(st^{-1/2n}).$$

Therefore if σ is isometric one obtains the bounds

$$\|H^m S_t^{(n)}\| \leqq t^{-m/2n} \max_{0 \leqq m \leqslant 2n} \int ds\,|(D^m \mu_1^{(n)})(s)|.$$

(Note that the integrand is bounded by $e^{-k|s|}$ for any $k \geqq 0$.) The general result follows similarly because σ satisfies bounds $\|\sigma_s\| \leqq M\exp\{\omega|s|\}$.

Next we are interested in operators satisfying bounds $\|Ka\| \leqq c\|a\|_p$ and it is convenient to re-express these as relative bounds with respect to powers of H.

LEMMA 2.2. *For each $\varepsilon > 0$ there is a $c_\varepsilon^{(p)}$ such that*

$$\|a\|_p \leqq \varepsilon\,\|H^{p+1} a\| + c_\varepsilon^{(p)}\,\|a\|, \quad a \in \mathscr{B}_{p+1}.$$

PROOF. First, since

$$\sigma_t a = a - tHa + \int_0^t ds(t-s)\,\sigma_s H^2 a, \quad a \in \mathscr{B}_2,$$

one readily obtains estimates

$$\|Ha\| \leqq \varepsilon\|H^2 a\| + k_\varepsilon^{(1)}\,\|a\|, \quad a \in \mathscr{B}_2. \tag{2.1}$$

Therefore

$$\|a\|_1 \leqq \varepsilon\|H^2 a\| + c_\varepsilon^{(1)}\,\|a\|, \quad a \in \mathscr{B}_2 \tag{2.2}$$

with $c_\varepsilon^{(1)} = 1 \vee k_\varepsilon^{(1)}$. Second, one argues by induction starting from (2.1) that

$$\|H^p a\| \leqq \varepsilon \|H^{p+1} a\| + k_\varepsilon^{(p)} \|a\| \tag{2.3}$$

for all $a \in \mathscr{B}_{p+1}$, $p = 1, 2, \ldots$ Finally one deduces the statement of the lemma by another inductive argument starting from (2.2) and using (2.3). We omit the details.

Lemma 2.2 provides the starting point of the proof of Theorem 1.1.

COROLLARY 2.3. *Let K be a closable operator from $\mathscr{B}_\infty$ into $\mathscr{B}$ with closure $\bar{K}$. Then there is an $n \geqq 1$ such that $\mathscr{B}_{2n} \subseteqq D(\bar{K})$ and the operators H_β; $\mathscr{B}_{2n} \mapsto \mathscr{B}$ defined for $\beta > 0$ by*

$$H_\beta = \beta(-H^2)^n + \bar{K}$$

are generators of holomorphic semigroups.

PROOF. It follows from the uniform boundedness theorem that there are $c, p \geqq 0$ such that

$$\|Ka\| \leqq c\|a\|_p, \quad a \in \mathscr{B}_\infty.$$

Then since $\mathscr{B}_\infty$ is $\|\cdot\|_p$-dense in $\mathscr{B}_p$, by a standard regularization argument, it follows that $\mathscr{B}_p \subseteqq D(\bar{K})$ and (2.4) extends to $\bar{K}$ on $\mathscr{B}_p$. Now choosing n such that $2n > p$ one deduces from Lemma 2.2 that

$$\|\bar{K}a\| \leqq \varepsilon c \|H^{2n} a\| + c c_\varepsilon^{2n-1} \|a\|, \quad a \in \mathscr{B}_{2n},$$

i.e. $\bar{K}$ is relatively bounded by $(-H^2)^n$ with relative bound zero. Hence the corollary follows from the theory of perturbation of generators of holomorphic semigroups (see, for example, [Paz] Theorem 3.2.1).

Our next aim is to obtain estimates on the resolvents and resolvent sets of operators H_β of the foregoing form. The principal tactic is to introduce a family of equivalent norms.

LEMMA 2.4. *Let $S^{(n)}$ denote the holomorphic semigroup generated by $(-H^2)^n$ and fix $M_n \geqq 1$, $\omega_n > 0$, such that $\|S_t^{(n)}\| \leqq M_n \exp\{\omega_n t\}$, $t \geqq 0$. For each $\beta > 0$ set $\gamma = \gamma_n(\beta) = \log M_n + \beta\omega_n$ and define $\|\cdot\|_\beta$ by*

$$\|a\|_\beta = \sup_{t \geqq 0} \|S_{\beta t}^{(n)} a\| e^{-\gamma t}, \quad a \in \mathscr{B}.$$

Then one has the following:

1. $\|a\| \leqq \|a\|_\beta \leqq M_n \|a\|$,
2. $\|a\|_\beta = \sup_{0 \leqq t \leqq 1} \|S_{\beta t}^{(n)} a\| e^{-\gamma t}$,
3. $\|S_{\beta t}^{(n)} a\|_\beta \leqq e^{\gamma t} \|a\|_\beta, \quad a \in \mathscr{B}$.

In particular $\beta(-H^2)^n + \gamma I$ is $\|\cdot\|_\beta$-dissipative.

PROOF. The proofs of 1 and 3 are elementary and 2 follows because γ is chosen such that

$$M_n e^{\beta\omega_n t} \leqq e^{\gamma t}, \quad t \geqq 1.$$

The last statements of the lemma follows since $\beta(-H^2)^n + \gamma I$ generates the $\|\cdot\|_\beta$-contractive semigroup $t \mapsto S^{(n)}_{\beta t} e^{-\gamma t}$.

Now let K be a dissipative, hence closable, operator from $\mathscr{B}_\infty$ into $\mathscr{B}$ and choose $n \geqq 1$ such that the conclusions of Corollary 2.3 are valid. The next idea is to use the assumption (1.1) to prove that $K + \omega_\beta I$ is $\|\cdot\|_\beta$-dissipative for some ω_β and hence deduce that $H_\beta + (\omega_\beta + \gamma)I$ is $\|\cdot\|_\beta$-dissipative. This then implies that the semigroup S^β generated by H_β satisfies the estimates $\|S_t\| \leqq M_n \exp\{t(\omega_\beta + \gamma)\}$, $t > 0$, and bounds on the resolvent of H_β follow by Laplace transformation. Unfortunately the first step in this procedure appears impossible unless one has stronger commutation hypotheses than (1.1). Nevertheless the procedure works if one replaces K by a regularized operator. But then it is critical that σ is isometric. Hence *throughout the rest of this section we adopt the assumption of Theorem* 1.1 *and choose n such that the conclusions of Corollary* 2.3 *are valid.*

Let $\alpha > 0$ and define the regularized operators $K^{(m)}_\alpha$ as linear operators from $\mathscr{B}_\infty$ into $\mathscr{B}$ by

$$K^{(m)}_\alpha = \alpha^{-m} \int_0^\alpha dt_1 \ldots \int_0^\alpha dt_m \, \sigma_{t_1 + \ldots + t_m} \bar{K} \sigma_{-t_1 - \ldots - t_m}.$$

Note that the strong integrals exist because σ is continuous and K satisfies the estimate (2.4). For brevity we also use the notation

$$K^{(m)}_\alpha = \left(\alpha^{-1} \int_0^\alpha dt\right)^m \sigma_{t^m} \bar{K} \sigma_{-t^m}.$$

LEMMA 2.5. *The operators* $K^{(m)}_\alpha$ *are dissipative and*

$$\|(ad\,\sigma_t)(K^{(m)}_\alpha)a\| \leqq k|t| \cdot \|a\|_1, \tag{2.5}$$

for all $a \in \mathscr{B}_\infty$ *and* $|t| \leqq 1$. *Moreover,* $K^{(m)}_\alpha \mathscr{B}_\infty \subseteq \mathscr{B}_m$ *and*

$$\|(K^{(m)}_\alpha - K)a\| \leqq (mk\alpha/2)\,\|a\|_1, \quad a \in \mathscr{B}_\infty. \tag{2.6}$$

PROOF. Since K is norm-densely defined and dissipative it follows that for every $a \in \mathscr{B}_\infty$ and $\omega \in \mathscr{B}^*$ such that $\omega(a) = \|\omega\| \cdot \|a\|$ one has $\operatorname{Re}\omega(Ka) \geqq 0$. But $(\omega \circ \sigma_t)(\sigma_{-t}a) = \|\omega\| \cdot \|a\| = \|\omega \circ \sigma_t\| \cdot \|\sigma_{-t}a\|$ because σ is isometric. Therefore $\sigma_t K \sigma_{-t}$ is dissipative and it follows immediately that the $K^{(m)}_\alpha$ are dissipative.

Next it follows from (1.1) that

$$\|(\operatorname{ad}\sigma_t)(K_\alpha^{(m)})a\| \leqq (\alpha^{-1}\int_0^\alpha ds)^m \|\sigma_{s^m}(\operatorname{ad}\sigma_t)(K)\sigma_{-s^m}a\|$$
$$\leqq (\alpha^{-1}\int_0^\alpha ds)^m k|t|\cdot\|\sigma_{-s^m}a\|_1$$
$$\leqq k|t|\cdot\|a\|_1, \quad a\in\mathscr{B}_\infty,$$

where we have again used the isometry of σ. Similarly

$$\|(K_\alpha^{(m)} - K)a\| \leqq \left(\alpha^{-1}\int_0^\alpha ds\right)^m \|(\operatorname{ad}\sigma_{-s^m})(K)a\|$$
$$\leqq \alpha^{-m}\int_0^\alpha ds_1 \ldots \int_0^\alpha ds_m(s_1 + \ldots + s_m)k\|a\|_1$$
$$= (mk\alpha/2)\|a\|_1.$$

Finally it follows by straightforward calculation that $K_\alpha^{(m)}\mathscr{B}_\infty \subseteqq \mathscr{B}_1$ and

$$HK_\alpha^{(m)}a = K_\alpha^{(m)}Ha + \alpha^{-1}(K_\alpha^{(m-1)} - \sigma_\alpha K_\alpha^{(m-1)}\sigma_{-\alpha})a \tag{2.7}$$

for all $a\in\mathscr{B}_\infty$. Thus by iteration one has $K_\alpha^{(m)}\mathscr{B}_\infty \subseteqq \mathscr{B}_m$.

The next result is the first crucial estimate in the proof of Theorem 1.1.

LEMMA 2.6. *Let α, $\beta\in]0,1]$ and fix $M_n \geqq 1$ such that $\|H^m S_t^{(n)}\| \leqq M_n t^{-m/2n}$, $t\geqq 0, m = 0,1,\ldots,2n$. Then $K_\alpha^{(2n)} + \omega I$ is $\|\cdot\|_\beta$-dissipative for all $\omega \geqq \omega_{\alpha,\beta}$ where*

$$\omega_{\alpha,\beta} = 8n^2kM_n^2(1 + (2/\alpha)(\beta/2)^{1/2})^{2n-1}.$$

PROOF. Since $K_\alpha^{(2n)}$ is dissipative, by Lemma 2.5, and $S_t^{(n)}\mathscr{B} \subseteqq \mathscr{B}_\infty$, $t > 0$, one has

$$(1+\varepsilon\omega)\|S_{\beta t}^{(n)}a\|e^{-\gamma t} \leqq \|(I + \varepsilon(K_\alpha^{(2n)} + \omega I))S_{\beta t}^{(n)}a\|e^{-\gamma t}$$
$$\leqq \|S_{\beta t}^{(n)}(I + \varepsilon(K_\alpha^{(2n)} + \omega I))a\|e^{-\gamma t}$$
$$+ \varepsilon\|(\operatorname{ad}S_{\beta t}^{(n)})(K_\alpha^{(2n)})a\|e^{-\gamma t}$$

for all $a\in\mathscr{B}_\infty$. Therefore setting $\gamma = \log M_n$ one has

$$(1+\varepsilon\omega)\|a\|_\beta \leqq \|(I + \varepsilon(K_\alpha^{(2n)} + \omega I))a\|_\beta + \varepsilon\sup_{0\leqq t\leqq 1}\|(\operatorname{ad}S_{\beta t}^{(n)})(K_\alpha^{(2n)})a\|.$$

Next we establish the bounds

$$\sup_{0\leqq t\leqq 1}\|(\operatorname{ad}S_{\beta t}^{(n)})(K_\alpha^{(2n)})a\| \leqq \omega_{\alpha,\beta}\|a\|_\beta \tag{2.8}$$

and consequently deduce that

$$\|a\|_\beta \leqq \|(I + \varepsilon(K_\alpha^{(2n)} + \omega I))a\|_\beta$$

for all $\varepsilon > 0$ and $\omega_{\alpha,\beta}$. Thus $K_\alpha^{(2n)} + \omega I$ is $\|\cdot\|_\beta$-dissipative for $\omega \geqq \omega_{\alpha,\beta}$.

The proof of (2.8) starts with the observation that $K_\alpha^{(2n)}\mathscr{B}_\infty \subseteq \mathscr{B}_{2n}$, by Lemma 2.5, and $K_\alpha^{(2n)}$ is $\|\cdot\|_{2n}$-continuous. Hence one can derive the Duhamel identity

$$(\operatorname{ad} S_{\beta t}^{(n)})(K_\alpha^{(2n)})a = -\beta(-1)^n \int_0^t ds\, S_{\beta s}^{(n)}(\operatorname{ad} H^{2n})(K_\alpha^{(2n)})\, S_{\beta(t-s)}^{(n)}\, a$$

for all $a \in \mathscr{B}_\infty$. Hence using the combinatorial relation

$$(\operatorname{ad} A)(B^{2n}) = \sum_{r=1}^{2n} (-1)^r \binom{2n}{r} B^{2n-r}(\operatorname{ad} B)^r(A)$$

one obtains

$$(\operatorname{ad} S_{\beta t}^{(n)})(K_\alpha^{(2n)})\, a = \beta \sum_{r=1}^{2n} (-1)^{n+r} \binom{2n}{r} \int_0^t ds \cdot$$
$$\cdot H^{2n-r} S_{\beta s}^{(n)}(\operatorname{ad} H)^r (K_\alpha^{(2n)})\, S_{\beta(t-s)}^{(n)}\, a$$

for all $a \in \mathscr{B}_\infty$. This immediately gives the estimates

$$\|(\operatorname{ad} S_{\beta t}^{(n)})(K_\alpha^{(2n)})\, a\| \leqq \beta \sum_{r=1}^{2n} \binom{2n}{r} \int_0^t ds\, \|H^{2n-r} S_{\beta s}^{(n)}\| \cdot$$
$$\cdot \|(\operatorname{ad} H)^r(K_\alpha^{(2n)})\, S_{\beta(t-s)}^{(n)}\, a\|$$
$$\leqq M_n \beta \sum_{r=1}^{2n} \binom{2n}{r} \int_0^t ds\, (\beta s)^{-1+r/2n} \cdot$$
$$\cdot \|(\operatorname{ad} H)^r (K_\alpha^{(2n)})\, S_{\beta(t-s)}^{(n)}\, a\|.$$

Now by iteration of (2.7) one finds

$$\|(\operatorname{ad} H)^r (K_\alpha^{(2n)})\, b\| \leqq \alpha^{-r+1} \sum_{s=0}^{r-1} \binom{r-1}{s} \|(\operatorname{ad} H)(K_\alpha^{(2n-r+1)})\sigma_{-s\alpha} b\|$$
$$\leqq (2/\alpha)^{r-1}\, k\, \|b\|_1$$

for all $b \in \mathscr{B}_\infty$ where the last bound follows from (2.5). But if $b = S_{\beta(t-s)}^{(n)} a$ then for $0 < t - s \leqq 1$

$$\|b\|_1 \leqq M_n(\beta(t-s))^{-1/2n}\, \|a\|$$

and hence combination of these estimates yields

$$\|(\operatorname{ad} S_{\beta t}^{(n)})(K_\alpha^{(2n)}) a\| \leqq k\, M_n^2 \sum_{r=1}^{2n} \binom{2n}{r} \cdot$$
$$\cdot (2/\alpha)^{r-1}\, \beta^{(r-1)/2n} \int_0^t ds\, s^{-1+r/2n} (t-s)^{-1/2n}\, \|a\|$$

for $t \in]0,1]$. But the integral is convergent, because $r \geqq 1$, and a crude estimate establishes that it is bounded by $4n(t/2)^{(r-1)/2n}$. Therefore

$$\|(\operatorname{ad} S_{\beta t}^{(n)})(K_\alpha^{(2n)})a\| \leqq 4nk\, M_n^2 \sum_{r=1}^{2n} \binom{2n}{r} (2/\alpha)^{r-1} (\beta/2)^{(r-1)/2n} \|a\|$$

$$\leqq 8n^2 k\, M_n^2 (1 + (2/\alpha)(\beta/2)^{1/2n})^{2n-1} \|a\|_\beta$$

for $t \in]0,1]$ where we have used $\|a\| \leqq \|a\|_\beta$ and the convexity inequality $(1 + x)^{2n} \leqq 1 + 2nx(1 + x)^{2n-1}$, $x \geqq 0$. This establishes the bounds (2.8) and completes the proof of the lemma.

Since $K_\alpha^{(2n)}$ is dissipative, by Lemma 2.5, it is closable and its closure is also dissipative and automatically satisfies the dissipativity property of Lemma 2.6. Moreover, since σ is isometric it follows easily that $K_\alpha^{(2n)}$ satisfies an estimate similar to (2.4), i.e.

$$\|K_\alpha^{(2n)}a\| \leqq c\|a\|_p, \quad a \in \mathscr{B}_\infty.$$

Therefore $\mathscr{B}_{2n} \subseteq \mathscr{B}_p \subseteq D(K_\alpha^{(2n)})$ where for simplicity we now use $K_\alpha^{(2n)}$ to denote the closed operator. But it follows from Lemma 2.2 that $K_\alpha^{(2n)}$ is relatively bounded by $(-H^2)^n$ with relative bound zero. Consequently, by Corollary 2.3, the operators

$$H_{\alpha,\beta} = \beta(-H^2)^n + K_\alpha^{(2n)}$$

are generators of holomorphic semigroups $S^{\alpha,\beta}$. But then it follows from Lemma 2.4, with $\omega_n = 0$, and Lemma 2.6 that $H_{\alpha,\beta} + \omega I$ is $\|\cdot\|_\beta$-dissipative for $\omega \geqq \omega_{\alpha,\beta} + \log M_n$. This implies, however, that

$$\|S_t^{\alpha,\beta} a\| \leqq \|S_t^{\alpha,\beta} a\|_\beta$$

$$\leqq e^{t(\omega_{\alpha,\beta} + \log M_n)} \|a\|_\beta \leqq M_n e^{t(\omega_{\alpha,\beta} + \log M_n)} \|a\|$$

for all $a \in \mathscr{B}$ and $t \geqq 0$. Now the most interesting feature of this estimate is that if $\beta \leqq \alpha^{2n}$ then $\omega_{\alpha,\beta}$ is uniformly bounded. Therefore one reaches the following conclusion.

COROLLARY 2.7. *If* $0 < \alpha \leqq 1$ *and* $0 < \beta \leqq \alpha^{2n}$ *the operators*

$$H_{\alpha,\beta} = \beta(-H^2)^n + K_\alpha^{(2n)}$$

generate holomorphic semigroups $S^{\alpha,\beta}$ *and there exist* $M \geqq 1, \omega \geqq 0$ *independent of* α *and* β *such that*

$$\|S_t^{\alpha,\beta}\| \leqq M\, e^{\omega t}$$

for all $t \geqq 0$. *Hence the resolvents* $(1 + \varepsilon H_{\alpha,\beta})^{-1}$ *exist for* $\varepsilon \in [0,\omega^{-1}[$ *and*

$$\|(I + \varepsilon H_{\alpha,\beta})^{-1}\| \leqq M(1 - \varepsilon\omega)^{-1}.$$

In fact the foregoing discussion establishes that one may choose $M = M_n = \sup\{t^{m/2n} \|H^m S_t^{(n)}\|; t \geqq 0, m = 0, 1, \ldots, 2n\}$ and $\omega = \omega_{\alpha, \alpha^{2n}} + \log M_n$.

Our next aim is to estimate $\|(I + \varepsilon H_{\alpha,\beta})^{-1}\|_m$ for $m = 1, 2, \ldots, 2n$. Now set $r_{\alpha,\beta}(\varepsilon) = (I + \varepsilon H_{\alpha,\beta})^{-1}$ and $D_t = (I - \sigma_t)/t$ then

$$H^m r_{\alpha,\beta}(\varepsilon) a = \lim_{t \to 0} D_t^m r_{\alpha,\beta}(\varepsilon) a$$

for all $a \in \mathscr{B}$ and $m = 1, 2, \ldots, 2n$ because $r_{\alpha,\beta}(\varepsilon)\mathscr{B} \subseteqq \mathscr{B}_{2n}$. But $D_t^m \mathscr{B}_{2n} \subseteqq \mathscr{B}_{2n}$ and hence

$$\begin{aligned} D_t^m r_{\alpha,\beta}(\varepsilon) a &= r_{\alpha,\beta}(\varepsilon) D_t^m a + (\operatorname{ad} D_t^m)(r_{\alpha,\beta}(\varepsilon))\, a \\ &= r_{\alpha,\beta}(\varepsilon) D_t^m a - \varepsilon r_{\alpha,\beta}(\varepsilon) (\operatorname{ad} D_t^{(m)}) (K_\alpha^{(2n)})\, r_{\alpha,\beta}(\varepsilon) a. \end{aligned}$$

Next we use the combinatorial relation

$$(\operatorname{ad} A)(B^m) = -\sum_{r=1}^{m} \binom{m}{r} (\operatorname{ad} B)^r (A)\, B^{m-r}$$

to deduce that

$$\begin{aligned} D_t^m r_{\alpha,\beta}(\varepsilon)\, a = r_{\alpha,\beta}(\varepsilon) D_t^m a - \varepsilon \sum_{r=1}^{m} \binom{m}{r} r_{\alpha,\beta}(\varepsilon) \cdot \\ \cdot (\operatorname{ad} D_t)^r (K_\alpha^{(2n)})\, D_t^{m-r}\, r_{\alpha,\beta}(\varepsilon)\, a \end{aligned}$$

for $t > 0$, and this leads to the estimates

$$\begin{aligned} \|H^m r_{\alpha,\beta}(\varepsilon)\, a\| \leqq \|r_{\alpha,\beta}(\varepsilon)\| \Big\{ \|H^m a\| + \varepsilon \sum_{r=1}^{m} \binom{m}{r} \cdot \\ \cdot \limsup_{t \to 0} \|(\operatorname{ad} D_t)^r (K_\alpha^{(2n)})\, D_t^{m-r}\, r_{\alpha,\beta}(\varepsilon) a\| \Big\} \end{aligned} \tag{2.9}$$

for $a \in \mathscr{B}_m$. Now we use a combination of three different estimates to bound the terms on the right.

First, if L denotes any closed operator with $\mathscr{B}_p \subseteqq D(L)$ then one can define similar operators $\tau_t(L)$, $t > 0$, by setting $\tau_t(L) = \sigma_t L \sigma_{-t}$. It then follows that

$$t^{-1}(I - \tau_t)(K_\alpha^{(m)}) = \alpha^{-1}(I - \tau_\alpha)\Big(t^{-1}\int_0^t ds\, \sigma_s K_\alpha^{(m-1)} \sigma_{-s}\Big).$$

Since $(\operatorname{ad} D_t)(L) = t^{-1}(I - \tau_t)(L)\sigma_t$ one immediately obtains the identity

$$(\operatorname{ad} D_t)^r (K_\alpha^{(m)}) = \alpha^{-r}(I - \tau_\alpha)^r \Big(t^{-1}\int_0^t ds\Big)^r (\sigma_{s^r} K_\alpha^{(m-r)} \sigma_{-s^r}) \sigma_{rs - q\alpha} b\|.$$

Second, it follows from Lemma 2.5 that

$$\|(\operatorname{ad} D_t)(K_\alpha^{(m)}) c\| \leqq k\, \|c\|_1.$$

Third, one has the Duhamel estimate

$$\|D_t^q d\|_1 \leqq \|H^q d\|_1.$$

These three estimates can also be established in combination. Then using the first with r replaced by $r-1$, $m = 2n$, and $L = (\operatorname{ad} D_t)(K)$, then the second, and finally the third with $q = m - r$, one obtains

$$\limsup_{t\to 0} \|(\operatorname{ad} D_t)^r (K_\alpha^{(2n)}) D_t^{m-r} b\| \leqq k(2/\alpha)^{r-1} \|H^{m-r} b\|_1. \tag{2.10}$$

Therefore (2.9) and (2.10) give

$$\begin{aligned}\|H^m r_{\alpha,\beta}(\varepsilon)a\| &\leqq \|r_{\alpha,\beta}(\varepsilon)\| \left\{\|H^m a\| + \varepsilon k \sum_{r=1}^{m} \binom{m}{r} (2/\alpha)^{r-1} \|H^{m-r} r_{\alpha,\beta}(\varepsilon)a\|_1\right\}\\ &\leqq \|r_{\alpha,\beta}(\varepsilon)\| \left\{\|a\|_m + \varepsilon k \sum_{r=1}^{m} \binom{m}{r} (2/\alpha)^{r-1} \|r_{\alpha,\beta}(\varepsilon)a\|_{m-r+1}\right\}.\end{aligned}$$

Since these bounds are valid for $m = 1, 2, \ldots, 2n$, one deduces that

$$\|r_{\alpha,\beta}(\varepsilon)a\|_m \leqq \|r_{\alpha,\beta}(\varepsilon)\| \left\{\|a\|_m + \varepsilon k \sum_{r=1}^{m} \binom{m}{r} \cdot (2/\alpha)^{r-1} \|r_{\alpha,\beta}(\varepsilon)a\|_{m-r+1}\right\} \tag{2.11}$$

for all $m = 1, 2, \ldots, 2n$. Now choose M and ω such that the bounds of Corollary 2.7 are and set $\omega_m = Mmk(1 - \varepsilon\omega)^{-1}$. If $\varepsilon\omega_1 < 1$ it follows directly from (2.11) that

$$\|r_{\alpha,\beta}(\varepsilon)a\|_1 \leqq M(1 - \varepsilon\omega)^{-1}(1 - \varepsilon\omega_1)^{-1} \|a\|_1.$$

More generally, if $\varepsilon\omega < 1$ and $\varepsilon\omega_m < 1$ one obtains

$$\|r_{\alpha,\beta}(\varepsilon)a\|_m \leqq M(1 - \varepsilon\omega_m)^{-1} \left\{\|a\|_m + \varepsilon k \sum_{r=2}^{m} \binom{m}{r} \cdot (2/\alpha)^{r-1} \|r_{\alpha,\beta}(\varepsilon)a\|_{m-r+1}\right\}$$

for $m = 2, 3, \ldots, 2n$. But as $\alpha \in]0, 1]$ this gives

$$\alpha^{m-1} \|r_{\alpha,\beta}(\varepsilon)a\|_m \leqq M(1 - \varepsilon\omega)^{-1}(1 - \varepsilon\omega_m)^{-1} \left\{\|a\|_m + \varepsilon k \sum_{r=2}^{m} \binom{m}{r} \cdot 2^{r-1} \alpha^{m-r} \|r_{\alpha,\beta}(\varepsilon)a\|_{m-r+1}\right\}$$

and iteration of these inequalities leads to a bound on $\alpha^{m-1} \|r_{\alpha,\beta}(\varepsilon)a\|_m$ which is independent of α and β. In particular one has the following:

LEMMA 2.8. *Choose* $M \geqq 1, \omega \geqq 0$, *such that* $\|S_t^{\alpha,\beta}\| \leqq M \exp\{\omega t\}$ *for all* $t \geqq 0$, $\alpha \in]0,1]$ *and* $\beta \in]0, \alpha^{2n}]$. *Then for* $\varepsilon \in [0, \omega^{-1}[$ *define* $\omega_m = \omega_m(\varepsilon) = Mmk(1 - \varepsilon\omega)^{-1}$. *If* $\varepsilon\omega_m < 1$, *where* $1 \leqq m \leqq 2n$, *there is a* $c_m(\varepsilon)$ *independent of* α *and* β *such that*

$$\alpha^{m-1}\|(1 + \varepsilon H_{\alpha,\beta})^{-1} a\|_m \leqq c_m(\varepsilon)\|a\|_m$$

for all $a \in \mathscr{B}_m$.

Note that it is not necessary to take a $2n$-fold regularization of K to obtain these bounds, it suffices to regularize $(2n - 1)$-times. One regularization allows one to reduce a double commutator estimate to a single commutator estimate and a $(2n - 1)$-fold regularization allows estimates of all commutators up to and including order $2n$ in terms of a simple commutator.

Now we are prepared to prove Theorem 1.1 by an elaboration of the method sketched in the introduction.

First, since K is dissipative $\bar{K}$ is also dissipative and then, by the Lumer-Philips version of the Hille-Yosida theorem, $\bar{K}$ generates a continuous contraction semigroup if, and only if, the range of $I + \varepsilon\bar{K}$ is dense in $\mathscr{B}$ for small $\varepsilon > 0$. Assume this is not the case then there is a non-zero $\phi \in \mathscr{B}^*$ such that

$$\phi((I + \varepsilon\bar{K})a) = 0$$

for all $a \in \mathscr{B}_{2n}$. Therefore

$$\phi((I + \varepsilon\bar{K})\, r_{\alpha,\beta}(\varepsilon)b) = 0$$

for all $b \in \mathscr{B}$ where once again $r_{\alpha,\beta}(\varepsilon) = (I + \varepsilon H_{\alpha,\beta})^{-1}$ and $H_{\alpha,\beta}$ denotes the operator introduced in Corollary 2.7. Consequently

$$\phi(b) = \varepsilon\,\phi((\beta(-H^2)^n + K_\alpha^{(2n)} - \bar{K})\, r_{\alpha,\beta}(\varepsilon)b)$$

which gives the estimate

$$|\phi(b)| \leqq \varepsilon\|\phi\|\,\{\beta\,\|r_{\alpha,\beta}(\varepsilon)b\|_{2n} + \alpha nk\,\|r_{\alpha,\beta}(\varepsilon)b\|_1\}$$

by dint of (2.6). But we may assume $\varepsilon\omega_{2n} < 1$, where ω_{2n} is defined in Lemma 2.8, and then the estimates of this lemma give

$$|\phi(b)| \leqq \varepsilon\|\phi\|\,\{\beta\alpha^{-2n+1}\, c_{2n}(\varepsilon)\,\|b\|_{2n} + \alpha nk\, c_1(\varepsilon)\,\|b\|_1\}$$

for all $b \in \mathscr{B}_{2n}$. Thus in the limit $\beta \to 0$ then $\alpha \to 0$ one concludes that $\phi(b) = 0$ for all $b \in \mathscr{B}_{2n}$ which forces the contradiction $\phi = 0$. There $R(I + \varepsilon\bar{K})$ is dense for small $\varepsilon > 0$ and $\bar{K}$ is a generator.

The proof of the second statement of Theorem 1.1 is not significantly different to the proof of the analogous statement in Theorem 2.1 of [BaR]. Hence we omit further details.

A slight simplification of the above proof also establishes that the regularized operators $K_\alpha^{(2n)}$, which are dissipative by Lemma 2.5, are generators of continuous contraction semigroups. In fact the holomorphic semigroups $S^{\alpha,\beta}$ generated by $H_{\alpha,\beta}$ converge strongly to the contraction semigroup S^α generated by $K_\alpha^{(2n)}$ as $\beta \to 0$. Moreover the S^α converge strongly to the contraction semigroups S generated by $\bar{K}$ as $\alpha \to 0$. These results can be established by proving resolvent convergence of the appropriate generators exactly as in [BaR]. This convergence argument carries through with the aid of the uniform estimates of Corollary 2.7 and Lemma 2.8, e.g. the usual resolvent identity gives the estimate

$$\begin{aligned}\|(r_{\alpha,\beta_1}(\varepsilon) - r_{\alpha,\beta_2}(\varepsilon))a\| &\leqq |\beta_1 - \beta_2| \cdot \|r_{\alpha,\beta_1}(\varepsilon)\| \cdot \|r_{\alpha,\beta_2}(\varepsilon)a\|_{2n} \\ &\leqq |\beta_1 - \beta_2| M(1 - \varepsilon\omega)^{-1} \alpha^{-2n+1} c_{2n}(\varepsilon) \|a\|_{2n}\end{aligned}$$

and strong convergence of $r_{\alpha,\beta}(\varepsilon)$ as $\beta \to 0$ follows for small $\varepsilon > 0$. Again we omit details.

We conclude this section by remarking that since σ is isometric $-\lambda H^2 + \mu H$ is dissipative for all $\lambda \geqq 0$ and $\mu \in \mathbb{R}$. Then replacing K by $-\lambda H^2 + \mu H + K$ does not affect the hypotheses of Theorem 1.1. Therefore one reaches the following conclusion.

COROLLARY 2.9. *Adopt the assumptions of Theorem* 1.1. *Then for each* $\lambda \geqq 0$ *and* $\mu \in \mathbb{R}$ *the closure of the operator* $-\lambda H^2 + \mu H + K$ *generates a* C_0*-semigroup of contractions* S *and if* $K\mathscr{B}_\infty \subseteq \mathscr{B}_1$ *and* $S|_{\mathscr{B}_1}$ *is a* $\|\cdot\|_1$*-continuous semigroup satisfying*

$$\|S_t a\|_1 \leqq e^{kt} \|a\|_1, \quad t \geqq 0, a \in \mathscr{B}_1.$$

3. Multi-commutator Theorems

The strength of Theorem 1.1 is that the action of K relative to H is only restricted by the simple commutator condition (1.1). But the proof nevertheless needs estimates on multi-commutators, and the regularization of K was used to estimate the higher order commutators in terms of the simple commutator. If, however, one has appropriate bounds on sufficient multiple commutators one can prove Theorem 1.1 without introducing the regularized K. This is of interest because isometry of σ is then inessential. The only delicate use of isometry in the foregoing proof was to establish dissipativity of the regularized K. If the regularization procedure is not used then the isometry condition can also be avoided.

THEOREM 3.1. *Let* σ *be a* C_0*-group on a Banach space* $\mathscr{B}$ *with infinitesimal generator* H *and* K *a dissipative operator from the Banach subspace* $\mathscr{B}_{2n}$ *into* $\mathscr{B}$ *where* $\mathscr{B}_m = D(H^m)$ *and* $\|a\|_m = \sup_{0 \leqq p \leqq m} \|H^p a\|$, $m = 1, 2, \ldots$ *Assume*

1. *for each* $\varepsilon \in]0, 1]$ *there is a* $k_\varepsilon > 0$ *such that*

$$\|Ka\| \leqq \varepsilon \|a\|_{2n} + k_\varepsilon \|a\|_{2n-1}, \quad a \in \mathscr{B}_{2n}$$

2. *there exist* $l_m \geqq 0$ *such that*

$$\|(\operatorname{ad}\sigma_t)^m(K)a\| \leqq l_m \|a\|_m |t|^m, \quad a \in \mathscr{B}_{2n}$$

for $|t| \leqq 1$ *and all* $m = 1, 2, \dots, 2n - 1$, *and*

$$\|(\operatorname{ad}\sigma_t)^{2n}(K)a\| \leqq (\delta\|a\|_{2n} + l_{2n}\delta^{-1}\|a\|_{2n-1})|t|^{2n}, \quad a \in \mathscr{B}_{2n}$$

for $|t| \leqq 1$ *and* $0 < \delta \leqq 1$.

It follows that the closure $\bar{K}$ *of* K *is the generator of a* C_0*-semigroup of contractions* S. *Moreover, for each* $m = 1, 2, \dots, 2n - 1$ *one has* $K\mathscr{B}_\infty \subseteq \mathscr{B}_m$, $S\mathscr{B}_m \subseteq \mathscr{B}_m$, *and the restrictions* $S|_{\mathscr{B}_m}$ *are* $\|\cdot\|_m$*-continuous semigroups satisfying*

$$\|S_t a\|_m \leqq \|a\|_m \exp\left\{t \sum_{p=1}^{m} \binom{m}{p} l_p\right\}.$$

PROOF. The proof of the generator property for $\bar{K}$ follows the outline given in the introduction and for this one needs modifications of the estimates of Lemmas 2.6 and 2.8.

First one proves that there is an ω_0 such that $K + \omega I$ is $\|\cdot\|_\beta$-dissipative for all $\omega \geqq \omega_0$ for all $\beta \in]0,1]$. The proof is along the lines of Lemma 2.6. It relies on an estimate

$$\sup_{0 \leqq t \leqq 1} \|(\operatorname{ad} S^{(n)}_{\beta t})(K)a\| \leqq \omega_0 \|a\|_\beta, \quad a \in \mathscr{B}_{2n}.$$

To obtain this estimate one begins with the Duhamel identity

$$\begin{aligned} S^{(n)}_{\beta r}(\operatorname{ad} S^{(n)}_{\beta t})(K)a &= \beta(-1)^{n+1}\int_0^t ds \Big\{H^{2n} S^{(n)}_{\beta(s+r)} K S^{(n)}_{t-s} a \\ &\quad - S^{(n)}_{\beta(s+r)} K H^{2n} S^{(n)}_{\beta(t-s)} a\Big\} \\ &= \beta(-1)^{n+1}\lim_{u\to 0}\int_0^t ds\, S^{(n)}_{\beta(s+r)}(\operatorname{ad} D^{2n}_u)(K) S^{(n)}_{\beta(t-s)} a \end{aligned}$$

where $D_u = (I - \sigma_u)/u$ and $a \in \mathscr{B}_\infty$. But this leads to a bound

$$\begin{aligned} \|S^{(n)}_{\beta r}(\operatorname{ad} S^{(n)}_{\beta t})(K)a\| \leqq \beta \sum_{m=1}^{n} \binom{2n}{m} \cdot \limsup_{u\to 0}\int_0^t ds\, \|H^{2n-m} S^{(n)}_{\beta(s+r)}\| \cdot \\ \cdot \|(\operatorname{ad}\sigma_u)^m(K) S^{(n)}_{\beta(t-s)} a\|/u^m. \end{aligned} \tag{3.1}$$

Now the integrals in this estimate can be bounded by combined use of Lemma 2.1 and Condition 2 of the theorem. First one has

$$(3.2)\qquad \int_0^t ds\, \|H^{2n-m} S^{(n)}_{\beta s}\| \cdot \|(\mathrm{ad}\,\sigma_u)^m (K) S^{(n)}_{\beta(t-s)} a\|/u^m$$

$$\leqq M_n^2 e^{\beta\omega_n t} l_m \beta^{-1} \int_0^1 ds\, s^{-1+m/2n}(1-s)^{-m/2n} \|a\|$$

for $m = 1,2,\ldots,2n-1$. In addition one has

$$(3.3)\qquad \int_0^t ds\, \|S^{(n)}_{\beta s}\| \cdot (\mathrm{ad}\,\sigma_u)^{2n} (K) S^{(n)}_{\beta(t-s)} a\|/u^{2n}$$

$$\leqq M_n^2 e^{\beta\omega_n t} \beta^{-1} \int_0^1 ds (\delta(1-s)^{-1} + l_{2n}\delta^{-1}(1-s)^{-1+1/2n}) \|a\|$$

$$\leqq M_n^2 e^{\beta\omega_n t}(1 + l_{2n}) \beta^{-1} \int_0^1 ds (1-s)^{-1+1/4n} \|a\|$$

where we have chosen $\delta = (1-s)^{1/4n}$. But the integrals in these last bounds are all convergent, and $\|a\| \leqq \|a\|_\beta$. Hence the desired bound follows for all $a \in \mathscr{B}_\infty$ by combining (3.1)-(3.3) and taking the limit $r \to 0$. The bound then extends to all $a \in \mathscr{B}_{2n}$ by continuity.

Next, if $H_\beta = \beta(-H^2)^n + K$ and $r_\beta(\varepsilon) = (I + \varepsilon H_\beta)^{-1}$ one has bounds $\|r_\beta(\varepsilon)\| \leqq M(1-\varepsilon\omega)^{-1}$ analogous to those of Corollary 2.7. These bounds are valid for all small $\varepsilon > 0$ and are uniform for $\beta \in]0,1]$. Using these bounds one can thus obtain uniform estimates on the norms $\|r_\beta(\varepsilon) a\|_m$ for $a \in \mathscr{B}_m$ and $m = 1,2,\ldots,2n$. First one has the identity

$$D_t^m r_\beta(\varepsilon) a = r_\beta(\varepsilon) D_t^m a + \sum_{p=1}^m (-1)^{p+1} \binom{m}{p} \varepsilon r_\beta(\varepsilon) \cdot$$

$$\cdot (\mathrm{ad}\,\sigma_t)^p (K) D_t^{m-p} r_\beta(\varepsilon) a/t^p$$

and using Condition 2 of the theorem one immediately obtains

$$\|H^m r_\beta(\varepsilon) a\| \leqq \|r_\beta(\varepsilon)\| \{\|a\|_m + \varepsilon\gamma_m \|r_\beta(\varepsilon) a\|_m\}$$

for $m = 1,2,\ldots,2n$ where

$$\gamma_m = \sum_{p=1}^m \binom{m}{p} l_p.$$

Solving these inequalities gives the desired bounds, e.g. one has

$$\|r_\beta(\varepsilon) a\|_1 \leqq M(1-\varepsilon\omega)^{-1} \|a\|_1 + \varepsilon\rho_1 \|r_\beta(\varepsilon) a\|_1$$

where $\rho_1 = \gamma_1 M(1-\varepsilon\omega)^{-1}$ and consequently

$$\|r_{\alpha,\beta}(\varepsilon) a\|_1 \leqq M(1-\varepsilon\omega)^{-1}(1-\varepsilon\gamma_1)^{-1} \|a\|_1$$

if $\varepsilon\gamma_1 < 1$.

The generator property of $\bar{K}$ now follows from the uniform resolvent bounds as before. But the proof of the second statement of the theorem has, however, a new element. In order to apply the arguments of [BaR] to obtain the smoothing properties $S\mathscr{B} \subseteqq \mathscr{B}_m, m = 1,2,\ldots,2n-1$ one must first prove that $K\mathscr{B}_\infty \subseteqq \mathscr{B}_{2n-1}$. But this follows from the bounds on the multiple commutators because of the following observation

LEMMA 3.2. *Let σ be a C_0-semigroup with generator H. The following conditions are equivalent for each $n \geqq 1$.*

1. $\sup\{\|(I-\sigma_t)^{m+1}a\|/t^{m+1}; 0 < t < 1, 1 \leqq m \leqq n\} < \infty$,
2. $a \in \mathscr{B}_n$ *and*

   $\sup\{\|(I-\sigma_t)H^n a\|/t; 0 < t < 1\} < \infty.$

This result was proved for $n = 1$ in [BaR] and the general case can be deduced from the special case by the arguments of [BaR]. In fact an even stronger result is true; the conditions of the lemma are equivalent to the condition

$$\sup_{0<t<1} \|(I-\sigma_t)^{n+1} a\|/t^{n+1} < \infty.$$

An elementary proof is given in [Bur] but the result can also be deduced from the general theory of Lipschitz spaces associated with a semigroup (see, for example, [BuB] Theorem 3.4.10).

Now let us return to the proof of Theorem 3.1. If $a \in \mathscr{B}_\infty$ and $D_t = (I-\sigma_t)/t$ then

$$D_t^m Ka = K\,D_t^m a + \sum_{p=1}^{m} \binom{m}{p} (-1)^p (\mathrm{ad}\,\sigma_t)^p (K)\, D_t^{m-p} a/t^p$$

and hence by Conditions 1 and 2 of the theorem

$$\|D_t^m Ka\| \leqq (1+k_1)\,\|a\|_{2n+m} + \sum_{p=1}^{m} \binom{m}{p} l_p\, \|a\|_m$$

for $m = 1,2,\ldots,2n$. Therefore $Ka \in \mathscr{B}_{2n-1}$ by Lemma 3.2. But once this is established the smoothing properties of the semigroup S are proved by a slight elaboration of the arguments of [BaR] using $K\mathscr{B}_\infty \subseteqq \mathscr{B}_{2n-1}$. First one proves that $K + \omega I$ is $\|\cdot\|_m$-dissipative for

$$\omega \geqq \sum_{p=1}^{m} \binom{m}{p} l_p$$

and then one argues that $(I + \varepsilon(K+\omega I))\mathscr{B}_\infty$ is $\|\cdot\|_m$-dense in $\mathscr{B}_m$, for $m = 1,2,\ldots,2n-1$. We will not give further details.

Note that if S^β denotes the semigroup generated by $H_\beta = \beta(-H^2)^n + K$ then it follows from the assumptions of the theorem that S^β converges strongly to the semigroup S generated by $\bar{K}$ as $\beta \to 0$. This is again established from the resolvent estimates as in [BaR].

Moreover, if $K\mathscr{B}_\infty \subseteq \mathscr{B}_{2n}$ then one also has $S\mathscr{B}_{2n} \subseteq \mathscr{B}_{2n}$ and $S|_{\mathscr{B}_{2n}}$ is $\|\cdot\|_{2n}$-continuous.

4. Holomorphic Semigroups

In this section we prove versions of Theorems 1.1 and 3.1 in which the group σ is replaced by a semigroup that is holomorphic in a suitably large sector.

Throughout this section σ will denote a holomorphic semigroup with generator H and sector of holomorphy $\sum(\theta) = \{z \in \mathbb{C};\ |\arg z| < \theta\}$ where $\theta \in]0, \pi/2]$. Now σ is at most of exponential growth in each closed subsector $\sum(\phi)$, where $\phi \in]0, \theta[$, and hence multiplication by a suitable exponential factor reduces it to a semigroup which is uniformly bounded in the closed subsectors of $\sum(\theta)$. Since properties of σ and those of the modified semigroup are related in an obvious way we will restrict our attention to uniformly bounded semigroups.

The first step in the proof of commutator theorems related to holomorphic semigroups is the derivation of an analogue of Lemma 2.1. In particular it is necessary to examine semigroups generated by powers of H.

Lemma 4.1. *Let σ be a uniformly bounded holomorphic semigroup with generator H and holomorphy sector $\sum(\theta)$. Assume that $\theta > (1 - 1/n)\pi/2$ for some positive integer n. It follows that H^n generates a uniformly bounded holomorphic semigroup with holomorphy sector $\sum(n\theta - (n - 1)\pi/2)$.*

This result was proved by Goldstein [Gol] in the special case $n = 2$. The general case was subsequently established by de Laubenfels [Lau]. Our proof is a simple extension of Goldstein's argument. Note that we use a different sign convention to both these authors, i.e. in our notation $\sigma_t = \exp\{-tH\}$ etc.

Proof. It follows from Hille's theorem that a closed densely defined operator H generates a bounded holomorphic semigroup σ, with holomorphy sector $\sum(\theta)$, if, and only if,

$$R(I + \varepsilon H) = \mathscr{B}$$

and

$$\|(I + \varepsilon H)a\| \geqq m_\phi \|a\|, \tag{4.1}$$

for some $m_\phi > 0$ and all ε, ϕ with $\phi \in]0, \theta[$ and $|\arg \varepsilon| \leqq \phi + \pi/2$. In fact one then has $\|\sigma_z\| \leqq M_\phi \leqq m_\phi^{-1}$ for all $z \in \sum(\phi)$.

Now if $a \in D(H^n)$ then

$$(4.2) \qquad (I + \varepsilon H^n)a = \prod_{m=1}^{n} (I + \varepsilon_m H)a$$

where $|\varepsilon_m| = |\varepsilon|^{1/n}$ and $\arg \varepsilon_m = ((n + 1 - 2m)\pi + \arg \varepsilon)/n$. Thus $|\arg \varepsilon_m| \leqq (1 - 1/n)\pi + |\arg \varepsilon|/n$ and if $|\arg \varepsilon| < n\theta - (n - 1)\pi/2 + \pi/2$ with $\theta > (n - 1)\pi/2n$ then $|\arg \varepsilon_m| < \theta + \pi/2$ for each $m = 1, 2, \ldots, n$. Therefore

$$\|(I + \varepsilon H^n)a\| \geqq m_\phi^n \|a\|, \quad a \in D(H^n),$$

by iteration of (4.1) and $R(I + \varepsilon H^n) = \mathscr{B}$ because

$$(I + \varepsilon H^n) \prod_{m=2}^{n} (I + \varepsilon_m H)^{-1} a = (I + \varepsilon_1 H)a, \quad a \in D(H).$$

The statements of the lemma then follows from Hille's criterion.

Next we have the analogue of Lemma 2.1.

LEMMA 4.2. *Let σ be a uniformly bounded holomorphic semigroup with generator H and holomorphy sector $\sum(\theta)$ where $\theta > (1 - 1/n)\pi/2$. Further let $S^{(n)}$ denote the holomorphic semigroup generated by H^n. Then there is a $M_n \geqq 1$ such that*

$$(4.3) \qquad \|H^m S_t^{(n)}\| \leqq M_n t^{-m/n}$$

for all $t > 0$ and $m = 1, 2, \ldots, n$.

PROOF. Since $S^{(n)}$ is a uniformly bounded holomorphic semigroup there is a $C_n > 0$ such that $\|S_t^{(n)}\| \leqq C_n$ and $\|H^n S_t^{(n)}\| \leqq C_n t^{-1}$ for all $t > 0$.

Next it follows from the factorization formula (4.2) that

$$(I + tH^n) = \prod_{m=1}^{n} (I + t_m H)a, \quad a \in D(H^n)$$

where $t_m = t^{1/n} \exp\{i(n + 1 - 2m)\pi/n\}$. Thus $|\arg t_m| \leqq (n - 1)\pi/n$ and choosing ϕ such that $\theta > \phi > (1 - 1/n)\pi/2$ one has $|\arg t_m| < \phi + \pi/2$ for each $m = 1, 2, \ldots, n$. Therefore using (4.1) one obtains the estimates

$$(4.4) \qquad \|(I + tH^n)a\| \geqq m_\phi^{n-1} \|(I + t_j H)a\|, \quad a \in D(H^n).$$

Consequently

$$\begin{aligned} t^{1/n} \|Ha\| &\leqq \|(I + t_1 H)a\| + \|a\| \\ &\leqq M_\phi^{n-1} \|(I + tH^n)a\| + \|a\| \\ &\leqq M_\phi^{n-1} t \|H^n a\| + (1 + M_\phi^{n-1}) \|a\| \end{aligned}$$

where $M_\phi = m_\phi^{-1}$. Replacing a by $S_t^{(n)} b$ one then has

$$\begin{aligned} t^{1/n} \|HS_t^{(n)} b\| &\leqq M_\phi^{n-1} t \|H^n S_t^{(n)} b\| + (1 + M_\phi^{n-1}) \|S_t^{(n)} b\| \\ &\leqq C_n(1 + 2M_\phi^{n-1}) \|b\| \end{aligned}$$

for all $b \in \mathscr{B}$ because $S_t^{(n)} \mathscr{B} \subseteqq D(H^n)$ for $t > 0$. This establishes (4.3) for $m = 1$ with M_1 replaced by $N_n = C_n(1 + 2M_\phi^{n-1})$. But then

$$\|H^m S_t^{(n)}\| \leqq \|HS_{t/m}^{(n)}\|^m \leqq N_n^m (m/t)^{m/n}$$

for $m \geqq 1$. Thus (4.3) is valid for $m = 1, 2, \ldots, n$ with $M_n = n N_n^n$.

After these preliminaries we can now formulate the third commutator theorem.

THEOREM 4.3. *Let σ be a uniformly bounded holomorphic semigroup on the Banach space $\mathscr{B}$ with generator H and holomorphy sector $\sum(\theta)$. Further let K be a dissipative operator from the Banach subspace $\mathscr{B}_n$ into $\mathscr{B}$, where $1 \leqq n \leqq \pi/2(\pi/2 - \theta)^{-1}$, and $\mathscr{B}_m = D(H^m)$ with $\|a\|_m = \sup_{0 \leqq p \leqq m} \|H^p a\|$.*

Assume

1. *for each $\varepsilon \in]0,1]$ there is a $k_\varepsilon > 0$ such that*

$$\|Ka\| \leqq \varepsilon \|a\|_n + k_\varepsilon \|a\|_{n-1}, \quad a \in \mathscr{B}_n,$$

2. *there exist $l_m \geqq 0$ such that*

$$\|(\operatorname{ad} \sigma_t)^m (K) a\| \leqq l_m \|a\|_m t^m, \quad a \in \mathscr{B}_n,$$

for $t \in]0, 1]$ and $m = 1, 2, \ldots, n - 1$, and

$$\|(\operatorname{ad} \sigma_t)^n (K) a\| \leqq (\delta \|a\|_n + l_n \delta^{-1} \|a\|_{n-1}) t^n, \quad a \in \mathscr{B}_n,$$

for $t, \delta \in]0, 1]$.

It follows that the closure $\bar{K}$ of K is the generator of a C_0-semigroup of contractions S. Moreover for each $m = 1, 2, \ldots, n$ one has $K\mathscr{B}_m, S\mathscr{B}_m \subseteqq \mathscr{B}_m$ and the restrictions $S|_{\mathscr{B}_m}$ are C_0-semigroups with

$$\|S_t a\|_m \leqq \|a\|_m \exp\left\{t \sum_{p=1}^{m} \binom{m}{p} l_p\right\}.$$

PROOF. First, it follows from Lemma 2.2, which is valid for the generator of a C_0-semigroup, that Condition 1 implies K is relatively bounded by H^n with relative bound zero. But H^n generates a holomorphic semigroup $S^{(n)}$, by Lemma 4.1, and hence the operators

$$H_\beta = \beta H^n + \bar{K}, \quad \beta \in]0,1]$$

also generate holomorphic semigroups by perturbation theory.

Second, if $r_\beta(\varepsilon) = (1 + \varepsilon H_\beta)^{-1}$ one has bounds $\|r_\beta(\varepsilon)\| \leqq M(1 - \varepsilon\omega)^{-1}$ analogous to those of Corollary 2.7. These bounds are again valid for all small $\varepsilon > 0$ and are uniform for $\beta \in]0, 1]$. The bounds are obtained by the arguments of Lemmas 2.4 and 2.6 but now $S^{(n)}$ is the semigroup generated by H^n. Thus one uses $S^{(n)}$ to define the equivalent norms $\|\cdot\|_\beta$ as in Lemma 2.4 and hence deduce that there is a $\gamma > 0$ and independent of β such that $\beta H^n + \gamma I$ is $\|\cdot\|_\beta$-dissipative. Then one uses the commutator bounds of the theorem to argue that there is an $\omega_0 > 0$ independent of β such that $K + \omega I$ is $\|\cdot\|_\beta$-dissipative for all $\omega \geqq \omega_0$. This argument follows the lines of the proof of Lemma 2.6 but modified as in the proof of Theorem 3.1. Then S^β denotes the semigroup generated by H_β one concludes that

$$\|S_t^\beta a\| \leqq \|S_t^\beta a\|_\beta \leqq e^{\omega t} \|a\|_\beta \leqq M_n e^{\omega t} \|a\|$$

where $\omega = \gamma + \omega_0$ and $M_n = \sup_{t>0} \|S_t^{(n)}\|$. The resolvent bounds follow by Laplace transformation.

Third, one proves that for small enough $\varepsilon > 0$ one has bounds

$$\|r_\beta(\varepsilon) a\|_m \leqq c_m \|a\|_m, \quad a \in \mathscr{B}_m$$

for $m = 1, 2, \ldots, n$ and these bounds are uniform in β. These bounds are obtained by the integral inequality method used to prove Lemma 2.8 but using the multi-commutator conditions as in the proof of Theorem 3.1.

The generator property of $\bar{K}$ then follows by use of the resolvent bounds exactly as in the proof of Theorem 1.1 and the smoothing properties of S are proved as in [BaR] with the extra argument outlined in the proof of Theorem 3.1.

Note that if $n = 1$ in Theorem 4.3 then there is no restriction on the angle of holomorphy θ but if $n > 1$ then there is a restriction. But if σ is holomorphic in the open right half plane, i.e. in $\sum(\pi/2)$ then all values of n are allowed.

If $n = 1$ then the semigroups S generated by $\bar{K}$ has no apparent smoothing properties unless one assumes that $K\mathscr{B}_\infty \subseteqq \mathscr{B}_1$ in which case $S\mathscr{B}_1 \subseteqq \mathscr{B}_1$ as in Theorem 1.1. In fact the $n = 1$ case is a single commutator theorem comparable to Theorem 1.1.

COROLLARY 4.4. *Let σ be a holomorphic semigroup with generator H and K a dissipative operator from $D(H)$ into $\mathscr{B}$. Assume*

1. *for each $\varepsilon \in]0, 1]$ there is a $k_\varepsilon > 0$ such that*

$$\|Ka\| \leqq \varepsilon \|Ha\| + k_\varepsilon \|a\|, \quad a \in D(H),$$

2. *there is an $l_1 \geqq 0$ such that*

$$\|(\operatorname{ad} \sigma_t)(K)a\| \leqq (\delta \|Ha\| + l_1 \delta^{-1} \|a\|)\, t, \quad a \in D(H),$$

for all $t, \delta \in]0,1]$.

It follows that $\bar{K}$ generates a C_0-semigroup of contractions S and if $K\mathscr{B}_\infty \subseteqq \mathscr{B}_1$ then $SD(H) \subseteqq D(H)$ and $\|H(S_t - I)a\| \to 0$ as $t \to 0$ for each $a \in D(H)$.

Note that a stonger version of this corollary can be proved [Rob2] if σ is a contraction semigroup, which is not necessarily holomorphic, and K is relatively bounded by H with relative bound zero. Then the commutator bound

$$\|(\mathrm{ad}\,\sigma_t)(K)\,a\| \leqq l_1\,\|a\|_1, \quad t \in]0,1[, a \in D(H),$$

suffices to prove that K is a generator.

Finally remark that if the semigroup σ in Theorem 4.3 is contractive then H is dissipative and satisfies estimates

$$\|Ha\| \leqq \varepsilon\|a\|_n + k_\varepsilon\,\|a\|_{n-1}, \quad a \in \mathscr{B}_n, \varepsilon \in]0,1[,$$

for all $n \geqq 2$. Therefore if $n \geqq 2$ Conditions 1 and 2 of the theorem are unchanged if one replaces K with $\lambda H + K$ where $\lambda \geqq 0$. Hence one reaches the following conclusion.

Corollary 4.5. *Adopt the hypotheses of Theorem* 4.3 *but further assume σ is contractive and $n \geqq 2$. Then for each $\lambda \geqq 0$ the closure of $\lambda H + K$ generates a C_0-semigroup of contraction S and for each $m = 1,2,\ldots,n-1$ one has $(\lambda H + K)\mathscr{B}_\infty \subseteqq \mathscr{B}_m$, $S\mathscr{B}_m \subseteqq \mathscr{B}_m$, and $S|_{\mathscr{B}_m}$ is a $\|\cdot\|_m$-continuous semigroup satisfying the bounds* (4.6).

5. Concluding Remarks

Commutator estimates play an important role in Nelson's theory [Nel] of analytic elements associated with representations of Lie groups. This theory gives commutator criteria for operators. As an illustration we mention the result of Goodman and Jørgensen, [GoJ] Theorem 2.1. This theorem has the corollary that if σ is a C_0-group with generator H and if $K; \mathscr{B}_\infty \mapsto \mathscr{B}$ satisfies the conditions

$$\|Ka\| \leqq l\|a\|_1, \tag{5.1}$$

$$\|(\mathrm{ad}\,\sigma_t)^m(K)\,a\| \leqq l\,k^m\,\|a\|_1\,|t|^m, \quad |t| \leqq 1, a \in \mathscr{B}_\infty, \tag{5.2}$$

for some $k, l \geqq 0$ and all $m = 1,2,\ldots$ then each analytic element of H is analytic for K. Thus if in addition K is dissipative then its closure is a generator. This type of result differs from the foregoing commutator theorem in several respects. In particular it only applies to operators K which are relatively bounded by H and this restriction appears difficult to relax. For example, the theorem cannot be expected to apply to $K = -H^2$, because H has analytic elements which are not analytic for H^2, but the commutator conditions are obviously satisfied.

It should also be emphasized that the C_1-norm $\|\cdot\|_1$ is essential in the commutator bound (1.1). Example 1 in Section X.5 of [ReS] demonstrates that this bound cannot be weakened without the validity of Theorem 1.1. Similarly Theorems 3.1 and 4.3 are sensitive to any change of order of the multicommutator bounds. There is, however, a weak relationship between these orders. For example, if one has bounds

$$\|Ka\| \leqq k_0\,\|a\|_{n_0}, \quad a \in \mathscr{B}_\infty,$$

and in addition

$$\|(\mathrm{ad}\,\sigma_t)^m(K)a\| \leqq k_m\,\|a\|_{n_m}|t|^m, \quad |t| \leqq 1, a \in \mathscr{B}_\infty,$$

for some fixed $m \geqq 1$ then it follows that

$$\|(\mathrm{ad}\,\sigma_t)^r(K)a\| \leqq k_r\,\|a\|_{n_r}|t|^r, \quad |t| \leqq 1, a \in \mathscr{B}_\infty,$$

for all $r = 1, 2, \ldots, m-1$ where $n_r = n_0 \vee n_m$ [Bur]. Hence if (5.1) is valid and (5.2) is valid for large m then (5.2) is valid for all m.

ACKNOWLEDGEMENTS. This paper arose as a direct result of an earlier collaboration with Charles Batty, and during the initial stages of its development I benefited from numerous discussions with Ola Bratteli and Trond Digernes.

REFERENCES

[BaR] Batty, C. J. K. and D. W. Robinson, *Commutators and generators*, Math. Scand. 62 (1988), 303–326.

[BuB] Butzer, P. L. and H. Berens, *Semigroups of Operators and Approximation*, Springer-Verlag, 1967.

[Bur] Burns, R. J.; *Commutators and Generators*, M. Sc. Thesis, Australian National University, 1987.

[GoJ] Goodman, F. M. and P. E. T. Jørgensen, *Lie algebras of unbounded derivations*, *J. Funct. Anal.* 52 (1983), 369–384.

[Gol] Goldstein, J. A., *Some remarks on infinitesimal generators of analytic semigroups*, Proc. Amer. Soc. 22 (1969), 91–93.

[Lau] de Laubenfels, R., *Powers of generators of holomorphic semigroups*, Proc. Amer. Math. Soc. 99 (1987), 105–108.

[Nel] Nelson, E., *Analytic vectors*, Ann. of Math. 70 (1959), 572–615.

[Paz] Pazy, A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, 1987.

[ReS] Reed, M., and B. Simon, *Methods of Modern Mathematical Physics II*, Academic Press, 1975.

[Rob1] Robinson, D. W., *Commutator theory on Hilbert space*, Can. J. Math. XXXIX (1987), 1235–1280.

[Rob2] Robinson, D. W., *Differential and integral structure of continuous representation of Lie groups*, J. Op. Theor. 19 (1988), 95–128.

DEPARTMENT OF MATHEMATICS
AUSTRALIAN NATIONAL UNIVERSITY
CANBERRA, ACT 2601,
AUSTRALIA