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TWISTED GROUP C*-ALGEBRAS CORRESPONDING TO NILPOTENT DISCRETE GROUPS

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Introduction.

Operator algebras formed from locally compact groups twisted by multipliers have been the object of study for several decades, much of this study stemming from the work of Mackey on unitary representations of group extensions [14]. The purpose of this paper is to present some simple techniques for examining some of the most basic of these operator algebras, those generated by (countable) discrete groups and their multipliers. If Γ is a discrete group and $\sigma\colon \Gamma \times \Gamma \to \mathsf{T}$ is a multiplier one can form the twisted group algebra $L^1(\Gamma, \sigma)$, an involutive Banach algebra, whose "left σ-regular" representation on $L^2(\Gamma)$ generates the von Neumann algebra $W^*(\Gamma, \sigma)$, first studied by Kleppner [13], then more extensively by Zeller-Meier [25]. The enveloping C*-algebra for $L^1(\Gamma, \sigma)$, $C^*(\Gamma, \sigma)$, first studied by Auslander and Moore [1] and also in [25], is currently of greater interest, and even in the simplest case, when $\Gamma = \mathsf{Z}^n$, has provided a wealth of examples on which one can fruitfully use non-commutative algebraic topology and differential geometry [3], [6], [21].

In this paper, we first review some known facts about the von Neumann algebra $W^*(\Gamma, \sigma)$, stating necessary and sufficient conditions (due to Kleppner [13]) which involve straightforward counting arguments, for $W^*(\Gamma, \sigma)$ to be a factor. Using this result in the case where Γ is nilpotent, one is able to determine very quickly, by applying a result of Poguntke [19], necessary and sufficient conditions that C*-algebra $C^*(\Gamma, \sigma)$ be simple. By modifying methods of Slawny [22] for countable abelian groups and of Howe [12] and Carey and Moran [2] on characters of nilpotent groups, these same conditions also can be used to tell us when the C*-algebra $C^*(\Gamma, \sigma)$ has a unique normalized trace:

THEOREM 1.7. *Let Γ be a countable nilpotent discrete group, with multiplier σ: $\Gamma \times \Gamma \to \mathsf{T}$, then $C^*(\Gamma, \sigma)$ is simple and has a unique normalized trace if and only if every non-trivial σ-regular conjugacy class of Γ is infinite in cardinality.*

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Here "σ-regular" means that the symmetrizer $\tilde{\sigma}$ of σ [10], [15], which has been shown to define a map from $x \in \Gamma$ to the characters of the centralizer of x in Γ, $\hat{C}_x$, is identically 1 on one (hence any) x in the conjugacy class in question. The idea of the proof of the theorem is to show that $C^*(\Gamma, \sigma)$ is a primitive quotient of $C^*(\Gamma')$ where Γ' is a countable central extension of Γ, and apply Poguntke's theorem on the maximality of primitive ideals for Γ'. The proof of uniqueness of the trace is not so immediate, but can be obtained by a combination of techniques used in the papers [12], [2], and [22] mentioned above.

Some twisted operator algebras arise naturally in the study of dynamical systems coming from the affine action of a (countable) discrete group G on a compact (s.c.) abelian group X; the discrete group in question will be a semi-direct product $\hat{X} \rtimes G$ ((special cases of this observation have been noted as far back as the 1950's in [4] and [20] and subsequently in [8], among many other references). In the second section of our paper we place some results involving these particular operator algebras in a more integrated framework and reprove some old results of Hahn [9] and Wieting [24] concerning ergodicity of discrete abelian groups acting affinely on compact abelian groups, by means of studying the von Neumann algebras involved, obtaining:

COROLLARY 2.3 ([9], [24]). *Let $(X, G)_{(B,\rho)}$ be an affine action of the countable discrete abelian group G on the compact (s.c.) abelian group X, where B: $G \to \operatorname{Aut} X$ is a homomorphism and ρ: $G \to X$ is a B-crossed homomorphism. Then the action of G on X is ergodic with respect to Haar measure if and only if for every $\eta \in \hat{X} \setminus \{e\}$ such that $\{B^*(g)(\eta) \mid g \in G\}$ is finite, there exists $g_1 \in G$ with $B^*(g_1)(\eta) = \eta$ and $\eta(\rho(g_1)) \neq 1$.*

When the discrete group $\hat{X} \times G$ associated to the affine action $(X, G)_{(B,\rho)}$ is nilpotent, the methods used to prove Theorem 1.7 allow us to prove that an ergodic affine action $(X, G)_{(B,\rho)}$ is actually minimal and uniquely ergodic; we then use this result in the situation where $(X, G) = (\mathrm{T}^n, \mathrm{Z})$ to reprove another result of F. Hahn.

Although a few of the results in this paper may be known to other workers in the field, we feel that the approach we use in the study of these results and in relating them to the Howe-Poguntke-Carey + Moran theory are new, and provide easy methods of determining whether or not certain group C*-algebras associated to nilpotent discrete groups are simple. Also, Corollary 2.3 and the results on minimal affine actions provide examples where operator algebras can be used to determine ergodicity and minimality of some dynamical systems; heretofore the roles of operator algebras and dynamical systems have been for the most part reversed in this exchange of information.

The material on von Neumann algebras in this paper first appeared in the author's docoral dissertation undertaken at Harvard University under the direction of Professor George Mackey, and to a great extent the results can be regarded as his inspirations. We sincerely thank him for his constant help and guidance in this and other areas of mathematics. We are also grateful to Iain Raeburn for many useful conservations on this topic, and for pointing out to us the reference [10]. Finally, we thank the referee for useful suggestions, and Jonathan Rosenberg for suggesting to us the references [2] and [19].

1. The operator algebras $W^*(\Gamma, \sigma)$ and $C^*(\Gamma, \sigma)$

Let Γ be a (countable) discrete group, and let $\sigma\colon \Gamma \times \Gamma \to \mathrm{T}$ be a multiplier, alternatively called a two-cocycle for Γ with values in T, $\sigma \in Z^2(\Gamma, \mathrm{T})$. Recall from [1] and [25] that the involutive Banach algebra $L^1(\Gamma, \sigma)$ is formed as follows: for $f, g \in L^1(\Gamma)$ set

$$f \circ g(\gamma) = \sum_{\gamma_1 \in \Gamma} \sigma(\gamma_1, \gamma_1^{-1}\gamma) f(\gamma_1) g(\gamma_1^{-1}\gamma)$$

and

$$f^*(\gamma) = \overline{\sigma(\gamma, \gamma^{-1})} f(\gamma^{-1})$$

The isomorphism class of $L^1(\Gamma, \sigma)$ depends only on the cohomology class of σ in $H^2(\Gamma, \mathrm{T})$. We can represent $L^1(\Gamma, \sigma)$ on $L^2(\Gamma)$ as follows:

$$T_f g(\gamma) = \sum_{\gamma_1 \in \Gamma} \sigma(\gamma_1, \gamma_1^{-1}\gamma) f(\gamma_1) g(\gamma_1^{-1}\gamma)$$

for $f \in L^1(\Gamma)$, $g \in L^2(\Gamma)$. We will denote the weak closure of $T(L^1(\Gamma, \sigma))$ in $B(L^2(\Gamma))$ by $W^*(\Gamma, \sigma)$; $W^*(\Gamma, \sigma)$ is called the von Neumann algebra generated by the *left σ-regular representation of Γ*. Thus $W^*(\Gamma, \sigma)$ is generated by operators of the form

$$\{U_\gamma \mid \gamma \in \Gamma\}$$

where

$$U_{\gamma_1} f(\gamma) = \sigma(\gamma_1, \gamma_1^{-1}\gamma) f(\gamma_1^{-1}\gamma).$$

We note that $U_{\gamma_1} U_{\gamma_2} = \sigma(\gamma_1, \gamma_2) U_{\gamma_1 \gamma_2}$, so that we are following the terminology of [25] rather than [14] (where what we have would be termed a σ^{-1} representation). The representation T of $L^1(\Gamma, \sigma)$ given above is unitarily equivalent to the one given in [25], and is more convenient for our purposes.

To state conditions that $W^*(\Gamma, \sigma)$ be a factor we recall some notation, mainly due to Kleppner [13] and Mackey [15]:

DEFINITION 1.1. For any $x \in \Gamma$ and multiplier $\sigma\colon \Gamma \times \Gamma \to \mathrm{T}$, let $\chi^{\sigma,x}\colon \Gamma \to \mathrm{T}$ be the function defined by

$$\chi^{\sigma,x}(y) = \sigma(x, y)\overline{\sigma(y, y^{-1}xy)}$$

The conjugacy class of x in Γ is termed *σ-regular* if $\chi^{\sigma,x}$ restricted to the centralizer of x in Γ, C_x, is identically 1. (Kleppner in [13, Lemma 3] and Mackey in the more general framework of [15, Theorem 6.1] show that whether or not $\chi^{\sigma,x} \equiv 1$ on C_x depends only on the conjugacy class of x in Γ).

REMARK 1.2. In [15, Lemma 5.2] Mackey shows that $\chi^{\sigma,x}$ is a character when restricted to C_x, and proves general identities involving elements of $Z^2(\Gamma, \mathscr{X})$ where $\mathscr{X}$ is an arbitrary l.c.s.c. abelian group. The function $\chi^{\sigma,x}$ is exactly the *symmetrizer* $\tilde{\sigma}$ discussed by Hannabuss in [10], and Proposition 1.2 there can be deduced from [15].

The following proposition, which is implicit in a theorem of Kleppner, was pointed out to us in the form below by G. Mackey and gives a straightforward method for determining whether or not $W^*(\Gamma, \sigma)$ is a factor:

PROPOSITION 1.3 [13, *Theorem* 2]. *Let σ be a multiplier for the countable discrete group Γ. Then the von Neumann algebra $W^*(\Gamma, \sigma)$ generated by the left σ-regular representation of Γ is a factor if and only if each non-trivial σ-regular conjugacy class in Γ is infinite in cardinality.*

The proof involves giving $W^*(\Gamma, \sigma)$ the structure of a Hilbert algebra with basis $\{U_\gamma \mid \gamma \in \Gamma\}$ and calculating what operators can appear in the center.

We now use the above results on twisted group von Neumann algebras to study the twisted group C*-algebras $C^*(\Gamma, \sigma)$, which are defined to be the enveloping C^*-algebras of the $L^1(\Gamma, \sigma)$. The reduced C*-algebra $C^*_{\text{red}}(\Gamma, \sigma)$ is the quotient of $C^*(\Gamma, \sigma)$ obtained by completing $L^1(\Gamma, \sigma)$ with respect to the norm given by the left σ-regular representation, so that $C^*_{\text{red}}(\Gamma, \sigma)$ may be viewed concretely as the C^*-algebra generated by the unitary elements $\{U_\gamma \mid \gamma \in \Gamma\}$ acting on $L^2(\Gamma)$. As such we may regard $C^*_{\text{red}}(\Gamma, \sigma)$ as a weakly dense $*$-subalgebra of $W^*(\Gamma, \sigma)$. If Γ is amenable, then $C^*(\Gamma, \sigma) \cong C^*_{\text{red}}(\Gamma, \sigma)$ [25, Section 5] so that in this case we may study $C^*(\Gamma, \sigma)$ in terms of its concrete representation on $L^2(\Gamma)$. It will be fruitful to consider $C^*(\Gamma, \sigma)$ as the quotient of some $C^*(\Gamma')$ for a central extension Γ' of Γ. Let

$$D = \left\{ \prod_{k=1}^{n} (\sigma(\gamma_{i_k}, \gamma_{j_k}))^{e_k} \mid \gamma_{i_k}, \gamma_{j_k} \in \Gamma,\ n \in \mathbf{N},\ e_k \in \{-1, 1\} \right\}.$$

Then D is a subgroup of the circle group $\mathbf{T}$ and since Γ is countable D will also be countable, though not necessarily closed in $\mathbf{T}$. Let $\Gamma' = \Gamma \times D$; then Γ' can be given the structure of a central extension of Γ by setting $(\gamma_1, d_1)(\gamma_2, d_2) = (\gamma_1\gamma_2, \sigma(\gamma_1, \gamma_2) d_1 d_2)$. Evidently Γ' is countable, and the injection $i: D \hookrightarrow \mathbf{T}$ may be viewed as a character $i \in \hat{D}$.

We consider the following unitary representation of Γ' on $\mathscr{H} = L^2(\Gamma)$:

$$W_{(\gamma_1, d_1)} f(\gamma) = \sigma(\gamma_1, \gamma_1^{-1}\gamma)\, i(d_1) f(\gamma_1^{-1}\gamma) \quad \gamma_1, \gamma \in \Gamma,\ d_1 \in D.$$

By inspection we see that $W_{(\gamma_1, d_1)} = i(d_1) U_{\gamma_1}$, so that the representation W of Γ' extends to give a homomorphism φ_W: $C^*(\Gamma') \to C^*(\Gamma, \sigma)$; in essence $C^*(\Gamma, \sigma)$ is a "twisted covariance algebra" of the form $C^*(\Gamma', \mathbf{C}, \tau_D)$ as described in [7]. (See [18] for a further exploration of this point of view). Then from Proposition 1.3 we immediately obtain the following

PROPOSITION 1.4. *Let Γ be a countable amenable group with multiplier σ: $\Gamma \times \Gamma \to \mathbf{T}$. Then $C^*(\Gamma, \sigma)$ is primitive if and only if every non-trivial σ-regular conjugacy class in Γ is infinite in cardinality.*

PROOF. If Γ contains a non-trivial finite σ-regular conjugacy class, then as in the proof of Proposition 1.3, we can show that the center of $C^*(\Gamma, \sigma) \cong C^*_{\text{red}}(\Gamma, \sigma) \subset W^*(\Gamma, \sigma)$ contains non-scalar elements so that $C^*(\Gamma, \sigma)$ is not primitive. As for the converse, if every non-trivial σ-regular conjugacy class is infinite, then by Proposition 1.3, $W^*(\Gamma, \sigma)$ is a factor. It follows that the representation W of Γ' described in the above paragraph is factorial, hence the kernel of the map φ_W: $C^*(\Gamma', \sigma) \to C^*(\Gamma, \sigma)$ is a prime ideal $\mathscr{I} \subset C^*(\Gamma')$. But Γ' is countable so that $\mathscr{I}$ will be primitive, and it follows that $C^*(\Gamma, \sigma) \cong C^*(\Gamma')/\mathscr{I}$ is primitive.

When Γ is nilpotent, by using results in Poguntke [29], which generalized results in Howe [12] and Moore-Rosenberg [16], we can say even more:

PROPOSITION 1.5. *Let Γ be a nilpotent discrete group, with multiplier σ: $\Gamma \times \Gamma \to \mathbf{T}$. Then $C^*(\Gamma, \sigma)$ is simple if and only if every non-trivial σ-regular conjugacy class of Γ is infinite in cardinality.*

PROOF. If Γ is nilpotent of degree k, it is easy to see that Γ' is nilpotent of degree $\leqq k + 1$. By Poguntke's result, in [19], every primitive ideal of $C^*(\Gamma')$ is in fact maximal. Thus if every non-trivial σ-regular conjugacy class in Γ is infinite, Proposition 1.4 shows that $C^*(\Gamma, \sigma) \cong C^*(\Gamma')/\mathscr{I}$, where $\mathscr{I}$ is a maximal ideal. Hence $C^*(\Gamma, \sigma)$ is simple. The other direction is clear.

We now consider the class of all finite normalized traces on $C^*(\Gamma, \sigma)$ for countable nilpotent groups Γ. Recall that a trace on $C^*(\Gamma, \sigma)$ may be viewed as a positive linear functional τ: $C^*(\Gamma, \sigma) \to \mathbf{C}$ satisfying $\tau(x^*x) = \tau(xx^*) \, \forall \, x \in C^*(\Gamma, \sigma)$, $\tau(\text{Id}) = 1$. Since $C^*(\Gamma, \sigma)$ is a quotient of $C^*(\Gamma')$, a finite normalized trace τ on $C^*(\Gamma, \sigma)$ lifts to give a trace $\tilde{\tau}$ on $C^*(\Gamma')$ and as such corresponds to a positive definite function ψ: $\Gamma' \to \mathbf{C}$ which is constant on conjugacy classes and satisfies $\psi(e) = 1$ (such a function ψ is also termed a *trace*). Howe showed in [12], Proposition 3, that if N is a finitely generated non-torsion nilpotent group, and $\mathscr{I}$ is a primitive ideal of $C^*(N)$, then $\mathscr{I}$ is maximal and there exists a unique normalized trace τ on $C^*(N)$ which vanishes on $\mathscr{I}$ (hence a unique normalized trace on $C^*(N)/\mathscr{I}$). Carey and Moran in [2] extended Howe's results on traces to the class of all "centrally inductive" nilpotent groups (i.e. those for which each character ψ vanishes on the infinite conjugacy classes of $G/K(\psi)$, where $K(\psi) = \{g \in G \mid \psi(g) = 1\}$), using in their proof deep results of Furstenberg involving unique ergodicity. These results of Carey and Moran allow us to deduce immediately that when Γ is a finitely generated nilpotent discrete group, then under the conditions of Proposition 1.5, $C^*(\Gamma, \sigma)$ is simple and has a unique trace. We wish to show that for *any* countable nilpotent group Γ, under the conditions of Proposition 1.5, $C^*(\Gamma, \sigma)$ is simple and has a unique trace. In this situation it turns out that we can modify relatively straightforward techniques of Carey and Moran [2], Howe [12] and Slawny [22] (not involving the work of Furstenberg) to prove the following lemma (we thank the referee for suggesting to us that our original results could be strengthened and their proofs simplified):

LEMMA 1.6. *Let Γ be a countable nilpotent group and let $\sigma\colon \Gamma \times \Gamma \to \mathrm{T}$ be a multiplier, and suppose that every non-trivial σ-regular conjugacy class of Γ is infinite in cardinality. Then $C^*(\Gamma, \sigma)$ has a unique normalized trace.*

PROOF: Let τ be a normalized trace on $C^*(\Gamma, \sigma)$ and consider without loss of generality the case where τ lifts to a character ψ on the group $\Gamma' = \Gamma \times D$ described above. Note that by Proposition 1.5 $C^*(\Gamma, \sigma) = C^*(\Gamma')/\mathscr{I}$ is simple; we will show any character ψ on $C^*(\Gamma')$ vanishing on $\mathscr{I}$ is unique, i.e. we will show that we must have

$$\psi((\gamma, d)) = \begin{cases} 0 & \gamma \neq e \\ i(d) & \gamma = e, \end{cases}$$

where $i\colon D \hookrightarrow \mathrm{T}$ is the natural injection. Using the notation given prior to the statement of Proposition 1.4, we see that $\varphi_W(U_{(\gamma,d)}) = i(d)\, U_\gamma$ so that $\forall\, (\gamma, d) \in \Gamma'$, $\psi((\gamma, d)) = \tau(i(d) U_\gamma) = i(d)\, \tau(U_\gamma) = i(d) \psi((\gamma, 1))$. Hence $\psi((e, d)) = i(d) \,\forall\, d \in D$. It is clear that since $C^*(\Gamma, \sigma)$ is simple, and ψ is never 1 on $\{e\} \times (D \setminus \{1\})$, ψ must be a fathful character on Γ'. Let $\mathrm{FC}(\Gamma)$ denote the normal subgroup of elements of Γ having finite conjugacy class in Γ. We first show that if $\gamma \in \mathrm{FC}(\Gamma) \setminus \{e\}$, then $\psi((\gamma, d)) = 0 \,\forall\, d \in D$. By hypothesis, if $\gamma \in \mathrm{FC}(\Gamma) \setminus \{e\}$, there exists $g \in C_\gamma$ with $\chi^{\sigma,\gamma}(g) \neq 1$. Then, following a method used by Slawny in the case where Γ is abelian, $\tau(U_g^{-1} U_\gamma U_g) = \tau(\varphi_W(U_{(g,1)^{-1}}) \varphi_W(U_{(\gamma,1)}) \varphi_W(U_{(g,1)})) = \psi((g, 1)^{-1} (\gamma, 1) (g, 1)) = \psi((g^{-1} \gamma g, \sigma(g^{-1}, \gamma g) \overline{\sigma(g, g^{-1})} \sigma(\gamma, g))) = \psi((\gamma, \chi^{\sigma,\gamma}(g)) = \chi^{\sigma,\gamma}(g)\, \psi((\gamma, 1)) = \chi^{\sigma,\gamma}(g)\, \tau(U_\gamma)$. But since τ is a trace on $C^*(\Gamma, \sigma)$ we must also have $\tau(U_g^{-1} U_\gamma U_g) = \tau(U_\gamma)$. Hence $\chi^{\sigma,\gamma}(g)\, \tau(U_\gamma) = \tau(U_\gamma)$ and since $\chi^{\sigma,\gamma}(g) \neq 1$ we must have $\tau(U_\gamma) = 0$. It follows that $\psi((\gamma, d)) = i(d)\, \psi((\gamma, 1)) = 0 \,\forall\, \gamma \in \mathrm{FC}(\Gamma) \setminus \{e\}, \forall\, d \in D$.

We now prove by induction on the length of the upper central series for Γ that $\psi((\gamma, d)) = 0 \,\forall\, \gamma \in \Gamma \setminus \{e\}, \forall\, d \in D$. Let $Z^1(\Gamma) \subset Z^2(\Gamma) \subset \ldots \subset Z^n(\Gamma) = \Gamma$ be an upper central series for Γ. Since $Z^1(\Gamma)$ is the center of Γ, $Z^1(\Gamma) \subset \mathrm{FC}(\Gamma)$, hence by the above paragraph $\psi((\gamma, d)) = 0 \,\forall\, \gamma \in Z^1(\Gamma) \setminus \{e\}$. Assume that $\forall\, \gamma \in Z^k(\Gamma) \setminus \{e\}$, $\psi((\gamma, d)) = 0 \,\forall\, d \in D$, and choose $h \in Z^{k+1}(\Gamma) \setminus Z^k(\Gamma)$. Following [2, Theorem

4.2], let $\alpha_{h,0}: \Gamma \to Z^k(\Gamma)/Z^{k-1}(\Gamma)$ be the homomorphism defined by $\alpha_{h,0}(g) = [h,g]\, Z^{k-1}(\Gamma)$. If the range of $\alpha_{h,0}$ is infinite, then we can choose an infinite sequence $\{g_1, g_2 \ldots\} \subset \Gamma$ such that $[h, g_m g_n^{-1}] \in Z^k(\Gamma)/Z^{k-1}(\Gamma)\ \forall\ m, n, m \neq n$, so that $\psi([(h,1),(g_m,1)(g_n,1)^{-1}]) = \psi(([h, g_m g_n^{-1}], d))$ (for some $d \in D) = 0\ \forall\ m, n, m \neq n$, and it follows by applying Lemma 4.1 of [2] that $\psi((h,1)) = 0$, hence $\psi((h,d)) = 0\ \forall\ d \in D$. If the range of Γ under $\alpha_{h,0}$ is finite, then set $\Gamma_1 = \ker \alpha_{h,0}$. Note Γ_1 is of finite index in Γ and contains C_h and $Z^k(\Gamma)$. Define the homomorphism $\alpha_{h,1}: \Gamma_1 \to Z^{k-1}(\Gamma)/Z^{k-2}(\Gamma)$ by $\alpha_{h,1}(g) = [h,g]\cdot Z^{k-2}(\Gamma)$ and continue the process outlined above, i.e. check if the range of $\alpha_{h,1}$ is infinite, if not, set $\Gamma_2 = \ker(\alpha_{h,1})$, etc. Following such a procedure we construct a chain of subgroups $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \ldots \supset \Gamma_j$ and homomorphisms

$$\alpha_{h,i}: \Gamma_i \to Z^{k-i}(\Gamma)/Z^{k-i-1}(\Gamma),\ 0 \leqq i \leqq j$$

where $\Gamma_i = \ker \alpha_{h,i-1}$, Γ_i has finite index in Γ_{i-1}, and $\alpha_{h,i}(g) = [h,g]\, Z^{k-i-1}(\Gamma)$, $g \in \Gamma_i$, $1 \leqq i \leqq j$. Clearly each Γ_i contains C_h and $Z^{k-i}(\Gamma)$. If the range of $\alpha_{h,j}$ is infinite, we can choose an infinite sequence $\{g_n\} \in \Gamma_j$ with $[h, g_m g_n^{-1}] \in Z^{k-j}(\Gamma) \setminus Z^{k-j-1}(\Gamma)\ \forall\ m, n, m \neq n$, so that $\psi([(h,1),(g_m,1)(g_n,1)^{-1}]) = \psi(([h, g_m g_n^{-1}], *)) = 0\ \forall\ m, n, m \neq n$, by the induction hypothesis. Hence applying Lemma 4.1 of [2] again we see that $\psi((h,d)) = i(d)\,\psi(h,1)) = 0\ \forall\ d \in D$. If $\alpha_{h,j}(\Gamma_j)$ is finite, set $\Gamma_{j+1} = \ker \alpha_{h,j}$ (note Γ_{j+1} has finite index in Γ_j), and continue as above. This process will terminate after at most k steps; for either the range of $\alpha_{h,j}$ is infinite for some $j < k$ (in which case we obtain $\psi((h,d)) = 0\ \forall\ d \in D$) or we obtain a chain of subgroups $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \ldots \supset \Gamma_{k-1}$, Γ_i of finite index in Γ_{i-1}, $1 \leqq i \leqq k-1$, $\Gamma_i \supset C_h$, $0 \leqq i \leqq k-1$ with $\alpha_{h,k-1} \to Z^1(\Gamma)/Z^0(\Gamma) = Z^1(\Gamma)$. If the range of $\alpha_{h,k-1}$ is infinite, then by Lemma 4.1 of [2] we get $\psi((h,d)) = 0\ \forall\ d \in D$. Otherwise $\Gamma_k = \ker \alpha_{h,k-1} = C_h$ has finite index in Γ_{k-1}. It follows in this case that C_h has finite index in Γ, so that $h \in \mathrm{FC}(\Gamma) \setminus \{e\}$, and by our original argument we get $\psi((h,d)) = 0\ \forall\ d \in \Gamma$. It follows that $\forall\ h \in Z^k(\Gamma) \setminus \{e\}$, $\psi((h,d)) = 0$. This proves the induction step, and since Γ is nilpotent $Z^n(\Gamma) = \Gamma$ for some n so that we have established

$$\psi((\gamma, d)) = \begin{cases} i(d) & \gamma = e \\ 0 & \gamma \in \Gamma \setminus \{e\}. \end{cases}$$

Thus the character ψ on Γ' is uniquely determined, so that the trace τ on $C^*(\Gamma, \sigma)$ is unique.

Lemma 1.6 and Proposition 1.5 combine to give

THEOREM 1.7. *Let Γ be a countable nilpotent discrete group, with multiplier $\sigma: \Gamma \times \Gamma \to \mathrm{T}$. Then $C^*(\Gamma, \sigma)$ is simple and has a unique normalized trace if and only if every non-trivial σ-regular conjugacy class is infinite in cardinality.*

PROOF. Lemma 1.4 and Proposition 1.5 give most of the above result. As for the direction $\Rightarrow$, if there exists a finite non trivial σ-regular conjugacy class for Γ, then $W^*(\Gamma, \sigma)$ is a finite von Neumann algebra which is not a factor, so that the trace on $C^*(\Gamma, \sigma)$ is not unique.

The combination of the above results give an alternative proof of the following well-known fact ([27], [7]).

COROLLARY 1.8. ([22], [7]) *Let Γ be a countable discrete abelian group, and let $\sigma: \Gamma \times \Gamma \to \mathrm{T}$ be a multiplier. Then $C^*(\Gamma, \sigma)$ is simple and has a unique normalized trace if and only if the symmetrier subgroup S_σ for σ is trivial, where*

$$S_\sigma = \{\gamma_1 \in \Gamma \mid \sigma(\gamma_1, \gamma)\overline{\sigma(\gamma_1, \gamma_1)} = 1 \,\forall\, \gamma \in \Gamma\}.$$

PROOF. Since Γ is abelian, each $x \in \Gamma$ is its own conjugacy class, $C_x = \Gamma$, and $\chi^{\sigma, x}(\gamma) = \sigma(x, \gamma)\overline{\sigma(\gamma, x)}$, hence x is σ-regular $\Leftrightarrow x \in S_\sigma$. Thus $C^*(\Gamma, \sigma)$ is simple and has a unique trace $\Leftrightarrow S_\sigma = \{e\}$. (One could also note that if $S_\sigma = \{e\}$, then the center of Γ' is exactly $\{1\} \times D$. By Howe's result [12, Prop. 3] any faithful character ψ on Γ' vanishes on $\Gamma' \setminus Z(\Gamma') = \Gamma' \setminus \{1\} \times D$ in this case. But any trace on $C^*(\Gamma, \sigma)$ uniquely determines the values of ψ on $\{1\} \times D$. Hence if $S_\sigma = \{e\}$, the trace on $C^*(\Gamma, \sigma)$ must be unique. We thank the referee for pointing out this simple proof of uniqueness of the trace to us, when we had originally used Theorem 5.2 of [2] to get this result).

EXAMPLE 1.9. Let Γ be the integer Heisenberg group $\{(m, n, p) \mid m, n, p \in \mathrm{Z}\}$ where $(m_1, n_1, p_1)(m_2, n_2, p_2) = (m_1 + m_2 + p_1 n_2, n_1 + n_2, p_1 + p_2)$. Cohomology classes in $H^2(\Gamma, \mathrm{T})$ are parametrized by T^2 [17], where for $(\lambda, \mu) \in \mathrm{T}^2$,

$$\sigma_{\lambda, \mu}((m_1, n_1, p_1)(m_2, n_2, p_2)) = \lambda^{m_2 p_1 + p_1 \frac{(p_1 - 1)n_2}{2}} \mu^{n_1(m_2 + p_1 n_2) + p_1 \frac{(n_2 - 1)n_2}{2}}.$$

The finite conjugacy classes in Γ are singletons consisting of elements in the center of Γ, $\{(m, 0, 0) \mid m \in \mathrm{Z}\}$. Then the centralizer of $(m, 0, 0)$ is all of Γ, and

$$\chi^{\sigma_{\lambda,\mu}, (m, 0, 0)}(m_1, n_1, p_1) = (\lambda^{p_1} \mu^{n_1})^{-m}.$$

Hence no non-trivial $(m, 0, 0)$ is σ-regular if and only if either λ and/or μ is of the form $e^{2\pi i \alpha}$ for irrational α, so that by Proposition 1.3 $W^*(\Gamma, \sigma_{\lambda,\mu})$ is a factor if and only if λ and/or μ is non-torsion in T, as shown in [17] by other methods. Thus by Theorem 1.7 we see that $C(\Gamma, \sigma_{\lambda,\mu})$ is simple and has a unique trace $\Leftrightarrow \lambda$ and/or μ is non-torsion, which provides a shorter proof of this result from [17].

EXAMPLE 1.10. Let Γ be a semi-direct product of the form $\mathrm{Z}^n \rtimes_A \mathrm{Z}$, where $A \in \mathrm{GL}(n, \mathrm{Z})$ and $(\bar{v}_1, n_1)(\bar{v}_2, n_2) = (\bar{v}_1 + A^{n_1}\bar{v}_2, n_1 + n_2)$. The group structure of $H^2(\mathrm{Z}^n \rtimes_A \mathrm{Z}, \mathrm{T})$ has been described in [18]; we may use Proposition 1.3 to determine when $W^*(\Gamma, \sigma)$ is a factor, and in the case where A-Id is nilpotent, Theorem 1.7 to determine when $C^*(\Gamma, \sigma)$ is simple and has a unique trace. Given $\sigma \in H^2(\Gamma, \mathrm{T})$ in the standard form of [18], let $\gamma = \sigma$ restricted to $\mathbf{Z}^n \rtimes_A \{0\}$, and S_γ = symmetrizer subgroup of γ in $\mathbf{Z}^n \rtimes_A \{\hat{0}\}$, and let

$$\mathrm{FAO}(\mathbf{Z}^n) = \{(\bar{v}, 0) : \{A^j(v) \mid j \in \mathbf{Z}\} \text{ is finite}\}.$$

Then we have two cases:

1) A has infinite order in $\mathrm{GL}(n, \mathbf{Z})$,
2) A has finite order in $\mathrm{GL}(n, \mathbf{Z})$.

Computations using Proposition 1.3 show that in case 1), $W^*(\Gamma, \sigma)$ is a factor if and only if $\forall (\bar{v}, 0) \in S_\gamma \cap \mathrm{FAO}(\mathbf{Z}^n) \setminus \{(\Gamma, 0)\}$, $\exists p \in \mathbf{Z}$ with $A^p \bar{v} = \bar{v}$ and $\sigma((\bar{0}, p), (\bar{v}, 0)) \neq 1$, and in case 2), $W^*(\Gamma, \sigma)$ is a factor if and only if the symmetrizer subgroup of σ restricted to $\Delta = \{(\bar{v}, m) \mid \bar{v} \in \mathbf{Z}^n,\ A^m = \mathrm{Id}\}$ is equal to the trivial subgroup $\{(\bar{0}, 0)\}$.

Thus when A–Id is nilpotent, by using Theorem 1.7 we see that $C^*(\mathbf{Z}^n \rtimes_A \mathbf{Z}, \sigma)$ is simple and has a unique trace if and only if the conditions given above for factoriality of $W^*(\Gamma, \sigma)$ hold. If A–Id is not nilpotent (hence $\mathbf{Z}^n \rtimes_A \mathbf{Z}$ is not nilpotent) it is still possible for $C^*(\mathbf{Z}^n \rtimes_A \mathbf{Z}, \sigma)$ to be simple; see [18] for necessary and sufficient conditions.

2. Relationship to affine group actions

In this section we restrict our study to semi-direct product groups of the form $\Gamma = N \rtimes G$ where N is discrete abelian and G is discrete (later we will consider only abelian G). Recall that in this case N is a normal subgroup of $N \rtimes G$ on which G acts by automorphisms: $(\eta_1, g_1)(\eta_2, g_2) = (\eta_1 A(g_1)\eta_2, g_1 g_2)$ where $A(g_1)\eta_2 = g_1 \eta_2 g_1^{-1}$. Consider a multiplier (two-cocycle) σ on $N \rtimes G$ such that σ when restricted to the subgroups $N \times N$ and $G \times G$ of $\Gamma \times \Gamma$ is identically one. Then by a results of Mackey [14, Cor. to Thm. 9.4], σ is cohomologous to a cocycle of the form

$$\sigma_{A,\rho}((\eta_1, g_1), (\eta_2, g_2)) = \langle \eta_2, \rho(g_1) \rangle$$

where $\rho : G \to \hat{N}$ is a "$\hat{A}$-crossed homomorphism" [24], [26], i.e., a map satisfying

$$(*) \quad \rho(g_1 g_2) = \hat{A}(g_2)(\rho(g_1))\rho(g_2).$$

Now G acts (on the left) on N via A as a group of automorphisms, so that by duality $\hat{A}$ gives a map of G into $\mathrm{Aut}(\hat{N})$ which is an antihomomorphism. Conversely, any map ρ' of G into $\hat{N}$ satisfying $(*)$ gives rise to a multiplier $\sigma_{\rho'}$ on $\Gamma \times \Gamma$ by setting

$$\sigma_{\rho'}((\eta_1, g_1), (\eta_2, g_2)) = \eta_2(\rho'(g_1))$$

Any $\hat{A}$-crossed homomorphism of G into $\hat{N}$ can be naturally associated to an affine action of G (on the right) on $\hat{N}$ as a group of homomorphisms:

$$\omega \cdot g = \rho(g)\,\hat{A}(g)(\omega),\ \omega \in \hat{N}.$$

Letting $X = \hat{N}$ and $B = \hat{A}$, we denote this dynamical system by $(X, G)_{(B,\rho)}$.

Consider the von Neumann algebra $W^*(X, G)_{(B,\rho)}$ formed via the group measure construction acting on the Hilbert space $L^2(X \times G, \nu_X \times \nu_G)$ where ν_X and ν_G are Haar measure on X and G respectively.

Recall (see [23]) that the von Neumann algebra crossed product $W^*(X, G)_{(B,\rho)}$ is generated by the operators

$$\{T_f \mid f \in L^\infty(X, \nu_X)\} \text{ and } \{V_g \mid g \in G\},$$

where

$$T_f\, l(\omega, g) = f(\omega)\, l(\omega, g)$$

and

$$V_{g_1} l(\omega, g) = l(\omega g_1, g_1^{-1} g)$$
$$= l(\rho(g_1)\, B(g_1)(\omega), g_1^{-1} g),\ \omega \in X,\ g_1, g \in G.$$

We now observe that $W^*(X, G)_{(B,\rho)}$ is spatially isomorphic to $W^*(N \rtimes_A G, \sigma_\rho)$. $W^*(N \rtimes_A G, \sigma_\rho)$. This is no doubt known to many people and in the case where either $\sigma_\rho \equiv 1$ or $A = \mathrm{Id}$ was first noticed in [20] and [8] respectively. We briefly indicate a proof for the general case.

Let $\mathscr{F}_N : L^2(N) \to L^2(X)$ be the Fourier transform, defined by $\mathscr{F}_N(f)(\omega) = \sum_{\eta \in N} f(\eta) \langle \omega, \eta \rangle$. Then $\mathscr{F}_N(\delta_\eta)(\omega) = \eta(\omega)$, $\eta \in \hat{X} = N$, $\omega \in X$. Let

$$\begin{array}{rcl} \mathscr{U} = \mathscr{F}_N \otimes \mathrm{Id}_{L^2(G)} : L^2(N \times G) & \to & L^2(X \times G) \\ \| & & \| \\ L^2(N) \otimes L^2(G) & \to & L^2(\hat{N}) \otimes L^2(G). \end{array}$$

One easily checks that $\mathscr{U}^{-1} V_g \mathscr{U} = U_{(e,g)}$, $g \in G$ and $\mathscr{U}^{-1} T_\eta \mathscr{U} = U_{(\eta,e)}$, $\eta \in N = \hat{X}$. Hence $\mathscr{U}^{-1} W^*(X, G_{(B,\rho)} \mathscr{U} \subset W^*(N \rtimes G, \sigma_\rho)$. Similarly $\mathscr{U}\, W^*(N \rtimes G, \sigma_\rho) \mathscr{U}^{-1} \subset W^*(X, G)_{(B,\rho)}$, so that $\mathscr{U}^{-1} W^*(X, G)_{(B,\rho)} \mathscr{U} = W^*(N \rtimes G, \sigma_\rho)$. Thus we have shown:

PROPOSITION 2.1. *If σ is a multiplier on the semi-direct product $\Gamma = N \rtimes_A G$ with N discrete abelian and G discrete, with σ identically one on the subgroups $N \times N$ and $G \times G$, then there exists an affine action of G on $\hat{N}$, $(\hat{N}, G)_{(B,\rho)}$ such that $W^*(N \rtimes_A G, \sigma) \cong W^*(\hat{N}, G)_{(B,\rho)}$. Conversely, if one is given an affine action $(\hat{N}, G)_{(B,\rho)}$, there is a multiplier σ_ρ on $N \rtimes_B G$ such that $W(\hat{N}, G)_{(B,\rho)} \cong W^*(N \rtimes G, \sigma_\rho)$.*

Now let G be a countable *abelian* group, acting on the compact Lebesgue space (X, μ) so as to leave the measure μ invariant. Then it is known that $W^*(X, G)$ is a factor if and only if G acts measure-theoretically freely and ergodically on (X, μ).

In the situation where $(X, G)_{(B,\rho)}$ is an affine action of G on the compact abelian group X, we can therefore deduce that the dynamical system $(X, G)_{(B,\rho)}$ is (m.t.) free and ergodic only when the discrete group-multiplier pair $(\hat{X} \rtimes G, \sigma_\rho)$ satisfies the conditions of Proposition 1.3. This allows us to deduce:

THEOREM 2.2. *Let $(X, G)_{(B,\rho)}$ be affine dynamical system where X is a compact (s.c.) abelian group, G is a (countable) discrete abelian group, $B: G \to Aut(X)$ is homomorphism, and $\rho: G \to X$ is a B-crossed homomorphism. Then the action of G on X is free and ergodic with respect to Haar measure if and only if*

1) (*Ergodicity condition*)
For every $\eta \in \hat{X} \setminus \{e\}$ such that $\{\hat{B}(g)(\eta) \mid g \in G\}$ is finite, there exits $g_1 \in G$ with $\hat{B}(g_1)(\eta) = \eta$ and $\eta(\rho(g_1)) \neq 1$.

and

2) (*Freeness condition*)
For every $g \in G \setminus \{e\}$ such that $\{\hat{B}(g)(\eta)\bar{\eta} \mid \eta \in \hat{X}\}$ is finite, there exists $\eta_1 \in \hat{X}$ with $\hat{B}(g)(\eta_1) = \eta_1$ and $\eta_1(\rho(g)) \neq 1$.

PROOF. Let $N = \hat{X}, A = \hat{B}: G \to \mathrm{Aut}(N)$. Then by Proposition 2.1, $W^*(X, G)_{(B,\rho)} \cong W^*(N \rtimes_A G; \sigma_\rho)$; thus to determine whether the system $(X, G)_{(B,\rho)}$ is m.t. free and ergodic it suffices to determine whether or not $W^*(N \rtimes_A G; \sigma_\rho)$ is a factor, by our remarks made prior to the statement of the theorem. Using Proposition 1.3, we must compute the non-trivial finite conjugacy classes of $N \rtimes_A G$ and determine whether they are σ-regular. If $(\eta, g) \in N \rtimes_A G$, then its conjugacy class in Γ will be $\{(\gamma A(g)(\gamma^{-1}) A(h)(\eta), g) \mid (\gamma, h) \in \Gamma\}$. We see that (η, g) is a finite conjugacy class (or $(\eta, g) \in \mathrm{FC}(\Gamma)$) if and only if (η, e) and (e, g) both have finite conjugacy classes; moreover $(\eta, e) \in \mathrm{FC}(\Gamma)$ if and only if $\{A(h)(\eta) \mid h \in G\}$ is finite, and $(e, g) \in \mathrm{FC}(\Gamma)$ if and only if the set $\{A(g)(\gamma^{-1})\gamma \mid \gamma \in N\}$ is finite. For $(\eta, e) \in \mathrm{FC}(\Gamma)$, $(\gamma, g) \in C_{(\eta,e)}$, we have

$$\begin{aligned}\chi^{\sigma_\rho,(\eta,e)}(\gamma, g) &= \sigma_\rho((\eta, e), (\gamma, g))\,\overline{\sigma_\rho((\gamma, g), (\gamma, g)^{-1}(\eta, e)(\gamma, g))} \\ &= 1 \cdot \overline{\sigma_\rho((\gamma, g), (\eta, e))} = \overline{\eta(\rho(g))}.\end{aligned}$$

Thus the class of (η, e) is not σ-regular if and only if there exists $g_1 \in G$ with $A(g_1)(\eta) = \eta$ and $\eta(\rho(g_1)) \neq 1$. Taking $(e, g) \in \mathrm{FC}(\Gamma), (\gamma, h) \in C_{(e,g)}$, then

$$\chi^{\sigma_\rho,(e,g)}(\gamma, h) = \sigma_\rho((e, g), (\gamma, h))\,\overline{\sigma_\rho((\gamma, h), (e, g))} = \gamma(\rho(g)).$$

Thus the class of (e, g) is not σ-regular if and only if there exists $\eta_1 \in N$ with $A(g)(\eta_1) = \eta_1$ and $\eta_1(\rho(g)) \neq 1$. Finally a calculation shows that if $(\eta, g) \in \mathrm{FC}(\Gamma)$ and $(\gamma, h) \in C_{(\eta,g)}$,

$$\begin{aligned}\chi^{\sigma_\rho,(\eta,g)}(\gamma, h) &= \sigma_\rho((\eta, g), (\gamma, h))\,\overline{\sigma_\rho((\gamma,h), (\eta, g))} \\ &= \gamma(\rho(g))\eta(\rho(h))\end{aligned}$$

Hence $(\eta, g) \in \mathrm{FC}(\Gamma)$ will not be σ-regular if and only if either (η, e) or (e, g) is not σ-regular. Hence statements 1) and 2) of the theorem guarantee that Γ has no non-trivial finite conjugacy classes, which implies by Proposition 1.3 that $W^*(N \rtimes_A G, \sigma_\rho)$ is a factor $\Rightarrow W^*(X, G)_{(B,\rho)}$ is a factor $\Rightarrow (X, G)_{(B,\rho)}$ is free and ergodic with respect to Haar measure. Conversely, if $(X, G)_{(B,\rho)}$ is free and ergodic, then $W^*(X, G)_{(B,\rho)}$ is a factor $\Rightarrow W^*(N \rtimes_A G, \sigma_\rho)$ is a factor $\Rightarrow$ Statements 1) and 2) hold.

George Mackey was the first to suggest to me that something like Theorem 2.2 could be proved via von Neumann algebra techniques, and that one could derive in an alternate fashion the following version of the well-known theorem of Hahn ([9], Theorem 4) (who dealt with the case $G = \mathbf{Z}$) which was generalized by Wieting to arbitrary l.c.s.c. abelian groups G [24]:

COROLLARY 2.3. *Let $(X, G)_{(B,\rho)}$ be an affine action of the countable abelian group G on the compact (s.c.) abelian group X. Then the action of G on X is ergodic with respect to Haar measure if and only if for every $\eta \in \hat{X} \setminus \{e\}$ such that $\{A(g)(\eta) \mid g \in G\}$ is finite, there exists $g_1 \in G$ with $A(g_1)(\eta) = \eta$ and $\eta(\rho(g_1)) \neq 1$. (Here $A = \hat{B}: G \to \operatorname{Aut} N$).*

PROOF. If the action of G on X satisfies the freeness condition 2) of theorem 2.2, the result follows immediately. We assume then that condition 1) but not condition 2) of Theorem 2.2 holds for $(X, G)_{(B,\rho)}$. Let $G_0 \subset G$ be the subgroup of $g \in G$ such that $\{A(g)(\gamma)\bar{\gamma} \mid \gamma \in N\}$ is finite *and* for every $\eta \in N$ with $A(g)(\eta) = \eta$, $\eta(\rho(g)) = 1$. If $g_0 \in G_0$, a calculation using 1) shows that $\mathcal{O}_{g_0} = \{A(g_0)(\eta)\bar{\eta} \mid \eta \in N\}$ must consist only of the trivial character $\gamma \equiv 1$. Hence the action of G_0 on X is trivial, since $B(g_0) = \mathrm{Id} \in \operatorname{Aut} X$ and $\rho(g_0) = e$ for $g_0 \in G_0$. Thus the action of G on X factors through to give an action of $G' = G/G_0$ on X. It is easy to check that $(X, G')_{(B,\rho)}$ will satisfy conditions 1) and 2) of Theorem 2.2 so that G' acts ergodically on X, which implies that G acts ergodically on X. The converse is extablished similarly.

We now discuss what the above results will tell us about the C^*-algebras corresponding to certain affine dynamical systems, $C^*(X, G)_{(B,\rho)}$. It is clear that the spatial automorphism between $W^*(N \rtimes_A G, \sigma_\rho)$ and $W^*(X, G)_{(B,\rho)}$ carries the weakly dense subalgebra $C^*_{\mathrm{red}}(N \times G, \sigma_\rho)$ onto a concrete faithful representation of the transformation group C^*-algebra $C(X) \times_{\mathrm{red}} G$ corresponding to the dynamical system $(X, G)_{(B,\rho)}$. If G is abelian, $C^*_{\mathrm{red}}(N \rtimes_A G, \sigma_\rho) \cong C^*(N \rtimes_A G, \sigma_\rho)$ and $C(X) \times_{\mathrm{red}} G \cong C(X) \ltimes G$, so that we have a $*$-isomorphism between $C^*(N \rtimes_A G, \sigma_\rho)$ and $C(X) \ltimes G$. Now it is known ([25]) that for a free action (X, G), (X, G) is minimal if and only if $C(X) \ltimes G$ is simple, and (X, G) is uniquely ergodic (i.e., there exists a unique G-invariant Borel probability measure on X), if and only if $C(X) \rtimes G$ has a unique normalized trace. Thus when $N \rtimes_A G$

is nilpotent, the results in Section 1 will yield information about the dynamics of $(X, G)_{(B,\rho)}$:

THEOREM 2.4. *Let $(X, G)_{(B,\rho)}$ be an affine action of the countable abelian group G on the compact abelian s.c. group X* and suppose that $\hat{X} \times_{\hat{B}} G$ *is nilpotent, i.e. suppose the action of G on $\hat{X}$ is nilpotent. Then $(X, G)_{(B,\rho)}$ is minimal and uniquely ergodic if and only if for every $\eta \in \hat{X} \setminus \{e\}$ such that $\{\hat{B}(g)(\eta) \mid g \in G\}$ is finite, there exists $g_0 \in G$ with $\hat{B}(g_0)(\eta) = \eta$ and $\eta(\rho(g_0)) \neq 1$.*

PROOF. ($\Leftarrow$): As in the proof of Corollary 2.3, we can find a quotient group G' of G with natural map $\psi: G \to G'$ such that (B, ρ) factors through G' as $(B' \circ \psi, \rho' \circ \psi)$, and with $(X, G')_{(B',\rho')}$ satisfying conditions 1) and 2) of Theorem 2.2. Then $\hat{X} \times_{\hat{B}} G'$ is a quotient of $\hat{X} \times_{\hat{B}} G$, hence is still nilpotent, and $W^*(X, G')_{(B',\rho')} \cong W^*(\hat{X} \times_{\hat{B}} G', \sigma_{\rho'})$ is a factor. Hence by Theorem 1.7 $C^*(\hat{X} \times_{\hat{B}} G', \sigma_\rho) \cong C^*(X) \times G'$ is simple and has a unique normalized trace, which implies that (X, G') is minimal and uniquely ergodic. This implies that (X, G) is minimal and uniquely ergodic.

($\Rightarrow$): If $(X, G)_{(B,\rho)}$ is minimal and uniquely ergodic, the stability subgroups of every $x \in X$ must be equal to some fixed subgroup G_0, and $(X, G)_{(B,\rho)}$ may be factored as before as $(X, G)_{(B' \circ \varphi, \rho' \circ \varphi)}$, where $\varphi: G \to G/G_0 = G'$ is the quotient map, and $(X, G')_{(B',\rho')}$ is an affine system in which G' acts freely and which is minimal and uniquely ergodic. This implies that $C(X) \times G' \cong C^*(X \times G', \sigma_{\rho'})$ is simple and has a unique normalized trace, which implies that the desired condition holds, from Theorem 1.7.

In the special case where $X = \mathbf{T}^n$ and $G = \mathbf{Z}$ we obtain as a corollary an alternate proof of the following theorem of Hahn [9]:

COROLLARY 2.5. *Let $(\mathbf{T}^n, \mathbf{Z})_{(B,\rho)}$ be an affine action of $\mathbf{Z}$ on the n-torus, with $T^n(\bar{z}) = \rho(n)B^n(\bar{z})$, for $B \in \mathrm{Aut}(\mathbf{T}^n) \cong \mathrm{GL}(n, \mathbf{Z})$ and ρ a B-crossed homomorphism of $\mathbf{Z}$ into $\mathbf{T}^n$. Suppose that B is unipotent. Then the action of $(\mathbf{T}^n, \mathbf{Z})_{(B,\rho)}$ is minimal and uniquely ergodic if and only if* (**) *for every $\bar{v} \in \hat{\mathbf{T}}^n \setminus \{e\} = \mathbf{Z}^n \setminus \{\bar{0}\}$ such that $\{\hat{B}(n)(\bar{v}) \mid n \in \mathbf{Z}\}$ is finite, there exists $n_0 \in \mathbf{Z}$ with $\hat{B}(n_0)(\bar{v}) = \bar{v}$ and $\langle \bar{v}, \rho(n_0) \rangle \neq 1$.*

PROOF. If B is unipotent, then $B^t - \mathrm{Id}$ is nilpotent so that $\mathbf{Z}^n \times_{\hat{B}} \mathbf{Z} = \mathbf{Z}^n \times_{\hat{B}} \mathbf{Z}$ is nilpotent. Hence we apply Theorem 2.4 to get the desired result.

REMARK 2.6. Hahn [9] and Hoare and Parry [11] have shown that all minimal affine actions of $\mathbf{Z}$ on the n-torus $(\mathbf{T}^n, \mathbf{Z})_{(B,\rho)}$ must satisfy condition (**) and must have B–Id nilpotent. Such systems have quasi-discrete spectrum (c.f. [11]; the "normal form" of these systems as given in [9] allows one to read off an ascending central series for $(\mathbf{Z}^n \rtimes \mathbf{Z}) \times_\sigma D$ quite easily). It follows that in order for $C^*(\mathbf{Z}^n \rtimes_A \mathbf{Z}, \sigma_\rho)$ to be simple it is necessary that A-Id be nilpotent and (A^t, ρ) satisfy (**); as mentioned at the end of Section 1, if we take more general $\sigma \in Z^2(\mathbf{Z}^n \rtimes_A \mathbf{Z}; \mathbf{T})$ this no longer is true [18].

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