MATH SCAND. 64 (1989), 123–132

THE S^p-CRITERION FOR HANKEL FORMS ON THE FOCK SPACE, $0 < p < 1$

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1. Introduction.

In [JPR] the authors study Hankel forms on the Hilbert space of analytic functions square integrable with respect to a given measure on a domain in $\mathbf{C}^n$. Among other results they obtain, under some conditions on the measure and the domain, necessary and sufficient S^p-criteria, $1 \leqq p \leqq \infty$, for Hankel forms (S^p = Schatten class). In particular the theory applies when the domain is $\mathbf{C}^n$ and the measure Gaussian, in which case results on decomposition, approximation and interpolation of norms and of values are given for the corresponding L^p-spaces, $1 \leqq p \leqq \infty$. In this paper we consider this special case and extend the mentioned results to the range $0 < p < 1$.

For a general discussion of the topics treated below, see [R], where further references may be found. See also [P2].

After reviewing some definitions and basic facts the decomposition theorem is stated and proved. Then, in the last part, the decomposition theorem merges with obvious modifications of known techniques, essentially from [CT], [JJ] and [S], to yield the remaining results.

2. Preliminaries.

2.1. *Function spaces.* For $\alpha > 0$ and $0 < p \leqq \infty$ let L^p_α denote the space of complex-valued measurable functions on $\mathbf{C}^n$ such that $f(z)e^{-\frac{1}{2}\alpha|z|^2} \in L^p(m)$, normed such that the constant function 1 has unit norm. The subspace of entire functions is denoted by F^p_α. The operator $C_\alpha(w)$, $w \in \mathbf{C}^n$, defined by

$$C_\alpha(w)f(z) = f(z-w)e^{\alpha\langle z,w\rangle - \frac{1}{2}\alpha|w|^2},$$

is an isometry of F^p_α (and L^p_α) onto itself. Moreover $C_\alpha(w_1 + w_2) = C_\alpha(w_2)e^{i\,\mathrm{Im}\langle w_1,w_2\rangle}$. Recalling the group law $(z,t)\circ(w,s) = (z+w, t+s-I_m\langle z,w\rangle)$

Received November 2, 1987.

of the Heisenberg group $H_n = \mathbf{C}^n \times \mathbf{R}$ one finds that $(w, t) \to e^{i\alpha t} C_\alpha(w)$ defines a unitary representation of H_n in F^2_α.

Using the isometries of $C_\alpha(w)$ one easily verifies that $F^p_\alpha \subset F^q_\alpha$ if $p \leqq q$.

When $p = 2$, we have $L^p_\alpha = L^2(\mu_\alpha)$, where $d\mu_\alpha = \left(\frac{\alpha}{\pi}\right)^n e^{-\alpha|z|^2}\, dm$. The space F^2_α is called the Fock space. With respect to the inner product $\langle f, g\rangle_\alpha = \int f\bar{g}\, d\mu_\alpha$ it has reproducing kernel $K_z(w) = e^{\alpha\langle w, z\rangle}$, i.e.

$$f(z) = \langle f, K_z\rangle_\alpha = \int e^{\alpha\langle z, w\rangle} f(w)\, d\mu_\alpha(w). \tag{2.1}$$

The Bergman projection P_α, defined by

$$P_\alpha f(z) = \int e^{\alpha\langle z, w\rangle} f(w)\, d\mu_\alpha(w),$$

is a bounded self-adjoint projection of L^p_α onto F^p_α, $1 \leqq p \leqq \infty$. For proofs, see [JPR].

2.2 *Hankel forms and Schatten classes of bilinear forms.* Let b be an entire function on $\mathbf{C}^n$. Then H^β_b, the Hankel form with symbol b with respect to μ_β, is defined by

$$H^\beta_b(f, g) = \int \bar{b} f g\, d\mu_\beta, \quad (f, g) \in F^2_{\alpha_1} \times F^2_{\alpha_2}.$$

In general, we say that a bilinear form H on $H_1 \times H_2$ is of Schatten class S^p if $\tilde{H}$: $H_2 \to H_1^*$ defined by

$$H(f, g) = \langle f, \tilde{H} g\rangle$$

belongs to the ordinary Schatten class S^p, cf. [P1] and the introduction in [JPR]. In the sequel the anti-linear identification of H_1 and H_1^* will be used without mention.

When $\beta = \alpha_1 = \alpha_2 = \alpha$, $\tilde{H}^\alpha_b$ is a convolution operator if it is bounded (see [JPR], sec. 10):

$$\tilde{H}^\alpha_b f(z) = \int b(z + w)\overline{f(w)}\, d\mu_\alpha(w).$$

For future use we note that

$$\tilde{H}^\alpha_b f(z) = b(z)\overline{f\left(\frac{w}{2}\right)}, \; b(z) = e^{\alpha/2\langle z, w\rangle - \alpha/4|z|^2}$$

and

$$\langle \tilde{H}_b^\alpha f, f\rangle_\alpha = b(2w)e^{-\alpha|w|^2}, \; f(z) = e^{\alpha\langle z,w\rangle - \frac{1}{2}\alpha|w|^2}$$

2.3. *Complex interpolation of quasi-Banach spaces.* Let $\bar{S} = \{z \in \mathbf{C} \mid 0 \leqq \operatorname{Re} z \leqq 1\}$ and let (X_0, X_1) be an interpolation couple of quasi-Banach spaces. Let $F = F(X_0, X_1)$ be a vector space of $X_0 + X_1$-valued analytic functions defined on $\bar{S}$ and quasi-normed by

$$\|F\|_F = \sup_{\substack{y \in \mathbb{R} \\ z \in S}} \{\|F(iy)\|_{X_0}, \|F(1+iy)\|_{X_1}, \|F(z)\|_{X_0+X_1}\}$$

If $F = F^s$ is the closure in norm of the space of functions $\sum_1^n f_k x_k$, where $x_k \in X_0 \cap X_1$ and f_k are bounded, continuous and analytic scalar functions, then for $0 < \theta < 1$ the quasi-norm on $[X_0, X_1]_\theta$, the (strong) complex interpolation space, is defined by

$$\|x\| = \inf\{\|F\|_{F^s} \mid F(\theta) = x\}.$$

By taking $F = F^w$ to be the space of $X_0 + X_1$-valued functions such that $\langle U, F(z)\rangle$ is analytic and continuous on $\bar{S}$ for any $U \in (X_0 + X_1)^*$ one similarly obtain the weak complex interpolation spaces, denoted by $[X_0, X_1]_\theta^w$.

The definition employed here is the same as in [JJ]. For the F_α^p-spaces to be treated below it is equivalent with the definition in [CMS]. The mentioned papers also contain further references on the subject. The basic facts from interpolation theory may be found in [BL].

3. The decomposition theorem.

A set $\{w_j\}$ in $\mathbf{C}^n$ will be called ε-dense if every ball with radius ε contains at least one of the w_j. If, in addition, any ball of radius 1 contains at most M points from $\{w_j\}$, the set will be called separated.

THEOREM 3.1. *Let $0 < p \leqq 1$ and $\alpha > 0$. There exists $\varepsilon_0 > 0$ such that if $\{w_j\} \subset \mathbf{C}^n$ is ε-dense with $\varepsilon < \varepsilon_0$ then $f \in F_\alpha^p$ iff*

$$f(z) = \sum_1^\infty \lambda_j e^{\alpha\langle z, w_j\rangle - \frac{1}{2}\alpha|w_j|^2} \tag{3.1}$$

with $\{\lambda_j\} \in \ell^p$. $\|f\|_{F_\alpha^p}$ is equivalent to $\inf\|(\lambda_j\}\|_{\ell^p}$ within constants depending on α, p and ε.

PROOF. Without loss of generality we may assume that $\alpha = 1$. To begin with we also assume that $\{w_j\} = \varepsilon\mathbf{Z}^{2n}$ in some order.

Let Q_j be cubes with centres w_j, side-length ε and such that $\bigcup Q_j = \mathbf{C}^n$. Let $\tilde{Q}_j$

be the cube with the same center as Q_j but with side-length 2ε. Let $m = \left[2n\left(\frac{1}{p} - 1\right)\right]$ and define for $|\beta| \leqq m, j \in \mathbf{Z}^+$

$$Sf_{j\beta} = \int_{Q_j} e^{i\,\mathrm{Im}\langle w, w-w_j\rangle - \frac{1}{2}|w-w_j|^2} \frac{(\bar{w} - \bar{w}_j)^\beta}{\beta!} f(w) e^{-\frac{1}{2}|w|^2} \pi^{-n}\, dV(w)$$

and

$$T\{\lambda_{j\beta}\} = \sum_{|\beta| \leqq m,\, j \in Z} \lambda_{j\beta}(z - w_j)^\beta e^{\langle z, w_j\rangle - \frac{1}{2}|w_j|^2}.$$

S maps F_1^p into ℓ^p, because

$$\left|\int_{Q_j} e^{i\,\mathrm{Im}\langle w, w-w_j\rangle - \frac{1}{2}|w-w_j|^2} \frac{(\bar{w} - \bar{w}_j)^\beta}{\beta!} f(w) e^{-\frac{1}{2}|w|^2} \pi^{-n}\, dV(w)\right|^p$$

$$= \left|\int_{Q_j} e^{i\,\mathrm{Im}\langle w, w-w_j\rangle - \frac{1}{2}|w-w_j|^2} \frac{(\bar{w} - \bar{w}_j)^\beta}{\beta!} f(w) e^{-\langle w-w_j, w_j\rangle}\right.$$

$$\left. e^{-\frac{1}{2}|w-w_j|^2 - \frac{1}{2}|w_j|^2 + i\,\mathrm{Im}\langle w-w_j, w_j\rangle} \pi^{-n}\, dV(w)\right|^p$$

$$\leqq C\left(e^{-\frac{1}{2}|w_j|^2} \int_{Q_j} |f(w) e^{-\langle w-w_j, w_j\rangle}|\, d(V)(w)\right)^p$$

$$\leqq C\varepsilon^{2n(p-1)} e^{-p\frac{1}{2}|w_j|^2} \int_{\tilde{Q}_j} |f(w) e^{-\langle w-w_j, w_j\rangle}|^p\, dV(w)$$

$$\leqq C\varepsilon^{2n(p-1)} \int_{\tilde{Q}_j} |f(w) e^{-\frac{1}{2}|w|^2}|^p\, dV(w)$$

by subharmonicity.

That T maps ℓ^p into F_α^p follows from the subadditivity of the p-th power of the norm.

By (2.1) we have

$$(I - TS)f(z) = \sum_j \int_{Q_j} \left[e^{\langle z, w\rangle - \frac{1}{2}|w|^2} - e^{i\,\mathrm{Im}\langle w_j, w-w_j\rangle - \frac{1}{2}|w-w_j|^2}\right.$$

$$\left(\sum_0^m \frac{\langle z - w_j, w - w_j\rangle^k}{k!}\right) e^{\langle z, w_j\rangle - \frac{1}{2}|w_j|^2}\Bigg] f(w) e^{-\frac{1}{2}|w|^2}\, dV(w)$$

$$= \sum_j \int_{Q_j} e^{i\,\mathrm{Im}\langle w_j, w - w_j\rangle - \frac{1}{2}|w - w_j|^2} \left(\sum_{m+1}^{\infty} \frac{\langle z - w_j, w - w_j\rangle^k}{k!}\right) f(w) e^{-\frac{1}{2}|w|^2} \pi^{-n}\, dV(w)$$

$$\cdot\, e^{\langle z, w_j\rangle - \frac{1}{2}|w_j|^2},$$

whence, using the estimate above,

$$\|f - TSf\|_{F_1^p}^p \leqq C \sum_j \int_{\mathbf{C}^n} e^{-p\frac{1}{2}|z - w_j|^2} \sup_{w \in Q_j} \left| \sum_{m+1}^{\infty} \frac{\langle z - w_j, w - w_j\rangle^k}{k!}\right|^p$$

$$\cdot \left| \int_{Q_j} f(w) e^{-\frac{1}{2}|w|^2}\, dV(w)\right|^p dV(z)$$

$$\leqq C \sum_j \int_{\mathbf{C}^n} e^{-p\frac{1}{2}|z - w_j|^2} \sum_{m+1}^{\infty} \frac{|z - w_j|^{kp} \varepsilon^{kp}}{(k!)^p} \varepsilon^{2n(p-1)} \int_{\tilde{Q}_j} |f(w) e^{-\frac{1}{2}|w|^2}|^p\, dV(w)\, dV(z)$$

$$\leqq C \sum_j \sum_{m+1}^{\infty} \frac{\left(\sqrt{\frac{2}{p}}\,\varepsilon\right)^{pk} \Gamma\left(\frac{pk}{2} + n\right)}{(k!)^p} \varepsilon^{2n(p-1)} \int_{\tilde{Q}_j} |f(w) e^{-\frac{1}{2}|w|^2}|^p\, dV(w)$$

$$\leqq C \varepsilon^{2n(p-1) + p(m+1)} \|f\|_{F_1^p}^p \leqq \tfrac{1}{2} \|f\|_{F_1^p}^p,$$

if ε has been chosen small enough using our choice of m. Hence TS is invertible and T is onto.

It follows that $f \in F_1^p$ iff

$$f(z) = \sum_{j \in \mathbf{Z},\, |\beta| \leq m} \lambda_{jb} (z - w_j)^{\beta} e^{\langle z, w_j\rangle - \frac{1}{2}|w_j|^2} \tag{3.2}$$

with $\|f\|_{F_1^p}$ equivalent to $\inf \|\{\lambda_{j\beta}\}\|_{\ell^p}$.

The functions $f(z) = z^{\beta}$, $|\beta| \leqq m$, all belong to F_1^1. What has been proved so far implies that $A_\varepsilon f \to f$ in measure as $\varepsilon \to 0$, where the ε is the used in the definition of T and S, and $A_\varepsilon = TS$ is the resulting approximation operator. Further

$$|(I - A_\varepsilon) f(z) e^{-\frac{1}{2}|z|^2}|^p \leqq C \left| \sum_j \int_{Q_j} |f(w)|\, e^{-\frac{1}{2}|w|^2}\, dV(w) e^{\varepsilon|z - w_j|} e^{-\frac{1}{2}|z - w_j|^2}\right|^p.$$

It is fairly easy to verify that, for the functions in question, this expression is dominated by an L^1-function independent of $\varepsilon < 1$. Dominated convergence yields that $A_\varepsilon f \to f$ in F_1^p. By choosing ε small we can find finite sequences $\{\lambda_j^\beta\}$, such that for $|\beta| \leqq m$

$$\|z^\beta - \sum_j \lambda_j^\beta e^{<z, w_j> - \frac{1}{2}|w_j|^2}\|_{F_1^p} \leqq \delta,$$

and thus, by translation,

$$\|(z - w_k)^\beta e^{<z, w_k> - \frac{1}{2}|w_k|^2} - \sum_j \lambda_j^\beta e^{i \operatorname{Im} <w_j, w_k>} e^{<z, w_j + w_k> - \frac{1}{2}|w_j + w_k|^2}\|_{F_1^p} \leqq \delta.$$

By the construction above, the coefficients in (3.2) depend linearly on f, we may define a linear operator A on F_1^p by

$$Af(z) = \sum_{j \in Z, |\beta| \leqq m} \lambda_{j\beta} \sum_k \lambda_k^\beta e^{i \operatorname{Im}\langle w_k, w_j\rangle} e^{\langle z, w_k + w_j\rangle - \frac{1}{2}|w_k + w_j|^2}.$$

We have $\|f - Af\|_{F_1^p}^p \leqq \sum |\lambda_{j\beta}|^p \delta^p \leqq C\delta^p \|f\|_{F_1^p}^p$. Thus, for δ small enough, A is invertible. This proves the theorem under the restriction $\{w_j\} = \varepsilon \mathbf{Z}^{2n}$.

Finally, let $\{w_j\}$ be an arbitrary ε-dense set in $\mathbf{C}^n$, which, after picking a subsequence and reindexing, may be assumed to satisfy $|w_j - u_j| \leqq \varepsilon$ for all j, where $\{u_j\} = \varepsilon\sqrt{\frac{2}{n}}\mathbf{Z}^{2n}$. Choose ε so small that every $f \in F_1^p$ can be written $f(z) = \sum \lambda_j e^{\langle z, u_j\rangle - \frac{1}{2}|u_j|^2}$, where λ_j are linear functionals of f and $\|\{\lambda_j\}\|_{\ell^p} \leqq D \cdot \|f\|_{F_1^p}^p$, and so small that

$$\|e^{\langle z, u_j\rangle - \frac{1}{2}|u_j|^2} - e^{i \operatorname{Im}\langle w_j, u_j\rangle} e^{\langle z, w_j\rangle - \frac{1}{2}|w_j|^2}\|_{F_1^p}^p \leqq \frac{1}{2D}$$

for all j. Define a linear operator B on F_1^p by

$$Bf = B\left(\sum_j \lambda_j e^{\langle z, u_j\rangle - \frac{1}{2}|u_j|^2}\right) = \sum_j \lambda_j e^{i \operatorname{Im}\langle w_j, u_j\rangle} e^{\langle z, w_j\rangle - \frac{1}{2}|w_j|^2}.$$

Then

$$\|f - Bf\|_{F_1^p}^p \leqq \left(\frac{1}{2D}\right)^p \|\{\lambda_j\}\|_{\ell^p}^p \leqq \left(\frac{1}{2}\right)^p \|f\|_{F_1^p}^p.$$

Thus, B is invertible and the theorem is proved.

4. Applications.

As the techniques for obtaining the results in this section are quite well-known, some details are omitted.

THEOREM 4.1. *Let* $\alpha, \beta > 0$ *and* $0 < p < 1$. *Then*

$$(F^p_\beta)^* \cong F^\infty_{\alpha^2/\beta}$$

with the pairing $\langle,\rangle_\alpha$.

PROOF. For $\alpha = \beta$ this follows from the decomposition theorem, since the closed convex hull in F^1_α of the F^p_α-ball contains an F^1_α-ball. The general case is then proved as in [JPR].

THEOREM 4.2. *Let* $\alpha_0, \alpha_1 > 0$, $0 < p_0 < 1$, $p_0 \leqq p_1 \leqq \infty$ *and* $0 < \theta < 1$. *Then*

$$[F^{p_0}_{\alpha_0}, F^{p_1}_{\alpha_1}]_\theta = F^p_\alpha$$

where $\dfrac{1}{p} = \dfrac{1-\theta}{p_0} + \dfrac{\theta}{p_1}$ *and* $\alpha = \alpha_0^{1-\theta}\alpha_1^\theta$.

PROOF. Let $f \in F^p_\alpha$ have norm 1. Then

$$f(z) = \sum \lambda_j e^{\alpha\langle z, w_j\rangle - \frac{1}{2}\alpha|w_j|^2}$$

with $\sum|\lambda_j|^p \leqq C$. Let $\dfrac{1}{p(\zeta)} = \dfrac{1-\zeta}{p_0} + \dfrac{\zeta}{p_1}$ and put

$$G(\zeta) = \sum |\lambda_j|^{p\,p(\zeta)^{-1}-1}\lambda_j e^{\alpha\langle(\alpha_1/\alpha_0)^{\frac{1}{2}|\zeta-\theta|}z, w_j\rangle - \frac{1}{2}\alpha|w_j|^2}$$

By a Taylor expansion of the exponentials, it follows that $G \in F^s$. Furthermore. $\|G\|_{F^s} \leqq C'$ and $G(\theta) = f$. It follows that $F^p_\alpha \subset [F^{p_0}_\alpha, F^{p_1}_{\alpha_1}]_\theta$.

For $\zeta \in \bar{S}$, define $T_\zeta f(z) = f\left(\left(\dfrac{\alpha_0}{\alpha_1}\right)^{(\zeta-\theta)/2} z\right)$. T_ζ is an isometry of $F^{p_j}_{\alpha_j}$ onto $F^{p_j}_\alpha$ when $\operatorname{Re}\zeta = j$, $j = 0, 1$. An application of the abstract version of the Stein interpolation theorem, theorem 0 in [JJ], yields $[F^{p_0}_{\alpha_0}, F^{p_1}_{\alpha_1}]_\theta \subset [F^{p_0}_\alpha, F^{p_1}_\alpha]^w_\theta$.

Finally, let $F_\zeta \in F^w(F^{p_0}_\alpha, F^{p_1}_\alpha)$ have norm 1, $0 < r < p_0$, $\ell(\zeta) = (1 - \operatorname{Re}\zeta)(1 - r/p_0) + \operatorname{Re}\zeta(1 - r/p_1)$ and set

$$I(\zeta) = \int_{|z| \leqq R} g(z)^{\ell(\zeta)} \left|F_\zeta(z) e^{-\frac{1}{2}\alpha|z|^2}\right|^r dV(z), \quad \zeta \in \bar{S},$$

where g is positive and continuous. By Hölder's inequality $I(\zeta) \leqq \displaystyle\int_{|z| \leqq R} g(z)\, dV(z)$

on δS. Since $\log I(\zeta)$ is subharmonic, it follows that

$$I(\theta) = \int\limits_{|z| \leqq R} g(z)^{1-r/p} |F_\theta(z) e^{-\frac{1}{2}\alpha|z|^2}|^r \, dV(z) \leqq \int\limits_{|z| \leqq R} g(z) \, dV(z)$$

which implies

$$\int\limits_{|z| \leqq R} |F_\theta((z) e^{-\frac{1}{2}\alpha|z|^2}|^p \, dV(z) \leqq 1.$$

Letting $R \to \infty$, we have $\|F_\theta\|^p_{F^p_\alpha} \leqq 1$ and thus $[F^{p_0}_\alpha, F^{p_1}_\alpha]^w_\theta \subset F^p_\alpha$.

THEOREM 4.3. *Suppose* $0 < p < 1$. *Then* $H^\alpha_b \in S^p(F^2_\alpha)$ *iff* $b \in F^p_{\alpha/2}$. *The respective norms are equivalent.*

PROOF. Without loss of generality let $\alpha = 1$. With the decomposition theorem at disposal half of the theorem is simple:

$$\|H_b\|^p_{S^b} = \|\sum \lambda_i H_{K_i}\|^p_{S^p} \leqq \sum |\lambda_i|^p \, \|H_{K_i}\|^p_{S^p} \leqq C \, \|b\|^p_{F^p_{\frac{1}{2}}},$$

where K_i are atoms in $F^p_{\frac{1}{2}}$, i.e. $K_i(z) = e^{\frac{1}{2}\langle z, w_i \rangle - \frac{1}{4}|w_i|^2}$ (c.f. sec. 2.2.).

For the reverse implication, choose an 1-dense and separated set $\{w_j\}$ such that

$$\sum_j |b(2w_j) e^{-|w_j|^2}|^p = C_n \, \|b\|^p_{F^p_{\frac{1}{2}}}.$$

The functions $f_j(z) = e^{\langle z, w_j \rangle - \frac{1}{2}|w_j|^2}$ are, by the decomposition theorem for F^2_1, almost orthogonal in the sense that they can be mapped, boundedly and invertibly, onto the orthonormal basis of a Hilbert space H.

Let (h_{ij}) denote the matrix of $\tilde{H}_b$ relative to the f_j. By our choice of the lattice $\{w_j\}$ we have

$$\sum_j |h_{ii}|^p = c \, \|b\|_{F^p_{\frac{1}{2}}}.$$

If (h_{ij}) were diagonal the proof would have ended here. However, (h_{ij}) is only close to diagonal. To remedy this, decompose $\{w_j\}$ into finitely many N-lattices, where N is to be chosen large. This corresponds to a direct dum decomposition of H. Let A be the direct sum of the compressions of (h_{ij}) to the summands of H. Write $A = D + F$, where D is the diagonal part. Then

$$\|D\|^p_{S^p} = \sum |h_{ii}|^p = C \, \|b\|_{F^p_{\frac{1}{2}}}.$$

Furthermore

$$\|F\|^p_{S^p} \leqslant \sum_{|w_i - w_j| \geqq N} |\langle \tilde{H}_b f_i, f_j \rangle|^p \leqq \sum_{|w_i - w_j| \geqq N} e^{-\frac{1}{2}p|w_i - w/2|^2} e^{-\frac{1}{2}p|w_j - w/2|^2},$$

if $b(z) = e^{\frac{1}{2}\langle z, w\rangle - \frac{1}{4}|w|^2}$. By the decomposition theorem we have

$$\|F\|_{S^p}^p = O(\|b\|_{F^p_{\frac{1}{2}}}), \ N \to \infty,$$

and therefore $\|b\|_{F^p_{\frac{1}{2}}} \leqq C \|\tilde{H}_b\|_{S^p}$, under the condition $b \in F^p_{\frac{1}{2}}$.

Assuming only $H_b \in S^p$, put $b_r(z) = b(rz)$ for $0 < r < 1$. Then $b_r \in F^p_{\frac{1}{2}}$ and $\|b_r\|_{F^p_{\frac{1}{2}}} \leqq C \|H_{b_r}\|_{S^p} \leqq C \|H_b\|_{S^p}$. Since $\|b\|_{F^p_{\frac{1}{2}}} = \lim \|b_r\|_{F^p_{\frac{1}{2}}}$, the theorem is proved.

Using the same method one can prove

THEOREM 4.4 *$H^\beta_b \in S^p(F^2_{\alpha_1} \times F^2_{\alpha_2})$ iff $b \in F^p_{\beta^2/(\alpha_1+\alpha_2)}$. The respective norms are equivalent.*

The proofs of the next two theorems are carried over almost verbatim from [JPR].

THEOREM 4.5. *Let $\alpha > 0$ and $0 < p < 1$. Then $f \in F^p_\alpha$ iff $f \in F^\infty_\alpha$ and $\{d_N\} \in \ell^p$, where* $d_N = \inf\left\{ \|f - g\|_{F^\infty_\alpha} \,\middle|\, g(z) = \sum_1^N a_j e^{\langle z, z_j\rangle} \right\}$.

PROOF. In one direction approximate f by the sum of the N terms in (3.1) having largest coefficients.

For the other direction observe that

$$S_N(H_f) \leqq \inf\left\{ \|H_f - H_g\|_{S^\infty(F^2_{2\alpha})} \,\middle|\, g(z) = \sum_I^N a_j e^{\langle z, z_j\rangle} \leqq C\, d_N \right\}.$$

Theorem 4.3. implies $\|f\|_{F^p_\alpha} \leqq C \|\{d_N\}\|_{\ell^p}$.

THEOREM 4.6. *Let $0 < p < 1$ and $\alpha > 0$. Then there exists $D(p, \alpha) < \infty$ such that if $\{z_j\}$ is a sequence in $\mathbf{C}^n$ with $\inf_{i=j} |z_i - z_j| > D(p, \alpha)$, then $\{a_j\} = \{f(z_j)\}$ for some $f \in F^p_\alpha$ iff $\{a_j e^{-\frac{1}{2}\alpha|z_j|^2}\} \in \ell^p$.*

PROOF. Define $Tf = \{f(z_j)e^{-\frac{1}{2}\alpha|z_j|^2}\}$ and $S\{a_j\} = \sum a_j e^{\alpha\langle z, z_j\rangle - \frac{1}{2}\alpha|z_j|}$. $T\colon F^p_\alpha \to \ell^p$ is continuous by subharmonicity (cf. the proof of theorem 3.1) and $S\colon \ell^p \to F^p_\alpha$ is bounded since the p-th power of the norm is subadditive. We have

$$TS\{a_j\} = \left\{ \sum_i a_i e^{\alpha\langle z_j, z_i\rangle - \frac{1}{2}\alpha|z_i|^2 - \frac{1}{2}\alpha|z_j|^2} \right\}$$

and therefore

$$\|I - TS\| \leqq \sup_i \left(\sum_{j \neq i} e^{-\frac{1}{2}p\alpha|z_i - z_j|^2} \right)^{1/p}.$$

Thus, if D is large enough, TS is invertible and T is onto.

ACKNOWLEDGEMENT. The author wishes to express his gratitude to Svante Janson for many valuable discussions.

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