MATH SCAND. 64 (1989), 161–218

ON THE CLASSIFICATION OF G-SPHERES II: PL AUTOMORPHISM GROUPS

IB MADSEN AND MELVIN ROTHENBERG

This paper studies the homotopy theory of groups of PL automorphisms of linear representations of finite groups. We calculate the connectivity of the PL Stiefel spaces $\mathrm{PL}_G(T)/\mathrm{PL}_G(V)$ for pairs of R-modules $V \subset T$; this is the homotopy theoretic basis for stable equivariant transversality in the locally linear PL category, via the results from [26].

As an application we obtain a G-version of the surgery exact sequence in the PL category for calculating the homotopy manifold structures of a homotopy type which satifies suitable gap conditions, provided G has odd order. This allows us to calculate the homotopy groups of $F_G(T)/\mathrm{PL}_G(T)$ in a stable range.

Call a G-CW complex X *topological stable* if is satisfies the following standard gap conditions:

$$\dim X^K > 2\dim X^H \geqq 12$$

for each pair of isotropy subgroups.

THEOREM A. *If V and T are topologically stable* RG*-modules and G has odd order then* $\mathrm{PL}_G(T)/\mathrm{PL}_G(V)$ *is* $(\dim V^G - 1)$*-connected, when V and T have the same set of isotropy groups.*

If G has even order, even for $G = \mathbf{Z}/2$, there is no connectivity for the Stiefel spaces above, and hence G transversality fails completely. It is an outstanding question to develop an effective obstruction theory for transversality for even order groups; the obstructions appear to be connected with generalizations of the Browder-Livesay invariants.

We next describe the equivariant surgery exact sequence in the locally linear PL category for $|G|$ odd.

Fix a 'model' RG-module Z. A locally linear (PL) G-$\mathbf{R}^n$ bundle ξ is *Z-restricted* if each fiber ξ_x is contained in some $Z^{\oplus k}$ an if ξ_x and Z have the same isotropy subgroups of G_x. Such bundles are classified by a G-space BPL. Similarly,

Received January 4, 1988

Z-restricted $G - S^n$ Hurewicz fibrations are classified by a G-space BF. Define $F/\mathrm{PL} = F/\mathrm{PL}(Z^\infty)$ to be the evident homotopy fiber. It has fixed sets

$$F/\mathrm{PL}(Z^\infty)^H = \operatorname{colim} F_H(Z^{\oplus k})/\mathrm{PL}_H(Z^{\oplus k})$$

with $F_H(\ )$ equal to the proper H-equivariant homotopy equivalences.

Let X be a Z-restricted locally linear PL G-manifold in the sense that its tangent bundle is Z-restricted.

Define $\tilde{\mathscr{S}}_G(X, \partial X)$ to be the equivalence classes of G-simple homotopy equivalences $h\colon M \to X$ with ∂h a PL-homeomorphism, *and* with $T_x M = T_{h(x)} X$ as $\mathbf{R}G$-modules. The equivalence relation is equivariant PL-homeomorphism in the domain.

Write Iso(X) for the set of isotropy groups which appear in X and (Iso (X)) for the conjugacy classes. When X is Z-restricted Iso(X) is closed under conjugation and inersection. Let $m\colon (\mathrm{Iso}(X)) \to \mathbf{Z}$ be the dimension funtion, $m(H) = \dim X^H$. Define

$$L_m(G; X) = \sum_{(H)}^{\oplus} L_{m(H)}(\mathbf{Z}[NH/H])$$

to be the sum of the simple, oriented L-groups over $(H) \in (\mathrm{Iso}(X))$.

THEOREM B. *Suppose $|G|$ is odd and X is a Z-restricted* PL *G-manifold with simply connected fixed sets. If $X \times D^k$ is topological stable then there is an exact sequence*

$$L_{k+m+1}(G; X) \xrightarrow{\alpha} \tilde{\mathscr{S}}_G(X \times D^k, \partial) \xrightarrow{\eta} [X \times D^k/\partial, F/\mathrm{PL}]^G \xrightarrow{\lambda} L_{k+m}(G; X).$$

For this sequence to be useful on needs a description of the normal invariant term. Let $\widetilde{\mathrm{KO}}_G(X)$ be the reduced equivariant KO-groups, which enumerates differences of isomorphism classes $[\xi] - [\eta]$ of G-vector bundles with $\xi_x \cong \eta_x$ as $\mathbf{R}G_x$-modules.

THEOREM C. *For G of odd order and any G-space,*

$$[X, F/\mathrm{PL}(Z^\infty)]^G \otimes \mathbf{Z}[1/2] = \sum_{(H)}^{\oplus} \mathrm{KO}_{WH}(X^H) \otimes \mathbf{Z}[1/2]$$

with $(H) \in (\mathrm{Iso}(Z))$.

At the prime 2, we do not know the equivariant homotopy type in general. The homotopy types of the fixed sets $(F/\mathrm{PL})^H_{(2)}$ have recently been worked out by M. Nagata, [43]. But only if G is abelian does this determine the equivariant homotopy type. One expects $F/\mathrm{PL}_{(2)}$ to classify Bredon cohomology (on 4-connected spaces). For the homotopy groups one has

THEOREM D. *If G is odd*, $\pi_k(F/\mathrm{PL}(Z^\infty)^G) = \mathscr{L}_k(G; Z)$.

In order to describe the final two resuls, write

$$\mathscr{K}\mathrm{O}_G^{-k}(X) = \sum_{(H)}^{\oplus} \widetilde{\mathrm{KO}}_{WH}^{-k}(X)$$

with $(H) \in (\mathrm{Iso}\,(SZ))$. Using transversality, implied by theorem A, one can define the *structure invariant*

$$\tilde{s}: \tilde{\mathscr{S}}_G(D^k \times SZ, \partial) \to 4\,\mathscr{K}\mathrm{O}_G^{-k}(\mathrm{pt})$$

THEOREM E. *If* $Z \oplus \mathrm{R}^k$ *is stable*, $k > 0$ *and* $|G|$ *is odd then* $\tilde{s}$ *is an isomorphism for* $k \not\equiv 2 \pmod 4$. *It is surjective with kernel an elementary abelian 2-group, if* $k \equiv 2 \pmod 4$.

It would be of considerable interest to generalize this result to $k = 0$, giving $\tilde{\mathscr{S}}_G(SZ)$. For a free representation and G cyclic this is done in [38]. The problem in general is that a join $M * SX$ for $M \in \tilde{\mathscr{S}}_G(SZ)$ is not a loally linear manifold.

THEOREM F. *Suppose* $|G|$ *is odd*, $Z \oplus \mathrm{R}^k$ *is topological stable and* $k > 0$. *After inverting 2 there is a commutative ladder of vertical isomorphisms*

$$\begin{array}{ccccccc}
\mathscr{L}_{k+m}(G; D^k \times SZ) & \xrightarrow{\alpha} & \tilde{\mathscr{S}}_G(D^k \times SZ, \partial) & \xrightarrow{\eta} & [D^k \times SZ/\partial, F/PL]^G & \to & \mathscr{L}_{k+m-1}(G, D^k \times SZ) \\
\downarrow \mathrm{sign} & & \downarrow \tilde{s} & & \downarrow \sigma & & \downarrow \mathrm{sign} \\
\mathscr{K}\mathrm{O}_G^{-k}(DZ, SZ) & \to & \mathscr{K}\mathrm{O}_G^{-k}(DZ) & \to & \mathscr{K}\mathrm{O}_G^{-k}(SZ) & \xrightarrow{\delta} & \mathscr{K}\mathrm{O}_G^{-k-1}(DZ, SZ)
\end{array}$$

The bottom sequence is the exact sequence for the pair (DZ, SZ).

The results above are interrelated, and are proved inductively. Given theorem A for all proper subgroups of a given group G, we get theorm B and C for manifolds without stationary points. One uses this to prove that the suspension of structure sets defines an isomorphism. This in turn implies theorem A for the group G itself and theorems B and C in general, as well as the rest of the mentioned results.

The paper is divided into sections as follows:

§1. Stability of homotopy equivalences.

Let $V \subseteq T$ be a pair of $\mathbb{R}G$-modules (with inner product) and let $SV \subseteq ST$ be the corresponding inclusion of the unit spheres. The spaces of G homotopy equivalences of SV and ST are denoted $F_G(SV)$ *and* $F_G(ST)$. In this section we study the connectivity of the suspension mapping (which takes join with the sphere of the complement U of V in T).

$$\Sigma\colon F_G(SV) \to F_G(ST), \Sigma(f) = f * \mathrm{id}_{SU}.$$

It is convenient to view $F_G(SV)$ as a subspace of the space $\mathrm{Map}_G(SV, SV)$ of all G self maps of SV. Let $\mathrm{Iso}(SV)$ be the set of isotropy subgroups of SV and consider the map

$$\mathrm{Deg}\colon \pi_0 \mathrm{Map}_G(SV, SV) \to \prod_{(H)} \mathbf{Z}$$

which to f assigns the set of degrees $\deg f^H$, where (H) varies over the conjugacy classes from $\mathrm{Iso}(SV)$.

The image of Deg is contained in the Burnside ring $A(G; \mathrm{Iso}(V))$ of formal differences of isomorphism classes of finite G-sets with isotropy groups in $\mathrm{Iso}(V)$, [44].

If $V^G \neq 0$, and V satisfies the following *weak gap conditions* (or codim 3 gap conditions):

(1.1) $\quad \dim V^K > \dim V^H + 2 \geqq 6$ for $K \subsetneqq H$ and $H, K \in \mathrm{Iso}\, SV)$,

then Deg maps onto $A(G; \mathrm{Iso}(SV))$, [14].

It follows from the equivariant Whitehead theorem [16] that $F_G(SV) \subseteq \mathrm{Map}_G$ (SV, SV) consists of the maps with $\deg f^H = \pm 1$ for $H \in \mathrm{Iso}(SV)$. Similarly we write $SF_G(SV)$ for the component with $\deg f^H = +1$ for all H.

The components of $F_G(SV)$ is in one to one correspondance with a subset of the units $A(G; \mathrm{Iso}(SV))^\times$. In general, this is a complicated group, but if G has odd order $A(G)^\times = \{\pm 1\}$. Thus $F_G(SV)$ has two components if $SV^G \neq \emptyset$, and just one component if $SV^G = \emptyset$.

If X and Y have base points, always assumed to be stationary under the action of G, then $\mathrm{Map}_G^*(X, Y)$ denotes the based G-maps. Its homotopy groups can be calculated as

$$\pi_k \mathrm{Map}_G^*(X, Y) = [S^k \wedge X, Y]_*^G$$

The space $\mathrm{Map}_G^* X, Y)$ is the fixed set of the G-space $\mathrm{Map}^*(X, Y)$ of all based maps.

In [16], Hauschild examined the equivariant suspension map

$$Y \to \mathrm{Map}^*(S^U, S^U \wedge Y)$$

where $S^U = U_+$ is the one point compactification of the RG-module U. From [16, Satz (2.4)] we have

PROPOSITION 1.2. *Suppose* $\dim SV^G > 0$ *and* $\mathrm{Iso}(SV) = Iso(ST)$. *Then*

$$\Sigma\colon \mathrm{Map}_G^*(SV, SV) \to \mathrm{Map}_G^*(ST, ST)$$

is k-connected when k satisfies the following two conditions
(i) $k \leqq \dim SV^H - 1$ *if* $H \in \mathrm{Iso}(SV)$ *and* $V^H \neq T^H$
(ii) $k \leqq \dim SV^K - \dim SV^H - 1$ *if* $K \subsetneqq H$ *and* $H, K \in \mathrm{Iso}(SV)$ *and* $V^K \neq T^K$.

When dim $SV^G > 0$ then all components of $\mathrm{Map}_G(SV, SV)$ are homotopy equivalent, so (1.2) above gives a similar result for the connetivity of $F_G(SV) \to F_G(ST)$· Unfortunately, in our applications we need to know the connectivity of Σ when SV and ST have no stationary points. In this case Hauschild's proof has to be modified slightly, but first we introduce the following terminology:

DEFINITION 1.3. An RG-module V is called topologically stable (or is said to satisfy the strong gap conditions) if

$$12 < 2\dim V^H < \dim V^K \text{ for } K \subset H \text{ and } H, K \in \mathrm{Iso}(SV).$$

THEOREM 1.4. *Let* $V \subseteq T$ *be a pair of* RG*-modules with* $\mathrm{Iso}(SV) = \mathrm{Iso}(ST)$. *If k satisfies the two conditions of* (1.2) *then*

$$\Sigma\colon SF_G(SV) \to SF_G(ST)$$

is k-connected. In particular, if V is topologically stable, Σ is k-connected with $k = \min\{\dim V^H - 2 \mid H \in \mathrm{Iso}(SV), V^H \neq T^H\}$.

PROOF. The argument follows closely the proof of [16, Satz 2.3] so we shall be relatively brief. We have

$$\pi_i SF_G(SV) = [D^i \times SV \operatorname{rel} \partial, SV]^G,$$

the G-homotopy classes of maps from $D^i \times SV$ into SV which is equal to the projection on the subspace $\partial = S^{i-1} \times SV$.

Let $X_0 \subset X_1$ be G-subspaces of SV. The obvious analogue of the usual Puppe sequence is the exact sequence

$$\ldots \to [D^i \times X_1 \operatorname{rel} \partial, SV]G \to [D^i \times X_0 \operatorname{rel} \partial, \mathrm{SV}]^G \to$$
$$[D^{i-1} \times X_1 \operatorname{rel}(\partial \cup D^{i-1} \times X_0), SV]^G \to$$

We can use a G-CW decomposition of SV along with the above version of the Puppe sequence. It becomes sufficient to examine Σ on the relative groups corresponding to situation: $X_1 = X_0 \cup G/H \times D^j$. But

$$[D^i \times X_1 \operatorname{rel}(D^i \times X_0 \cup \partial), SV]^G \cong [D^{i+j} \operatorname{rel} \partial, SV]^H$$
$$\cong [S^{i+j}, SV]^H_* = \pi_{i+j}(SV^H)$$

where the subscript $*$ indicates based H-homotopy classes. We note that $j \leqq \dim SV^H$.

Let U be a complement of V in T so $ST = SV * SU$. We choose a G-CW decomposition of SV and let $\{X_\alpha\}$ be the associated filtration of SV, $X_\alpha = X_{\alpha-1} \cup D^j \times G/H$. Taking join with SU defines a filtration $\{X_\alpha * SU\}$ of ST.

We show that Σ is an isomorphism step by step, beginning with the set of G-cells of dimension 0. We have

$$[D^i \times G/H \operatorname{rel} S^{i-1} \times G/H, SV]^G \cong \pi_i(SV^H) = 0$$
$$[D^i \times (G/H * SU) \operatorname{rel} S^{i-1} \times (G/H * SU), ST]^G = 0$$

for $i < \dim V^H - 1$. The first equation is obvious; the second follows by equivariant obstruction theory because, for every $K \subseteq G$,

$$\dim(D^i \times (G/H * SU)^K) < \operatorname{Hur}(ST^K) = \dim T^K - 1. \tag{1.5}$$

Here $\operatorname{Hur}(X)$ denotes the maximal integer k with $\pi_{k-1}(X) = 0$.

For the inductive step, suppose $X_\alpha = X_{\alpha-1} \cup D^j \times G/H, j > 0$. From above,

$$D^i \times X_\alpha \operatorname{rel} D^i \times X_{\alpha-1} \cup \partial, SV]^G \cong \pi_{i+j}(SV^H)$$

and we must calculate the corresponding term when X_α is replaced by $X_\alpha * SU$. There is the excision

$$(D^i \times (X_\alpha * SU), D^i \times (X_{\alpha-1} * SU) \cup \partial)$$
$$\sim (D^i \times (e_j * SU), D^i \times (\partial e_j * SU) \cup S^{i-1} \times (e_j * SU))$$

where $e_j = G/H \times D^j$ and $j = 0$.

We use the model of the join $X * Y = X \times cY/\equiv$, where cY is the (closed) cone on Y and $(x, y) \equiv y$. There is a natural mapping

$$p: X \times (Y * Z) \to (X \times Y) * Z$$

which we can apply to the space $D^j \times (e_j * SU)$ and its two subspaces above:

$$p: (D^i \times (e_j * SU), D^i \times (\partial e_j * SU) \cup S^{i-1} \times (e_j * SU))$$
$$\to (D^{i+j} \times G/H, S^{i+j-1} \times G/H) * SU$$

Since $j > 0$, p^K is a homotopy equivalence. Hence

$$[D^i \times X_\alpha * SU \operatorname{rel} D^i \times X_{\alpha-1} * SU \cup \partial, ST]^G \cong$$
$$[(D^{i+j} \times G/H) * SU \operatorname{rel}(S^{i+j-1} \times G/H) * SU, ST]^G \cong [S^{i+j} * SU, ST]^H_*$$

We must examine the composition

$$[S^{i+j}, SV]^H_* \xrightarrow{\Sigma_1} [S^{i+j} * SU^H, SV * SU^H]^H_* \xrightarrow{\Sigma_2} [S^{i+j} * SU, SV * SU]^H_*.$$

The first mapping Σ_1 is an isomorphism for $i + j < 2 \dim SV^H - 1$, hence for $i < \dim SW^H - 1$, by the usual suspension theorem. It is an epimorphism for $i \leqq \dim SW^H - 1$. The second map, which takes the join with the identity mapping of SU_H (where $U_H \oplus U^H = U$), is always monomorphic. Indeed taking H-fixed points provides a one-sided inverse. It is onto, if and only if

$$\mathrm{Fix}_H: [S^{i+j} * SU, SV * SU]^H_* \to [S^{i+j} * SU^H, SV^H * SU^H]_*$$

is injective. This is the case provided the Bredon cohomology groups

$$H^l_H(S^{i+j} * SU, S^{i+j} * SU^H; \pi_l(ST)) = 0;$$

$(\pi_l(ST)(G/H) = \pi_l(ST^H)$; compare with [8])
This gives the condition $i + j < \dim SW^K$ for $K \subsetneqq H$, $K \in \mathrm{Iso}(SU)$, i.e. $i \leqq \dim SV^K - \dim SV^H - 1$, and completes the proof.

It is illustrative to compare this result with the corresponding result for the space of orthogonal maps. We restrict for convenience to the case where G has odd order. Decompose V and T into their irreducible components

$$V = \sum_{i=1}^{r} n_i \chi_i, \quad T = \sum_{i=1}^{r} (n_i + l_i) \chi_i$$

Assuming $V^G = T^G = 0$, Schur's lemma gives

$$O_G(V) = \sum_{i=1}^{r} U(n_i), \quad O_G(T) = \sum_{i=1}^{r} U(n_i + l_i)$$

Thus the inclusion

$$\Sigma: O_G(V) \to O_G(T)$$

is $\min\{2n_i \mid l_i > 0\}$-connected.

Suppose G is cyclic of odd order. The maximal connectivity of $O_G(T)/O_G(V)$ occurs when the eigenvalues for V of the same order are equal. In this case

(1.6) $$V = \sum n_H \chi_H, \quad T = \sum m_H \chi_H$$

where χ_H is a faithful complex 1-dimensional representation of G/H. The connectivity of $O_G(T)/O_G(V)$ is equal to $\min\{2n_H \mid m_H > n_H\}$. Using the Möbius inversion formula, one can express the connectivity in terms of the dimensions of the fixed sets (cf. [25]):

(1.7) $$\dim V^H = \sum_{H \subseteq K} 2n_K; \quad 2n_H = \sum_{H \subseteq K} \mu(|K:H|) \dim V^K$$

One might suspect that the connectivities of $F_G(T)/F_G(V)$ and $O_G(T)/O_G(V)$ would agree when V and T are as in (1.6) but this is far from being the case, as one can see by comparing (1.4) and (1.7); $F_G(T)/F_G(V)$ is far more connected than one would have any right to believe from the corresponding orthogonal quotient.

Finally consider an $\mathsf{R}G$-module W with $W^G = 0$ and let $T = W \oplus \mathsf{R}^{k+1}$ with trivial action on R^{k+1}. There is a split fibration

$$F_G(ST, S^k) \to F_G(ST) \to F(S^k)$$

so that $F_G(ST) \simeq F(S^k) \times F_G(ST, S^k)$. For orthogonal automorphisms the fibre can be replaced with the automorphisms of SW. In a range of dimensions the same is case for homotopy automorphisms:

PROPOSITION 1.8. *For an* $\mathsf{R}G$-*module* W *with* $W^G = 0$,

$$\Sigma\colon F_G(SW) \to F_G(SW * S^k, S^k)$$

is k-*connected if* $k \leqq \dim W^H - 2$ *for all* H.

The proof is quite similar to that of (1.4) and is left for the reader.

§2 Reduction to surgery – block theory.

Let X be a triangulated space with a PL action of a finite group G. There are Δ-groups $\mathrm{PL}_G(X) \subseteq \widetilde{\mathrm{PL}}_G(X)$ of equivariant automorphisms resp. block automorphisms of X, cf. [34], [29]. Below we often do not distinguish between a Δ-space and its topological realization.

A k-simplex of $\widetilde{\mathrm{PL}}_G(X)$ consists of an equivariant PL automorphism

$$\sigma\colon \Delta^k \times \mathrm{X} \to \Delta^k \times X$$

which preserves the face structure in the sense that σ maps $\Delta_i^k \times X$ onto $\Delta_i^k \times X$ where $\Delta_i^k \subset \Delta^k$ in any face. The k-simplex σ belongs to $\mathrm{PL}_G(X)$ if it commutes with the projection onto Δ^k. If $A \subseteq X$ is a G subcomplex, $\mathrm{PL}_G(X, A)$ is the subset of simplices which is the identity on A. The identity map is the base point if not otherwise indicated.

Let V be an orthogonal representation of G. We decompose it as $V = W \oplus \mathsf{R}^k$ where $W^G = 0$ and suppose that $k \geqq 6$. Our first objective is to relate $\mathrm{PL}_G(V)$ to $\widetilde{\mathrm{PL}}_G(SW)$ where SW denotes the unit sphere. Since $\Delta^k = \Lambda_i^k \times I$ where Λ_i^k is the i'th horn, the Δ-groups above are Kan Δ-groups, and we can do homotopy theory as usual, [34].

Let $\mathrm{PL}_G^{\gamma}(V)$ denote the Δ-group of germs at 0 of automorphisms. A k-simplex is an equivalence class of equivariant embeddings

$$\begin{array}{ccc} \Delta^k \times \mathring{D}_\varepsilon(V) & \xrightarrow{\ \sigma\ } & \Delta^k \times V \\ & \searrow \quad \swarrow & \\ & \Delta^k & \end{array}$$

where $\mathring{D}_\varepsilon(V)$ is the open ε-ball. Two such σ are counted equal if they agree on some smaller ball.

We can take restriction to the fixed set

$$\mathrm{PL}_G(V) \to \mathrm{PL}(V^G).$$

This is a Kan fibration with fiber $\mathrm{PL}_G(V, V^G)$, and since there is an obvious section,

$$\mathrm{PL}_G(V) \simeq \mathrm{PL}_G(V, V^G) \times \mathrm{PL}(V^G).$$

We use the following conventions: $S^{-1} = \emptyset$ and $\emptyset * X = X$.

There are restriction maps

$$r: \mathrm{PL}_G(S(V \oplus \mathsf{R}), S(V^G \oplus \mathsf{R})) \to \mathrm{PL}_G^{\gamma}(V, V^G)$$

$$r': \mathrm{PL}_G(V, V^G) \to \mathrm{PL}_G^{\gamma}(V, V^G)$$

LEMMA 2.1. *The restrictions r, r' are homotopy equivalences, so* $\mathrm{PL}_G(S(V \oplus \mathsf{R}), S(V^G \oplus \mathsf{R})) \simeq \mathrm{PL}_G(V, V^G)$.

PROOF. The maps r and r' have the Kan extension property, and their fibers are contractible by the Alexander trick. The images of r and r' are full sets of components in the base space, cf. [41, p. 125], so it suffices to check that each component is hit. Let

$$f: (D_\varepsilon V, D_\varepsilon V^G) \to (V, V^G)$$

be a G embedding. We show that the complement $M = V_+ - f(\mathring{D}_\varepsilon V)$ is standard in V_+; then f can be extended to a PL automorphism of V_+ by coning ($V_+ = S(V \oplus \mathsf{R})$ is the one-point compactification).

Locally linear PL G-disc's such as M, are classified by their Whitehead torsion invariant, [30]. Thus, we must calculate $\tau_G(M)$. By definition, this is the equivariant torsion of the h-cobordism which arises from M by removing a small linear disc around an interior fixed point. Since f is PL, $\tau_G(f(D_\varepsilon V)) = \tau_G(D_\varepsilon V) = 0$. Moreover,

$$\tau_G(M) + \tau_G(f(D_\varepsilon V)) = \tau_G(DV \times I) = 0,$$

and $\tau_G(M) = 0$ as claimed.

Let $\mathscr{P}\mathrm{PL}_G(SV)$ be the (realization of the) Δ-set of equivariant pseudo-isotopies. A k-simplex is a G-automorphism

$$\sigma: \Delta^k \times SV \times I \to \Delta^k \times SV \times I$$

which commutes with the projection onto Δ^k, is the identity on $\Delta^k \times SV \times 0$ and maps $\Delta^k \times SV \times 1$ into itself.

LEMMA 2.2. *For each orthogonal representation V, the suspension*

$$\Sigma\colon \mathrm{PL}_G(SV, SV^G) \to PL_G(S(V \oplus \mathsf{R}), S(V^G \oplus \mathsf{R}))$$

has homotopy theoretic fiber $\mathscr{P}\mathrm{PL}_G(SV, SV^G)$

PROOF. Consider the diagram

$$\begin{array}{ccc} \mathrm{PL}_G(SV, SV^G) & \xrightarrow{\Sigma} & \mathrm{PL}_G(S(V \oplus \mathsf{R}), S(V^G \oplus \mathsf{R})) \\ \simeq \uparrow r_1 & & \simeq \downarrow r \\ \mathrm{PL}_G(DV, DV^G) & \xrightarrow{r_2} & \mathrm{PL}_G^{\gamma}(V, V^G) \end{array} \tag{2.3}$$

The map r_1 is a homotopy equivalence (by the Alexander trick) with inverse c (coning). The restriction r_2 *is* a Kan fibration (onto) and its fibre is homotopy equivalent to $\mathscr{P}\mathrm{PL}_G(\mathrm{SV})$. This follows from Lemma 2.1. Since $\mathscr{P}\mathrm{PL}(\mathsf{R}^k) \simeq \Omega(\mathrm{PL}(\mathsf{R}^{k+1})/\mathrm{PL}(\mathsf{R}^k))$, and

$$\mathrm{PL}(\mathsf{R}^{k+1})/\mathrm{PL}(\mathsf{R}^k) \simeq \mathrm{O}(k+1)/\mathrm{O}(k) * C_k \tag{2.4}$$

with $\pi_i(C_k) = 0$ $i \leqq k+2$, cf. [11], [18], $\mathscr{P}\mathrm{PL}(\mathsf{R}^k)$ is $(k-2)$-connected. The following is a simple application of the regular neighbourhood theorem cf. [2, §11] or [27, Remark (5.4)].

LEMMA 2.5. *For any representation V, $\pi_k\mathscr{P}\mathrm{PL}_G(V, V^G) = 0$ when $k \leqq \dim V^G$.*

Consider the Kan fibration

$$\mathrm{PL}_G(SV \times I, SV^G \times I \cup \partial) \to PL_G(SV, SV \cup DV^G) \to \mathrm{PL}_G^{\gamma}(V, V^G)$$

Since $\mathrm{PL}_G(DV, SV \cup DV^G)$ is contractible by the Alexander trick, and since

$$r\colon \mathrm{PL}_G(S(V \oplus \mathsf{R}), S(V^G \oplus \mathsf{R})) \to \mathrm{PL}_G^{\gamma}(V, V^G)$$

is a homotopy equivalence by (2.3), it follows that

$$\Omega\mathrm{PL}_G(S(V \oplus \mathsf{R}), S(V^G \oplus \mathsf{R})) \simeq \mathrm{PL}_G(SV \times I, SV^G \times I \cup \partial).$$

The same argument gives the following loop lemma (cf. [1]):

LEMMA 2.6. *There are homotopy equivalences*

$$\Omega\mathrm{PL}_G(S^k * SW \times D^l, S^k \times D^l \cup \partial) \simeq \mathrm{PL}_G(S^{k-1} * SW \times D^{l+1}, S^{k-1} \times D^{l+1} \cup \partial),$$

$$\Omega\mathscr{P}\mathrm{PL}_G(S^k * SW \times D^l, S^k \times D^l \cup \partial) \simeq \mathrm{PL}_G(S^{k-1} * SW \times D^{l+1}, S^{k-1} \times D^{l+1} \cup \partial),$$

for $k \geqq 0$.

The next result is analogous to Proposition 1.8, but note the change from automorphisms to block automorphisms.

PROPOSITION 2.7. *For* $i \leqq \dim V^G$

$$\pi_i \widetilde{\mathrm{PL}}_G(SW) \cong \pi_i \mathrm{PL}_G(V, V^G)$$

where $W \oplus V^G = V$.

PROOF. By definitions we have the exact sequence

$$\pi_0 \mathscr{P}\mathrm{PL}_G(X) \to \pi_0 \mathrm{PL}_G(X) \to \pi_0 \widetilde{\mathrm{PL}}_G(X) \to 0. \tag{2.8}$$

Lemma 2.2 and Lemma 2.5 give

$$\pi_i \mathrm{PL}_G(V, V^G) \cong \pi_i \mathrm{PL}_G(W \oplus \mathsf{R}^i, \mathsf{R}^i)$$

for each i. In the exact sequence

$$\pi_i \mathscr{P}\mathrm{PL}_G(\mathsf{R}^{i-1} \oplus W, \mathsf{R}^{i-1}) \to \pi_i \mathrm{PL}_G(\mathsf{R}^{i-1} \oplus W, \mathsf{R}^{i-1}) \xrightarrow{\Sigma} \pi_i \mathrm{PL}_G(\mathsf{R}^i \oplus W, \mathsf{R}^i),$$

the image of Σ is equal to $\pi_0 \widetilde{\mathrm{PL}}_G(SW \times D^i, \partial)$ by (2.8). But Σ is also onto, so

$$\pi_0 \widetilde{\mathrm{PL}}_G(SW \times D^i, \partial) \cong \pi_i \mathrm{PL}_G(W \oplus \mathsf{R}^i, \mathsf{R}^i).$$

Finally the definition of homotopy groups in a Kan complex gives directly $\pi_0 \widetilde{\mathrm{PL}}_G(SW \times D^i, \partial) = \pi_i \widetilde{\mathrm{PL}}_G(SW)$. This completes the proof.

The cone map

$$c\colon \widetilde{\mathrm{PL}}_G(SV, SV^G) \to \widetilde{\mathrm{PL}}_G(DV, DV^G)$$

is defined by an iterated coning process on each simplex: if c is already defined on k-simplices then for a $(k+1)$-simplex $\sigma\colon \Delta^{k+1} \times SV \to \Delta^{k+1} \times SV$, $c(\sigma)$ is partially given on $\partial(\Delta^{k+1} \times DV)$ and one more coning defines $c(\sigma)$ on all of $\Delta^{k+1} \times DV$. There is an obvious product

$$\mu\colon \widetilde{\mathrm{PL}}_G(DV_1, DV_1^G) \times \widetilde{\mathrm{PL}}_G(DV_2, DV_2^G) \to \widetilde{\mathrm{PL}}_G(D(V_1 \oplus V_2), D(V_1 \oplus V_2)^G)$$

We may use the cone map and restriction to the boundary to get a corresponding

$$\mu\colon \widetilde{\mathrm{PL}}_G(SV_1 SV_1^G) \times \widetilde{\mathrm{PL}}_G(SV_2, SV_2^G) \to \widetilde{\mathrm{PL}}_G(S(V_1 \oplus V_2), S(V_1 \oplus V_2)^G) \tag{2.9}$$

In particular we get the k fold suspension map, and from the above

PROPOSITION 2.10. $\Sigma^k\colon \widetilde{\mathrm{PL}}_G(SW) \to \widetilde{\mathrm{PL}}_G(SW * S^k, S^k)$ *is a homotopy equivalence for each* $k \geqq 0$.

The propositions (2.1), (2.7) and (2.10) together give.

COROLLARY 2.11. *The inclusion map*

$$\mathrm{PL}_G(S^k * SW, S^k) \to \widetilde{\mathrm{PL}}_G(S^k * SW, S^k)$$

is $(k+1)$*-connected.*

The inverse of

$$\pi_i \mathrm{PL}_G(S^k * SW, S^k) \to \pi_i \widetilde{\mathrm{PL}}_G(S^k * SW, S^k), \quad i \leqq k$$

is given by the Haefliger-Poenaru differential along the fixed set,

$$d\colon \pi_i \widetilde{\mathrm{PL}}_G(S^k * SW, S^k) \to \pi_i \mathrm{PL}_G(S^k * SW, S^k)$$

cf. [17], [33, (5.1)].

The traditional way to study block automorphism spaces is to compare them with homotopy automorphism, and use the fact that the quotient space can be studied via surgery theory. This can be done also in the equivariant setting; where the reduction to surgery theory is based upon the equivariant s-cobordism theorem, [30].

Recall first that for homotopy automorphisms there is no difference between the block space and the ordinary space, cf. [31, Theorem 5.8]:

$$F_G(SW) \xrightarrow{\simeq} \tilde{F}_G(SW).$$

The k'th homotopy group of the quotient $\tilde{F}_G(SW)/\widetilde{\mathrm{PL}}_G(SW)$ consists of equivalence classes of G-homotopy equivalences

$$f\colon D^k \times SW \longrightarrow D^k \times SW$$

such that ∂f is a PL G-homeomorphism. In fact, we can assume, and this will often be covenient, that the restriction of ∂f to $D^{k-1}_- \times SW$ is the identity where $D^{k-1}_- \subset S^{k-1}$ is the lower hemisphere.

Two maps $f_0, f_1\colon D^k \times SW \to D^k \times SW$ represent the same homotopy class if there exists an equivariant PL homeomorphism h of $D^k \times SW$ such that $f_0 \simeq {}_G f_1 \circ h$.

On the other hand, the object one can evaluate by surgery theory is the structure set $\tilde{\mathscr{S}}_G(D^k \times SW, \partial)$. An element of this is represented by a pair (M, t) consisting of a locally linear PL G-manifold M and a G-simple homotopy equivalence

$$t\colon M \to D^k \times SW$$

with $\partial t\colon \partial M \to S^{k-1} \times SW$ a PL-homeomorphism. Two such elements (M_1, t_1) and (M_2, t_2) are equivalent if there exists a PL-homeomorphism $h\colon M_1 \to M_2$ with $t_2 \circ h$ and t_1 G-homotopy equivalent.

PROPOSITION 2.12. *Suppose W satisfies the* codim 3 *gap conditions. Then*

$$\pi_k(\tilde{F}_G(SW)/\widetilde{\mathrm{PL}}_G(SW)) \cong \tilde{\mathscr{S}}_G(D^k \times SW, \partial), \qquad k \geqq 1$$
$$= 0 \qquad , \qquad k = 0$$

PROOF. An element of the left hand side is represented by *a* G-homotopy equivalence $f\colon D^k \times SW \to D^k \times SW$ which is a PL-homeomorphism on the boundary. We can assume $f \mid D^{k-1}_- \times SW = \mathrm{id}$ and must show that f is G-simple (i.e. has vanishing G Whitehead torsion). Consider

$$0 \to C_*(D^k \times SW, D^{k-1}_- \times SW)$$
$$\to C_*(f, \partial_- f) \to C_{*-1}(D^k \times SW, D^{k-1}_- \times SW) \to 0$$

Since f_- is a PL G-homeomorphism its torsion $\tau(C_*(f_-)) = 0$ and hence $\tau(f) = \tau(C_*(f, f_-))$. But the exact sequence gives

$$\tau(C_*(f, f_-)) = \tau(C_*(D^k \times SW, D^{k-1}_- \times SW))$$
$$- \tau(C_{*-1}(D^k \times SW, D^{k-1}_- \times SW)) = 0.$$

It remains to show that every element of $\tilde{\mathscr{S}}_G(D^k \times SW, \partial)$ is represented by an automorphism. Let

$$t\colon (M, \partial M) \to (D^k \times SW, S^{k-1} \times SW)$$

be a G-simple homotopy equivalence. Let $\partial_+ M = t^{-1}(D^{k-1}_+ \times SW)$ and $\partial_- M = t^{-1}(D^{k-1}_- \times SW)$ and view M as an h-cobordism from $\partial_- M$ to $\partial_+ M$, trivial on the boundary. One has

$$\tau(C_*(M, \partial_- M)) = \tau(C_*(t, \partial_- t)) = \tau(C_*(t)) = 0$$

It follows from the equivariant s-cobordism theorem in the locally linear PL-category that M is PL-homeomorphic with $\partial_- M \times I$, hence that $M \cong_G D^k \times SW$. This completes the proof.

The restriction of μ in (2.9) to $\widetilde{\mathrm{PL}}_G(SV_1, SV_1^G) \times (*)$ defines a suspension map

$$\Sigma\colon \widetilde{\mathrm{PL}}_G(SV_1, SV^G{}_1) \to \widetilde{\mathrm{PL}}_G(S(V_1 \oplus V_2), S(V_1 \oplus V_2)^G)$$

compatible with

$$\Sigma\colon F_G(SV_1, SV_1^G) \to F_G(S(V_1 \oplus V_2), S(V_1 \oplus V_2)^G)$$

under the identification $F_G \simeq \tilde{F}_G$.

This gives a map from $\tilde{F}_G(SV_1)/\widetilde{\mathrm{PL}}_G(SV_1)$ to $\tilde{F}_G(S(V_1 \oplus V_2))/\widetilde{\mathrm{PL}}_G(S(V_1 \oplus V_2))$.

We can evaluate on homotopy groups and use Proposition 2.12 to get the suspension map of the structure set

$$\Sigma\colon \tilde{\mathscr{S}}_G(D^k \times SV_1, \partial) \to \tilde{\mathscr{S}}_G(D^k \times S(V_1 \oplus V_2), \partial), \quad k > 0. \tag{2.13}$$

The Σ in (2.13) will play an important role in this work; it has the following more direct definition. Let

$$f\colon D^k \times SV_1 \to D^k \times SV_1$$

represent an element $[f] \in \tilde{\mathscr{S}}_G(D^k \times SV_1, \partial)$; suppose $\partial_- f = f \mid D_-^{k-1} \times SV_1 = \text{id}$ and $\partial_+ f = f \mid D_+^{k+1} \times SV_1$ is a PL G-homeomorphism. Write

$$D^k \times DV_1 = \text{Cone}(D^k \times SV_1 \cup D_-^{k-1} \times DV_1 \cup \text{cone}(\partial D_+^{k-1} \times DV_1 \cup D_+^{k-1} \times SV_1))$$

Two conings extend f to a G-map

$$Df: D^k \times DV_1 \to D^k \times DV_1 \tag{2.14}$$

which is a PL-homeomorphism on $S^{k-1} \times DV_1$. Then $\Sigma(f)$ is the restriction of $Df \times DV_2$ to $D^k \times S(V_1 \oplus V_2)$.

§3 The special case of relatively free representations.

This section presents a special case of our main theorem about the connectivity of the Stiefel spaces $\text{PL}_G(T)/\text{PL}_G(V)$.

We suppose that G is cyclic of odd order and that V and T are relatively free $\mathbf{R}G$-modules, say $V = W \oplus \mathbf{R}^l$ with G acting freely on $W - \{0\}$ and similarly for T. The special case is easier than the general case because it uses only non-equivariant surgery. In fact, the relevant calculations are very similar to the ones made in by C. T. C Wall in [38, chapter 14E].

Throughout the section the following gap conditions are assumed

$$\dim W \geqq 6, \ l = \dim V^G > 2 \tag{3.1}$$

From Proposition 2.12, and since SW is G-free

$$\begin{aligned}\pi_k(\tilde{F}_G(SW)/\widetilde{\text{PL}}_G(SW)) &= \tilde{\mathscr{S}}_G(D^k \times SW, S^{k-1} \times SW)\\ &= \tilde{\mathscr{S}}(D^k \times LW, S^{k-1} \times LW)\end{aligned}$$

where LW is the lens space, $LW = SW/G$. Wall examined in [38] $\tilde{\mathscr{S}}(D^k \times LW, S^{k-1} \times LW)$ for $k = 0$; we generalize to $k > 0$.

The main tool is the surgery exact sequence (in the simple category):

$$\begin{aligned}\ldots \xrightarrow{\lambda} L_{k+w}(\mathbf{Z}G) \xrightarrow{\alpha} \tilde{\mathscr{S}}_G(D^k \times SW, \partial)\\ \xrightarrow{\eta} [D^k \times LW/\partial, F/\text{PL}] \xrightarrow{\lambda} \ldots\end{aligned} \tag{3.2}$$

The term $[D^k \times LW/\partial, F/\text{PL}]$ has odd torsion depending on W but its 2-torsion is only a function of k, and often it is convenient to do calculations with (3.2) 'at 2' and 'away from 2' separately. We use the notation

$$\tilde{\mathscr{S}}_G(D^k \times SW, \partial)_{\text{odd}} = \tilde{\mathscr{S}}_G(D^k \times SW, \partial) \otimes \mathbf{Z}[\tfrac{1}{2}].$$

$$\tilde{\mathscr{S}}_G(D^k \times SW, \partial)_{(2)} = \tilde{\mathscr{S}}_G(D^k \times SW, \partial) \otimes \mathbf{Z}_{(2)}.$$

Since $SK_1(ZG) = 0$ for cyclic groups the term $L_i(ZG)$ is equal to the group $L_i'(ZG) = L_i'(ZG, \alpha, 1)$ with α the usual anti-involution, and it is calculated in [39]. We recall the result.

If i is odd $L_i(ZG) = 0$. If *i is even we have the G*-signature homomorphism

$$\mathrm{sign}_G : L_{2k}(ZG) \to RG.$$

Is is injective for k even, has kernel Z/2 (Arf-invariant) when k is odd and has image

$$\mathrm{Im}(\mathrm{sign}_G) = 4(1 + (-1)^k \psi^{-1})RG. \tag{3.3}$$

where ψ^{-1} is complex conjugation. This follows from [39, Corollary 2.4.3] and [39, Theorem 2.2.1]. Since the right hand side in (3.3) will appear very often in this paper we give it a special name,

$$RO_{2k}(G) = (1 + (-1)^k \psi^{-1})RG, \tag{3.4}$$

and note that

$$RO_{2k}(G) \otimes Z[\tfrac{1}{2}] = KO_G^{-2k}(pt) \otimes Z[\tfrac{1}{2}] = KO_G^{-2k}(pt; Z[\tfrac{1}{2}]).$$

The term $[D^k \times LW/\partial, F/PL]$ in (3.2) can also be expressed in terms of K-theory away from 2 by a theorem of D. Sullivan:

There is an isomorphism

$$\sigma: [D^k \times LW/\partial; F/PL]_{\mathrm{odd}} \xrightarrow{\cong} \widetilde{KO}^{-k}(LW, Z[\tfrac{1}{2}]),$$

and $\widetilde{KO}^{-k}(LW; Z[\tfrac{1}{2}]) \cong \widetilde{KO}_G^{-k}(SW)$.

LEMMA 3.5. *For $k > 0$, $\tilde{\mathscr{S}}_G(D^{2k-1} \times SW, \partial) = 0$. There is an exact sequence*

$$0 \longrightarrow \tilde{L}_{2k+m}(ZG) \xrightarrow{\alpha} \tilde{\mathscr{S}}_G(D^{2k} \times SW, \partial) \xrightarrow{\eta} [D^{2k} \times LW/\partial, F/PL] \longrightarrow 0$$

where $m = \dim_R W$ and $\tilde{L}_*(ZG) = \mathrm{cok}\,(L_*(Z) \to L_*(ZG))$.

PROOF. Since $L_{2i-1}(ZG) = 0$, the result claimed follows from (3.2) once we prove that

$$\lambda: [D^{2k+1} \times LW/\partial, F/PL] \to L_{2k+m}(ZG)$$

has image $L_{2k+m}(Z)$. The source of λ is equal to $Z \oplus T$ or $Z/2 \oplus T$, according to the parity of $k + m/2$ with T a torsion group of odd order.

The Z or Z/2 is represented as the Milnor or Kervaire surgery problems, and their surgery obstructions generate $L_{2k+m}(Z) \subseteq L_{2k+m}(ZG)$.

It remains to determine the extension in Lemma 3.5. This uses the ρ-invariant, which we redefine below.

By Proposition 2.12, an element of $\tilde{\mathscr{S}}_G(D^k \times SW, \partial)$ is represented as a homotopy equivalence

$$f: D^k \times SW \to D^k \times SW$$

which is a PL-homeomorphism on $D^{k-1}_+ \times SW$ and the identity on $D^{k-1}_- \times SW$.

By coning, f extends to a map (cf. (2.13))

$$Df: D^k \times DW \to D^k \times DW.$$

We form the PL G-manifolds

$$X(f) = (D^k \times SW) \cup_{\partial f} (D^k \times SW);$$

$$\text{(3.6)}\quad DX(f) = (D^k \times DW) \cup_{\partial_1(Df)} (D^k \times DW); \partial_1(Df) = Df \mid S^{k-1} \times DW.$$

One can view $X(f)$ as an SW-block bundle over S^k and $DX(f)$ as its corresponding disc block bundle; $\partial DX(f) = X(f)$.

The action of G on $X(f)$ is free, so it determines a bordism class in $\Omega^{\mathrm{PL}}_{k+m-1}(BG)$. Since PL/$O$ is rationally a point, $\Omega^{\mathrm{PL}}_*(BG) \otimes \mathbf{Q} \cong \Omega^{SO}_*(BG) \otimes \mathbf{Q}$, and in particular $\Omega^{\mathrm{PL}}_{2i-1}(BG)$ is finite. Assuming k is even, a suitable number of the manifold $X(f)$ bounds a free G-manifold, say $\partial N_r = r \cdot X(f)$.

Let $\mathrm{sign}_G(N_r)$ be the G-signature of the cup product form defined on the image of $H^n(N_r, \partial N_r; \mathbf{R})$ in $H^n(N_r; \mathbf{R})$ where $2n = \dim N_r$. It is an element of $\mathbf{R}G$. The traditional way to define the ρ-invariant is to take $1/r\, \mathrm{sign}_G(N_r)$ as an element of $\tilde{\mathbf{R}}G = \mathrm{cok}(\mathbf{R}1 \to \mathbf{R}G)$. However, we prefer to work with a slightly different definition, better related to the problems at hand.

We fix an orientation for W and let $e(W) \in \mathrm{KO}_G^{-m}(\mathrm{pt}; \mathbf{Z}[\frac{1}{2}])$ be the Euler class of W considered as an oriented G-bundle over a single point. This Euler class depends on the choice of Thom class. We use the one related to the symbol class of the index operator, called Δ_M in [22].

If W is considered a $\mathbf{C}G$-bundle, compatible with the chosen orientation, then

$$e(W) = \prod \frac{1 - \chi_i}{1 + \chi_i} \quad \text{when} \quad W = \sum{}^{\oplus} \chi^i$$

with $\dim_{\mathbf{C}} \chi_i = 1$. We define

$$\text{(3.7)}\qquad \tilde{\rho}_G([f]) = \frac{1}{r} \mathrm{sign}_G(N_r) \cdot e(W)$$

The multiplication with $e(W)$ in (3.7) guarantees that $\tilde{\rho}_G[f]$ is a well-defined element of $\mathbf{R}G \otimes \mathbf{Q}$ rather than in the quotient $\widetilde{\mathbf{R}G} \otimes \mathbf{R}$. Indeed, the possible variation in N_r is a closed PL G-manifold. Since G is finite,

$$\Omega^{\mathrm{PL}}_*(\mathrm{pt}) \otimes \mathbf{Q} = \Omega^{\mathrm{PL}}_*(BG) \otimes \mathbf{Q},$$

and $\mathrm{sign}_G(M \times G) = \mathrm{sign}(M) \cdot [\mathbf{R}G], \mathrm{sign}_G(N_r)$ is determined in $\mathbf{R}G \otimes \mathbf{Q}$ up to a multiple of the regular representation. But $e(W)\cdot[\mathrm{R}G] = 0$, so $\tilde{\rho}_G([f])$ is well-defined.

LEMMA 3.8. *For $k + m = 2n$, the diagram*

$$\begin{array}{ccc} \tilde{L}_{2n}(\mathbf{Z}G) & \xrightarrow{\alpha} & \tilde{\mathcal{S}}_G(D^k \times SW, \partial) \\ \downarrow \mathrm{sign}_G & & \downarrow \tilde{\rho}_G \\ \tilde{\mathrm{R}}G & \xrightarrow{e(W)\cdot} & \mathrm{R}G \otimes \mathbf{Q} \end{array}$$

is commutative.

PROOF. Let $x \in \tilde{L}_{2n}(\mathbf{Z}G)$ be represented by the G-free normal cobordism class $(F, \hat{F})$. Then

$$F:(U; U_-, U_+, V) \to (D^k \times SW \times I, D^k \times SW \times 0, D^k \times SW \times 1, S^{k-1} \times SW \times I)$$

with $F \mid U_- = \mathrm{id}$ and $F \mid V = f$, a PL G-homeomorphism, and $F \mid U_+$ is a G-simple homotopy equivalence, cf. [38, Theorem 6.5]. By the s-cobordism theorem we may assume that $U_+ = D^k \times SW$. The manifold

$$\hat{U} = U \cup_f D^k \times SW \times I$$

is a cobordism from $X(f)$ to $S^k \times SW$, so we can calculate

$$\begin{aligned} \tilde{\rho}_G([f]) &= \mathrm{sign}_G(\hat{U} \cup_\partial D^{k+1} \times SW \times \{0\})e(W) \\ &= \mathrm{sign}_G(U)\cdot e(W). \end{aligned}$$

We have assumed that $(F, \hat{F})$ represents $x \in L_{2n}(\mathbf{Z}G)$. Hence $\mathrm{sign}_G(U) = \mathrm{sign}_G(x)$. Since $\alpha(x) = (U_+, F \mid U_+)$ by the definitions, the result follows.

We next study the suspension from (2.13).

LEMMA 3.9. *Let W_1 be a second free representation of G. The diagram*

$$\begin{array}{ccc} \tilde{\mathcal{S}}_G(D^{2k} \times SW, \partial) & \xrightarrow{\Sigma} & \tilde{\mathcal{S}}_G(D^{2k} \times S(W \oplus W_1), \partial) \\ & \tilde{\rho}_G \searrow \quad \swarrow \tilde{\rho}_G & \\ & \mathrm{R}(G) \otimes \mathbf{Q} & \end{array}$$

is commutative.

PROOF. We start with the block G-homotopy equivalence $f: D^{2k} \times SW \to D^{2k} \times SW$ with ∂f a block PL-homeomorphism and form $X(f)$ *and* $X(\Sigma(f))$. From the paragraph following (2.13) it follows that

$$X(\Sigma(f)) = DX(f) \times SW_1 \cup X(f) \times DW_1.$$

Let $\partial N = r \cdot X(f)$ and $\partial N_1 = t \cdot SW_1$, and define

$$U = (N \cup r \cdot DX(f)) \times (N_1 \cup t \cdot DW_1) - rt \cdot \operatorname{int}(DX(f) \times DW_1).$$

Then $\partial U = rt \cdot \partial(D\xi_f \times DW_1) = rt \cdot S\xi_{\Sigma(f)}$, so we can use U to calculate the $\tilde{\rho}$-invariant of $\Sigma(f)$. From the additivity theorem [7, (7.1)],

$$\operatorname{sign}_G(U) = \operatorname{sign}_G(N) \cdot \operatorname{sign}_G(N_1)$$

and the result follows because $\frac{1}{t} \operatorname{sign}_G(N_1) = e(W_1)^{-1}$ by [38, chapter 14C].

We now prove our variant of Wall's theorem from [38, ch 14E]:

THEOREM 3.10. *Let $U \subseteq W$ be two relatively free* R*G-modules with $U^G = W^G = 0$ and dimension at least 6. Then*

$$\Sigma: \tilde{S}_G(D^{2k} \times SU, \partial) \to \tilde{S}_G(D^{2k} \times SW, \partial)$$

is an isomorphism.

PROOF. We have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{L}_{2k+m}(ZG) & \xrightarrow{\alpha^U} & \tilde{S}_G(D^{2k} \times SU, \partial) & \xrightarrow{\eta} & [D^{2k} \times LU/\partial, F/\mathrm{PL}]^G & \longrightarrow & 0 \\ & & & & \Big\downarrow \Sigma & & \Big\uparrow i^* & & \\ 0 & \longrightarrow & \tilde{L}_{2k+n}(ZG) & \xrightarrow{\alpha^W} & \tilde{S}_G(D^{2k} \times SW, \partial) & \xrightarrow{\eta} & [D^{2k} \times LW/\partial, F/\mathrm{PL}]^G & \longrightarrow & 0 \end{array}$$

with $i: LU \to LW$, so to check Σ is injective, it suffices to see that $\Sigma \circ \alpha$ is injective. But

$$\rho_G(\Sigma \circ \alpha(x)) = \rho_G(\alpha(x)) = e(U) \cdot \operatorname{sign}_G(\mathrm{x})$$

by (3.8) and (3.9), and

$$\tilde{L}(ZG) \xrightarrow{\operatorname{sign}_G} \tilde{\mathrm{R}}(G) \xrightarrow{e(U)} \mathrm{R}(G) \otimes \mathbb{Q}$$

is a composition of injective maps.

We check surjectivity at 2 and away from 2 separately. At 2, i^* is an isomorphism, so we must show that the image of α^W is contained in the image of $\Sigma \circ \alpha^U$. Since $\tilde{\rho}_G$ is injective on these images, it suffices to see that the image of $\tilde{\rho}_G \circ \Sigma \circ \alpha^U$ contains the image of $\tilde{\rho}_G \circ \alpha^W$. By (3.3), (3.8) and (3.9) this is equivalent to the inclusion

$$4\, e(U)\, \mathrm{RO}_{2k+m}(G)_{(2)} \supseteq 4e(W) \cdot \mathrm{RO}_{2k+n}(G)_{(2)}$$

which holds trivially because $e(W) = e(U)e(\chi)$ when $U \oplus \chi = W$.

Away from 2, i^* has kernel equal to the quotient of the ideals $\langle e(U) \rangle$ and $\langle e(W) \rangle$. This follows from the isomorphisms

$$[D^{2k} \times LU/\partial, F/\mathrm{PL}]^G_{\mathrm{odd}} \cong \mathrm{KO}_G^{-2k}(SU;\mathrm{Z}[\tfrac{1}{2}]) = (1 + (-1)^k\psi^{-1})K_G^{-2k}(SU;\mathrm{Z}[\tfrac{1}{2}])$$
$$K_G^{-2k}(SU;\mathrm{Z}[\tfrac{1}{2}]) \cong \mathrm{R}G/\langle e(U)\rangle.$$

We have $\mathrm{Im}(\Sigma \circ \alpha^U)_{\mathrm{odd}} \supseteq \mathrm{Im}(\alpha^W)_{\mathrm{odd}}$ just as in the 2-local case, so it remains to be seen that

$$\eta(\mathrm{Im}(\Sigma \circ \alpha^U))_{\mathrm{odd}} = (\mathrm{Ker}\, i^*)_{\mathrm{odd}}.$$

This in turn follows by a counting argument, because

$$\tilde{\rho}_G\colon \mathrm{Im}(\Sigma \circ \alpha^U)/\mathrm{Im}(\alpha^W) \to \langle e(U)\rangle/\langle e(W)\rangle$$

is onto by (3.8) and (3.9) on the one hand, and because

$$\eta\colon \mathrm{Im}(\Sigma \circ \alpha^U)/\mathrm{Im}(\alpha^W) \to \langle e(U)\rangle/\langle e(W)\rangle$$

is injective by (3.5) on the other hand; coefficients in $\mathrm{Z}[\frac{1}{2}]$ are understood for both maps.

Finally, $\tilde{S}_G(D^{2k+1} \times SW, \partial) = 0$ since

$$\lambda\colon [D^{2k+1} \times LW/\partial, F/\mathrm{PL}] \to L_{2k+m}(\mathrm{Z}G)$$

is a monomorphism (check at 2 and away from 2).

ADDENDUM 3.11. *For U as in* (3.10), *$\tilde{S}_G(D^{2k} \times SU, \partial)_{\mathrm{odd}}$ is torsion free.*

Proof. If suffices to show that the kernel of

$$\eta\colon \tilde{\mathscr{S}}_G(D^{2k} \times SU, \partial)_{\mathrm{odd}} \to [D^{2k} \times LU/\partial, F/\mathrm{PL}]_{\mathrm{odd}} \to \pi_{2k}(F/\mathrm{PL})_{\mathrm{odd}}$$

is torsion free. We have the exact sequence

$$0 \to \tilde{L}_{2k+m}(\mathrm{Z}G)_{\mathrm{odd}} \to \mathrm{Ker}\, \tilde{\eta}_G \to \mathrm{Tor}\, \widetilde{\mathrm{KO}}_G^{-2k}(SU)_{\mathrm{odd}} \to 0,$$

so we must prove that

$$\tilde{\rho}_G\colon \mathrm{Ker}\, \tilde{\eta}_G \to \mathrm{IO}_{2k}(G)_{\mathrm{odd}}$$

detects the image of $\mathrm{Ker}\, \tilde{\eta}_G$ in $\mathrm{Tor}\, \mathrm{KO}_G^{-2k}(SU)_{\mathrm{odd}}$. Indeed, if this is true for any given U, then it will be true for any larger $W \supseteq U$ by the argument given at the end of (3.10). If $k \geqq 2$ then we can get the argument started with $SU = S^1$. For $k = 1$, we have the usual low dimensional problems with the surgery exact sequence, but can use the trick from [38, chapter 14.E] to start the induction.

THEOREM 3.12. *Let $V \subseteq T$ be relatively free* $\mathrm{R}G$*-modules which satisfies* (3.1). *Then $\pi_{k-1}(\mathrm{PL}_G(T)/\mathrm{PL}_G(V)) = 0$ for $k \leqq \min(\dim V^G, \dim V - \dim V^G - 1)$.*

Proof. Let $T = W \oplus T^G$, $V = U \oplus V^G$. By (2.7) it suffices to show

$$\pi_{k-1}(\widetilde{\mathrm{PL}}_G(SW)/\widetilde{\mathrm{PL}}_G(SU)) = 0$$

in the stated range. Consider the diagram

$$\begin{array}{ccccccc}
\dots \to \pi_{k-1}\widetilde{\mathrm{PL}}_G(SU) & \to & \pi_{k-1}\tilde{F}_G(SU) & \to & \tilde{\mathscr{S}}_G(D^{k-1} \times SU, \partial) \to \dots \\
\downarrow \Sigma_2 & & \downarrow \Sigma_1 & & \downarrow \Sigma \\
\dots \to \pi_{k-1}\widetilde{\mathrm{PL}}_G(SW) & \to & \pi_{k-1}\tilde{F}_G(SW) & \to & \tilde{\mathscr{S}}_G(D^{k-1} \times SW, \partial) \to \dots
\end{array}$$

We have just seen that Σ is an isomorphism for all k; Σ_1 is an isomorphism for $k \leqq \dim SU - 1$ and an epimorphism for $k = \dim SU$ by Theorem 1.4, hence the same is true for Σ_2.

If V in (3.12) is topologically stable in the sense of (1.3) then the minimum is equal to $l = \dim V^G$, and we see that $\mathrm{PL}_G(T)/\mathrm{PL}_G(V)$ is $(\dim V^G - 1)$-connected (when $T^G \neq V^G$). Indeed, the connectivity is larger than or equal to $\dim V^G$ by (3.12), and it cannot be larger because $\mathrm{PL}(T^G)/\mathrm{PL}(V^G)$ has the same connectivity as $O(T^G)/O(V^G)$.

REMARK 3.13. Given $[f] \in \tilde{\mathscr{S}}_G(D^{2k} \times SW, \partial)$ we can define a class function on G by

$$\begin{aligned}
\tilde{s}_G([f])(g) &= \tilde{\rho}_G([f](g) \qquad & g \neq 1 \\
&= \tilde{\eta}_G([f]) \qquad & g = 1
\end{aligned}$$

where $\chi(g)$ denotes the character value of χ at g. We prove in section 9 that

$$\tilde{s}_G\colon \tilde{\mathscr{S}}_G(D^{2k} \times SW, \partial) \otimes \mathbf{Z}[\tfrac{1}{2}] \to \mathrm{RO}_{2k}(G) \otimes \mathbf{Z}[\tfrac{1}{2}]$$

is an isomorphism.

§4 Non-transversality for $G = \mathbf{Z}/2$.

The main result of the last section, Theorem 3.11, implies stable G-transversality for $G = \mathbf{Z}/p$ when p is an odd prime, cf. [26]. We shall see in this section that this result cannot be generalized to $G = \mathbf{Z}/2$. Thus the question of transversality for even order groups is much more subtle. It would be of considerable interest to understand the failure of stable transversality for even order groups.

In the rest of this section $G = \mathbf{Z}/2$ and $U \subseteq W$ will be $\mathbf{R}G$-modules with $U^G = W^G = 0$. We consider

$$\Sigma\colon \tilde{F}_G(SU)/\widetilde{\mathrm{PL}}_G(SU) \to \tilde{F}_G(SW)/\widetilde{\mathrm{PL}}_G(SW)$$

from (2.13).

LEMMA 4.1. *For* $i \leqq \dim U - 2$, $\pi_{i+1}(\Sigma) \cong \pi_i(\widetilde{\mathrm{PL}}_G(SW)/\widetilde{\mathrm{PL}}_G(SU))$.

PROOF. We have the square

$$\begin{array}{ccc} \tilde{F}_G(SU) & \longrightarrow & \tilde{F}_G(SU)/\widetilde{\mathrm{PL}}_G(SU) \\ \downarrow \Sigma_1 & \Phi & \downarrow \Sigma \\ \tilde{F}_G(SW) & \longrightarrow & \tilde{F}_G(SW)/\widetilde{\mathrm{PL}}_G(SW) \end{array}$$

and corresponding exact homotopy sequences

$$\ldots \to \pi_{i+1}(\Phi) \to \pi_i \widetilde{\mathrm{PL}}_G(SU) \to \pi_i \widetilde{\mathrm{PL}}_G(SW) \to \ldots$$

$$\ldots \to \pi_{i+1}(\Phi) \to \pi_{i+1}(\Sigma_1) \to \pi_{i+1}(\Sigma) \to \ldots$$

From (1.4), Σ_1 is $(\dim U - 2)$-connected, so

$$\pi_{i+1}(\Sigma) \cong \pi_i(\Phi) \text{ for } i \leqq \dim U - 2$$

and the first sequence gives $\pi_i(\Phi) \cong \pi_i(\widetilde{\mathrm{PL}}_G(SW)/\widetilde{\mathrm{PL}}_G(SU))$

The quotient $P^n = SW/G$ is a real projective space; the classification of homotopy manifold structures on P^n has been examined by Wall and Lopez de Medrano in [38] and [21]. We need the corresponding results for $\tilde{\mathscr{S}}_G(D^k \times SW, \partial) = \tilde{\mathscr{S}}(D^k \times P^n, \partial)$. Here are some preliminary facts.

Browder and Livesay defined an invariant

$$\beta : \tilde{\mathscr{S}}(D^k \times P^n, \partial) \to BL_{k+n}((-1)^{n+1});$$

the groups $BL_i(\pm)$ have the values:

$$BL_{4i-1}(+) = \mathbf{Z}, BL_{4i+1}(+) = \mathbf{Z}/2, BL_{2i}(+) = 0$$

and $BL_i(+) = BL_{i+2}(-)$.

Let $f: M \to D^k \times P^n$ represent an element of $\tilde{\mathscr{S}}(D^k \times P^n, \partial)$. By transversality f induces a normal map $f_0: M_0^{k+n-1} \to D^k \times P^{n-1}$ and $\beta(f)$ is the obstruction to perform ambient surgery on $f_0 = f \mid M_0$ to obtain a homotopy equivalence. The ordinary surgery obstruction of f_0 is related to $\beta(f)$ by the formula

$$\lambda(f_0) = l(\beta(f)),$$

where l is a certain map

$$l = l_{n+1}^{\pm} : BL_{n+1}(\mp) \to L_n(G, \pm).$$

Here we have written $L_n(G, \pm)$ for the groups $L_n(\mathbf{Z}G, \alpha^{\pm}, 1)$ where α^+ is the usual oriented anti-involution and α^- the non-oriented one. The values of the groups are

$$L_{2i}(G, -) = \mathbf{Z}/2, L_{2i+1}(G, -) = 0,$$

$$L_{4i}(G, +) = \mathbf{Z} \oplus \mathbf{Z}, L_{4i+1}(G, +) = 0,$$

$$L_{4i+2}(G, +) = \mathbf{Z}/2, L_{4i+3}(G, +) = \mathbf{Z}/2$$

and $l_n^{\pm}$ is onto where it can be, except in two cases: l_{4i+1}^{+} has image $L_{4i}(1) \subseteq L_{4i}(G, +)$ and $l_{4i-1}^{-} = 0$.

The invariant β is related to the suspension homeomorphism by the Browder-Livesay exact sequence:

$$(4.2) \qquad \tilde{\mathscr{S}}(D^k \times P^n, \partial) \xrightarrow{\Sigma} \tilde{\mathscr{S}}(D^k \times P^{n+1}, \partial) \to BL_{n+k+1}((-1)^{n+1}).$$

For proofs of the cited fracts we refer the reader to [21] and [38].

We shall also need a few facts about the surgery exact sequence,

$$\ldots \to L_{k+n+1}(G, \varepsilon) \xrightarrow{\alpha} \tilde{\mathscr{S}}(D^k \times P^n, \partial) \xrightarrow{\eta}$$
$$[D^k \times P^n/\partial, F/\mathrm{PL}] \xrightarrow{\lambda_{k+n}} L_{k+n}(G, \varepsilon) \to \ldots$$

where $\varepsilon = (-1)^n$. The surgery obstruction λ_{k+n} can be determined by the following surgery formulae, cf. [9]. There are cohomology classes

$$K_{2i} \in H^{2i}(F/\mathrm{PL}; L_{2i}(1));$$

$(L_{4i+2}(1) = \mathbf{Z}/2, L_{4i}(1) = \mathbf{Z})$ such that

(i) if $n + k \equiv 2 (\mathrm{mod}\, 4)$, or $n + k \equiv 0 (\mathrm{mod}\, 4)$ and $\varepsilon = -1$, then

$$\lambda_{k+n}(g) = \langle g^*(K_{4*-2}), [D^k \times P^n] \rangle$$

(ii) if $n + k \equiv 0 (\mathrm{mod}\, 4)$ and $\varepsilon = +1$, then

$$\lambda_{k+n}(g) = \langle g^*(K_{4*}) V^2(P^n), [D^k \times P^n] \rangle$$

(iii) if $n + k \equiv 3 (\mathrm{mod}\, 4)$ and $\varepsilon = +1$, then

$$\lambda_{k+n}(g) = \langle g^*(K_{4*-2}) V^2(P^n) \theta, [D^k \times P^n] \rangle$$

where $\theta \in H^1(P^n; \mathbf{Z}/2)$ is the generator. Moreover,

$$K_{2*}: F/\mathrm{PL} \to \prod_{n \geqq 1} K(\mathbf{Z}, 4n) \times K(\mathbf{Z}/2, 4n-2)$$

induces an isomorphism on 2-local homotopy groups except on π_4 where K_4 gives multiplication by 2; the bottom 2-stage Postnikov system of F/PL has k-invariant $\beta \mathrm{Sq}^2(K_2)$, [24].

PROPOSITION 4.3. *For* $n \geqq 2$

(i) $\tilde{\mathscr{S}}(D^{2k+1} \times P^{2n-1}, \partial) = \mathbf{F}_2\{\bar{\beta}_{2k+1} \mid i \equiv (-1)^{k+1} (4), 1 \leqq i \leqq 2n-3\}$.

(ii) $\tilde{\mathscr{S}}(D^{2k} \times P^{2n-1}, \partial) = L_{2k}(1) \oplus \langle \beta_{2k+2n-1} \rangle \oplus \mathbf{F}_2\{\bar{\beta}_{2k+i} \mid i \equiv 1 (2), 3 \leqq i \leqq 2n-3\}$, *where* $\langle \beta_{2k+2n-1} \rangle = \mathbf{Z}$ *if* $k + n \equiv 0 (2)$ *and* $\langle \beta_{2k+2n-1} \rangle = \mathbf{Z}/2$ *if* $k + n \equiv 1 (2)$.

(iii) *The double suspension* $\Sigma^2: \tilde{\mathscr{S}}(D^j \times P^{2n-1}, \partial) \to \tilde{\mathscr{S}}(D^j \times P^{2n+1}, \partial)$ *maps* $\beta_{2k+2n-1}$ *to* $\bar{\beta}_{2k+2n-1}$ *and preserves* $\bar{\beta}_{2k+i}$ *and* $L_{2k}(1)$. *It is injective unless* $j = 2k$ *and* $k + n \equiv 0 (2)$ *where the kernel is* $2\langle \beta_{2k+2n-1} \rangle$.

PROOF. We give the proof in case (ii), leaving the entirely similar case (i) to the reader. The composite

$$\tilde{\mathscr{S}}(D^{2k} \times P^{2n-1}, \partial) \xrightarrow{\ \eta\ } [D^{2k} \times P^{2n-1}/\partial, F/\mathrm{PL}] \longrightarrow \pi_{2k}(F/\mathrm{PL})$$

is a surjection. This gives the summand $L_{2k}(1)$ in $\tilde{\mathscr{S}}(D^{2k} \times P^{2n-1}, \partial)$. We now proceed by introduction over n. Suppose $k + m$ is even and consider the diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow \tilde{L}_0(\mathbf{Z}/2,+) & \xrightarrow{\alpha_0} & \tilde{\mathscr{S}}(D^{2k} \times P^{2m-1}, \partial) & \xrightarrow{\eta_0} & [S^{2k} \wedge P_+^{2m-1}, F/\mathrm{PL}] & \xrightarrow{\lambda_3} & L_3(\mathbf{Z}/2,+) \\
 & & \downarrow \Sigma_0 & & \uparrow \mathrm{Res}_0 & & \\
 & 0 \longrightarrow & \tilde{\mathscr{S}}(D^{2k} \times P^{2m}, \partial) & \xrightarrow{\eta_1} & [S^{2k} \wedge P_+^{2m}, F/\mathrm{PL}] & \xrightarrow{\lambda_1} & L_0(\mathbf{Z}/2,-) \\
 & & \downarrow \Sigma_1 & & \uparrow \mathrm{Res}_1 & & \\
 & 0 \longrightarrow & \tilde{\mathscr{S}}(D^{2k} \times P^{2m+1}, \partial) & \xrightarrow{\eta_2} & [S^{2k} \wedge P_+^{2m+1}, F/\mathrm{PL}] & \xrightarrow{\lambda_1} & 0 \\
 & & \downarrow \Sigma_2 & & \uparrow \mathrm{Res}_2 & & \\
 & 0 \longrightarrow & \tilde{\mathscr{S}}(D^{2k} \times P^{2m+2}, \partial) & \xrightarrow{\eta_3} & [S^{2k} \wedge P_+^{2m+2}, F/\mathrm{PL}] & \xrightarrow{\lambda_2} & L_2(\mathbf{Z}/2,-) \\
 & & \downarrow \Sigma_3 & & \uparrow \mathrm{Res}_3 & & \\
0 \longrightarrow \tilde{L}_0(\mathbf{Z}/2,+) & \xrightarrow{\alpha_0} & \tilde{\mathscr{S}}(D^{2k} \times P^{2m+3}, \partial) & \xrightarrow{\eta_0} & [S^{2k} \wedge P_+^{2m+3}, F/\mathrm{PL}] & \xrightarrow{\lambda_3} & L_3(\mathbf{Z}/2,+)
\end{array}$$

We have used the surgery formulae to show that

$$\lambda_0 : [S^{2k+1} \wedge P_+^{2m-1}, \mathrm{F}/PL] \to L_0(\mathbf{Z}/2,+)$$

$$\lambda_2 : [S^{2k+1} \wedge P_+^{2m+1}, F/\mathrm{PL}] \to L_2(\mathbf{Z}/2,+)$$

maps onto $L_*(1)$ with cokernel $\tilde{L}_*(\mathbf{Z}/2,+)$; this group is equal to $\mathbf{Z}$ if $* \equiv 0\,(4)$ and 0 otherwise.

The restriction maps Res_i are all surjective. It is an easy consequence of the sugery formulae (i), (ii) and (iii) above that

$$\mathrm{Res}_i : \mathrm{Ker}\,\lambda_i \to \mathrm{Ker}\,\lambda_{i-1}$$

is surjective for all i; it has kernel $\mathbf{Z}/2$ except for $i = 2$ where Res_2 is an isomorphism.

From the Browder-Livesay sequence (4.2), Σ_0 and Σ_2 are epimorphisms and α_0 is split injective. It follows that

$$\eta_1 \circ \Sigma_0 \circ \alpha_0 : \tilde{L}_0(\mathbf{Z}/2,+) \to [S^{2k} \wedge P_+^{2m}, F/\mathrm{PL}]$$

has image $\mathbf{Z}/2$ corresponding to the top part, $H^{2k+2m}(S^{2k} \wedge P_+^{2m}; \mathbf{Z})$ of the range.

The surgery formulae and the formula $l_n^{\pm} \circ \beta_n = \lambda_{n-1}$ give a calculation of the cokernels of Σ_{2i+1}:

$$\operatorname{coker} \Sigma_1 \cong BL_1(+),$$

$$\operatorname{coker} \Sigma_3 \cong BL_3(+).$$

This completes the inductive step; (i) is similar, and (iii) follows.

By (2.7), (4.1) and (4.3) we have

COROLLARY 4.4. *Let $V \subset T$ be $\mathsf{R}G$-modules with free parts* $\dim U = 2n$, $\dim W = 2n + 2r$. *Suppose V and T satisfy the stability conditions*, $\dim V^G \leqq \dim U - 2$, $\dim T^G \leqq \dim W - 2$. *With the conventions of* (4.3) *and in degrees less than* $\dim V^G$ *we have,*

(i) $\pi_{2k}(\mathrm{PL}_G(T)/\mathrm{PL}_G(V)) =$

$$2\langle \beta_{2k+2n-1} \rangle \oplus \mathsf{F}_2\{\bar\beta_{2k+i} \mid i \equiv (-1)^{k+1}(4), -1 \leqq i - 2n \leqq 2r - 3\}$$

(ii) $\pi_{2k-1}(\mathrm{PL}_G(T)/\mathrm{PL}_G(V)) =$

$$\langle \beta_{2k+2n+2r-1} \rangle \oplus \mathsf{F}_2\{\bar\beta_{2k+i} \mid i \equiv 1(2), 1 \leqq i - 2n \leqq 2r - 3\}$$

The non-vanishing of the homotopy groups in Corollary 4.4 prevents stable equivariant transversality for $G = \mathsf{Z}/2$ in the locally linear PL-category. Examples can be constructed as follows:

Let $U \subset W$ be free G-representations with spheres $S^{2n-1} = S(U)$, $S^{2n+2r-1} = S(W)$ and let $\alpha \in \pi_i \widetilde{\mathrm{PL}}_G(S^{2n+2r-1})$ $(i < 2n - 3)$ be an element which projects non-trivially to $\pi_i(\widetilde{\mathrm{PL}}_G(S^{2n-1}))$. Let ξ_α be the PL block over S^{i+1} associated to α, $D\xi_\alpha$ the corresponding block disc bundle. The total space of $D\xi_\alpha$ is a compact PL G-manifold M with boundary and $M^G = S^{i+1}$. The restriction of the tangent bundle M to M^G can be identified with the stabilization of ξ_α,

$$TM \mid M^G \cong \xi_\alpha \oplus TS^{i+1}.$$

Let χ be the complement of U in W and consider the constant map $f\colon M \to \chi$ with $f(M) = 0$. We claim that f is not G-homotopic to a G-transversal map. This follows from [26] where we proved that f is G-homotopic to a G-transversal map if and only if there is a G-section in the bundle $\mathrm{Epi}(TM, W)$. But a G-section implies a section over the fixed point $M^G = S^{i+1}$, that is, a section of

$$\mathrm{PL}_G(T)/\mathrm{PL}_G(V) \to \mathrm{Epi}_G(\xi_\alpha \oplus TS^{i+1}, W) \to S^{i+1}$$

where $T = W \oplus \mathsf{R}^{i+1}$ and $V = U \oplus \mathsf{R}^{i+1}$. The obstruction to a section is

$$\partial_*([S^{i+1}]) \in \pi_i(\mathrm{PL}_G(T)/\mathrm{PL}_G(V)).$$

Moreover,

$$\partial_*([S^{i+1}]) = J_*(\alpha)$$

where $J\colon \widetilde{\mathrm{PL}}_G(S(W)) \to \mathrm{PL}_G(T, T^G) \to \mathrm{PL}_G(T)/\mathrm{PL}_G(V)$. In the commutative diagram

$$\begin{array}{ccc}
\pi_i\,\widetilde{\mathrm{PL}}_G(SW) & \xrightarrow{\tilde{J}_*} & \pi_i(\widetilde{\mathrm{PL}}_G(SW)/\widetilde{\mathrm{PL}}_G(SU)) \\
\downarrow\cong & \searrow^{J_*} & \downarrow\cong \\
\pi_i\,\mathrm{PL}_G(T, T^G) & \longrightarrow & \pi_i(\mathrm{PL}_G(T)/\mathrm{PL}_G(V)),
\end{array}$$

the vertical maps are isomorphisms by Proposition 2.7. Since $\tilde{J}_*(\alpha) \neq 0$ it follows that $\partial_*([S^{i+1}]) \neq 0$. We have proved:

THEOREM 4.5. *For $G = \mathbf{Z}/2$, G-transversality does not hold in the locally linear* PL-*category, even stably.*

§5 The equivariant surgery exact sequence.

As detailed in [13], existence of a good surgery exact sequence is based upon three requirements. One needs a transversality theorem in order to relate cobordism classes of normal maps to homotopy theory; a stability result for the bundle theory in question in order to determine the actual normal bundles to the embedded spheres one wants to surger out, given the stable ones and finally; one needs a suitable $\pi - \pi$ theorem in order to give faithful obstructions in the relevant surgery obstruction groups.

In the equivariant setting the $\pi - \pi$ theorem is due to Dovermann, Petrie and Rothenberg [13], so is at hand. The transversality theorem is only available in some stable setting, and the bundle desuspension resuls depend on the precise connectivity of certain "Stiefel manifolds". In the smooth G-category, the Stiefel manifolds are the usual linear ones, in general not sufficiently connected; one must make unpleasant assumptions on the structure of the normal bundles to the fixed sets (cf. [13]). In the (locally linear) topological category transversality fails, even stably. But in the locally linear PL-category we have all three requiremens available, when $|G|$ is odd and under the strong gap conditions (1.3).

In this section we set up the surgery sequence under the assumption (5.1) below, which will in turn be proved inductively over the group order $|G|$ when it is odd.

ASSUMPTION 5.1. Let $V \subseteq T$ be topogogical stable $\mathbf{R}G$-modules in the sense of (1.3). Then $\mathrm{PL}_G(T)/\mathrm{PL}_G(V)$ is ($\dim V^G - 1$)-connected.

We first derive the suitable bundle desuspension results from (5.1). All bundles considered will be locally linear $G - \mathbf{R}^n$ bundles in the PL category (abbreviated G-bundles), cf. [19] and [26] for a discussion. Any such bundle is classified by a G-map $X \to \mathrm{BPL}_n(G)$. If X^G is connected the fibers ξ_x for $x \in X^G$ are all PL-equivalent, hence linearly equivalent to a fixed $\mathbf{R}G$-module W by [12], and ξ is classified by a G-map $X \to \mathrm{BPL}(W)$.

PROPOSITION 5.2. *Let ξ be a G-bundle over a G-CW complex X and V an $\mathbf{R}G$-module. Supose V and each fibre ξ_x are topological stable and that $V \subset \xi_x$. Then $\xi \cong \eta \oplus V$ for some G-bundle η provided* $\dim X^H \leqq \dim \eta_x^H$ *for each $H \subset G_x$.*

PROOF. We want to construct the dotted arrow

$$\begin{array}{ccc} & & \mathrm{BPL}_n(G) \\ & \overset{\eta}{\nearrow} & \big\downarrow \oplus V \\ X & \longrightarrow & \mathrm{BPL}_{n+|V|}(G) \end{array}$$

by equivariant obstruction theory. Suppose η given on $A \subset X$. Let B be the union of A and a G-cell $D^i \times G/H$ with isotropy group H. The obstruction to extend η over B lies in

$$\pi_i(\mathrm{BPL}_H(\eta_x) \to \mathrm{BPL}_H(\eta_x \oplus V)) \cong \pi_{i-1}(\mathrm{PL}_H(\eta_x \oplus V)/\mathrm{PL}_H(\eta_x))$$

where η_x is the $\mathbf{R}H$-module defined by $\eta \mid \partial(D^i \times G/H)$. Apply (5.1).

There is a similar result for desuspending PL G-bundle isomorphisms which we now formulate. First we make

DEFINITION 5.3. A PL G-bundle ξ over the G-complex X is called G-stable if for each $\Gamma \subseteq G$ and $x \in X^\Gamma$ its fibre ξ_x is topological stable as $\mathbf{R}\Gamma$-module, and $\dim \xi_x^\Gamma > \dim X^\Gamma$ for $x \in X^\Gamma$.

PROPOSITION 5.4. *Let ξ and η be stable* PL *G-bundles over X. Suppose f: $\xi \oplus \zeta \xrightarrow{\cong} \eta \oplus \zeta$ is a* PL *G-homeomorphhism for some stable G-bundle ξ. Then $\xi \cong \eta$ via a map g such that $g \oplus \mathrm{id}$ is G-isotopic to f.*

PROOF. A G-bundle equivalence $\xi \to \eta$ can be considered as a section of the G-fibration $\mathrm{PL}(\xi, \eta)$ over X with fibre $\mathrm{PL}(\xi_x, \eta_x)$, the space of PL-homeomorphisms. There is an obvious inclusion of fibrations.

$$\begin{array}{ccc} \mathrm{PL}(\xi, \eta) & \xrightarrow{\Sigma} & \mathrm{PL}(\xi \oplus \zeta, \eta \oplus \zeta) \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

We check by equivariant obstruction theory that a G-section of $\mathrm{PL}(\xi \oplus \zeta, \eta \oplus \zeta)$ can be compressed into a G-section of $\mathrm{PL}(\xi, \eta)$. The obstructions lie in

$$H_G^i(X; \pi_i(\mathrm{PL}_{G_x}(\xi_x \oplus \zeta_x, \eta_x \oplus \zeta_x)/\mathrm{PL}_{G_x}(\xi_x, \eta_x))),$$

which all vanish by (5.1).

If one sharpens the stability conditions in (5.3) to read $\dim X^\Gamma < \dim \xi_x^\Gamma - 1$ then the constructed PL G-homeomorphism g between ξ and η in (5.4) will be unique up to isotopy, by a similar argument.

For the rest of the section we *fix a representation* Z and consider only G-bundles ξ which are subordinate Z in the sense that each fibre ξ_x is an $\mathrm{R}G_x$ submodule of $Z^{\oplus k}$.

DEFINITION 5.5. A G-bundle ξ is called Z-restricted if it is subordinate to Z and $\mathrm{Iso}(Z, G_x) = \mathrm{Iso}(\xi_x, G_x)$ for all x. A manifold is called Z-restricted of its tangent bundle is.

It is easy to see that each Z-restricted G-bundle ξ over a finite $G-\mathrm{CW}$ complex has a Z-restricted complement, $\xi \oplus \zeta = V$ with $V \subset Z^{\oplus k}$ being Z-restricted (and stable).

Let $\mathcal{N}_G(X)$ denote the equivalence classes of triples (ξ, t, η) where ξ and η are PL G-bundles over X and $t: \xi_x \to \eta_x$ is a proper G_x-homotopy equivalence for all x. Two triples (ξ_i, t_i, η_i) are equivalent if

$$(\xi_1 \oplus \zeta_1, t_1 \oplus 1, \eta_1 \oplus \zeta_1) \cong_G (\xi_2 \oplus \zeta_2, t_2 \oplus 1, \eta_2 \oplus \zeta_2)$$

for PL G-bundles ζ_1 and ζ_2. Here $\cong_G$ indicates a G-bundle PL homeomorphism compatible with the two proper homotopy equivalences.

We need a restricted version $\tilde{\mathcal{N}}_G(X)$ of $\mathcal{N}_G(X)$. It consists of equivalence classes of triples (ξ, t, η) such that

(5.6) (i) ξ and η are Z-restricted
(ii) $\xi_x \cong \eta_x$ as $\mathrm{R}G_x$-modules.

Each element of $\tilde{\mathcal{N}}_G(X)$ can be represented by a triple of the form (ξ, t, V) with ξ Z-restricted and $V (= X \times V)$ the trivial bundle of a Z-restricted $\mathrm{R}G$-module.

Forgetting t we can consider stable equivalence classes of pairs (ξ, V), with $\xi_x \cong_{G_x} V_x$, or what is the same, equivalence classes of virtual bundls of the form $\xi - V$. They are classified by the G-space

$$\mathrm{BPL}(G; Z^\infty) = \mathrm{colim}\, \mathrm{BPL}(V)$$

where V varies over the Z-restricted representations.

We can weaken the equivalence relation and consider the fiber-wise one-point compactifications $\xi^c = S(\xi \oplus \mathrm{R})$, $V^c = S(V \oplus \mathrm{R})$ up to equivariant fibre homotopy equivalence. The corresponding classifying space is $BF(G; Z^\infty)$, and we define $F/\mathrm{PL}(G; Z^\infty)$ to be the homotopy fiber in the G-fibration sequence

$$F/\mathrm{PL}(G; Z^\infty) \to \mathrm{BPL}(G; Z^\infty) \to BF(G; Z^\infty). \tag{5.7}$$

The fixed point sets are

$$\mathrm{BPL}(G; Z^\infty)^G = \mathrm{colim}\, \mathrm{BPL}_G(V)$$

$$BF(G; Z^\infty)^G = \mathrm{colim}\, BF_G(V)$$

$$F/\mathrm{PL}(G; Z^\infty)^G = \operatorname{colim} F_G(V)/\mathrm{PL}_G(V),$$

where V runs through the Z-restricted representations. Often G and Z will be clear from the context and we will write BPL, BF instead of $\mathrm{BPL}(G; Z^\infty)$, $BF(G; Z^\infty)$. Similarly the fixed sets will sometimes be abbreviated to BPL_G, BF_G etc.

The spaces $\mathrm{BPL}_G(V)$, hence BPL_G and BF_G, are connected because they are classifying spaces. Moreover

$$\pi_1(BF_G(V)) = \pi_0(F_G(V)) \subseteq A(G)^\times = \{\pm 1\}$$

since $|G|$ is odd. From (5.7) it follows that $F_G(V)/\mathrm{PL}_G(V)$ is connected. Therefore the spaces F/PL, BPL and BF are G-connected.

The usual argument shows that $\tilde{\mathcal{N}}_G(X)$ is classified by $F/\mathrm{PL}(G; Z^\infty)$ in the sense that we have

PROPOSITION 5.8. $\tilde{\mathcal{N}}_G(X) \cong [X, F/\mathrm{PL}]^G, F/\mathrm{PL} = F/\mathrm{PL}(G; Z^\infty)$.

If X has a base point $x_0 \in X^G$ then we can consider also the based homotopy set, $[X, F/\mathrm{PL}]^G_*$. The free homotopy set in (5.8) is a quotient of the based set by the action of $\pi_1(F_G/\mathrm{PL}_G)$, but the action is trivial since F_G/PL_G has a multiplication. Thus $[X, F/\mathrm{PL}]^G_* = [X, F/\mathrm{PL}]^G$.

Under special circumstances there is a more intricate interpretation of $\tilde{\mathcal{N}}_G(X)$ than (5.8). Suppose X is a Z-restricted G-manifold, and that it is G-oriented in the sense that each fixed set is oriented.

DEFINITION 5.9. A Z-reduced G-normal map $(f, \hat{f})$ over X is a G-bundle diagram

$$\begin{array}{ccc} TM \oplus U & \xrightarrow{\hat{f}} & T(X) \oplus \zeta \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X, \end{array}$$

where M is a Z-restricted G-oriented PL G-manifold, U is an $\mathsf{R}G$-module, f has degree 1 on each fixed point set and $T_xM \cong T_{f(x)}X$ as $\mathsf{R}G_x$-modules for each $x \in M$. If M, X have boundaries $f|\partial M$ is assumed to be an equivariant PL homeomorphism.

REMARK 5.10. If M and X are topological stable then each reduced normal map is the stabilization of $(f, \hat{f})$ with $\hat{f}$: $TM \oplus \mathsf{R} \to \xi \oplus \mathsf{R}$. This follows from (5.2) and (5.4). Note also, if each component of each fixed set X^H has non-empty boundary that the local conditions $T_xM \cong T_{f(x)}X$ in (5.9) are automatically satisfied, since PL-homeomorphic representations are linearly equivalent by a result of deRham, [30].

THEOREM 5.11. *Let X be a Z-restricted G-oriented* PL *G-manifold which satisfies* (5.5) *and is topological stable.*

Then $\tilde{\mathcal{N}}_G(X/\partial X)$ is in $1-1$ correspondance with normal cobordism classes of Z-reduced G-normal maps over X.

PROOF. We refer to [38, page 110] for $G = 1$. The argument in general is similar. The map from $\tilde{\mathcal{N}}_G(X)$ to $[X/\partial X, F/\mathrm{PL}]^G$ is induced by collapsing onto a regular neighbourhood of an embedding of M in ζ, homotopic to f. The map in the other direction requires G-transversality which is available from [26, Theorem 4.4] under the assumption (5.1).

Define $\tilde{\mathcal{S}}_G(X, \partial X)$ to be the set of equivalence classes of G-simple G-homotopy equivalences $f\colon (M, \partial M) \to (X, \partial X)$ with

(i) ∂f PL-homeomorphism
(ii) $T_x M \cong T_{f(x)} X$ as $\mathrm{R}G_x$-modules.

There is a map as usual

$$\eta_G\colon \tilde{\mathcal{S}}_G(X, \partial X) \to \tilde{\mathcal{N}}_G(X/\partial X)$$

and we are interested in its kernel and cokernel.

Let m: Iso $(X) \to \mathrm{Z}$ be the dimension function of X, $m(H) = \dim X^H$. Define the equivariant simple L-groups by

$$\mathcal{L}_m(G; X) = \sum\nolimits^{\oplus} L_{m(H)}(\mathrm{Z}[NH/H]), (H) \in (\mathrm{Iso}\,(X)),$$

the direct sum over the conjugacy classes (H) of the usual oriented (simply) surgery obstruction groups from [38]. In [42], a geometric definition of $\mathcal{L}_m(G; X)$ is given in the spirit of [38, chapter 9], and the above sum decomposition is derived (for $|G|$ odd).

THEOREM 5.12. *Let X be a Z-restricted,* PL *G-manifold. If X^H is simply connected for all H and $X \times I$ is topologically stable then* (*under assumption* (5.1)) *there is an exact sequence*

$$\tilde{\mathcal{S}}_G(D^1 \times X, \partial) \xrightarrow{\eta} \tilde{\mathcal{N}}_G(D^1 \times X/\partial) \xrightarrow{\lambda} \mathcal{L}_{m+1}(G; X) \xrightarrow{\alpha}$$
$$\xrightarrow{\alpha} \tilde{\mathcal{S}}_G(X, \partial) \xrightarrow{\eta} \tilde{\mathcal{N}}_G(X/\partial) \xrightarrow{\lambda} \mathcal{L}_m(G; X)$$

The proof of (5.12) is a formal consequence of the geometric definition of equivariant L-theory, and the $\pi - \pi$ theorem [13]. We refer the reader to [42], Part II, where a more general theorem is proved.

ADDENDUM 5.13. Let G be an odd order group and suppose (5.1) satisfied for all proper subgroups. Then (5.2), (5.4), (5.11) and (5.12) remains valid provided $X^G = \phi$.

PROOF. The first two Propositions are based on obstruction theory and when $X^G = \phi$ the only homotopy groups which appear are $\pi_*(\mathrm{PL}_\Gamma(T)/\mathrm{PL}_\Gamma(V))$ with $\Gamma \in \mathrm{Iso}(X)$, hence $\Gamma \neq G$. The theorems (5.11) and (5.12) are based on transversality, hence by [26] on obstruction theory. Again it only involves the above homotopy groups with $\Gamma \neq G$.

§6 Normal invariants: The Sullivan mapping.

This section relates the set of Z-restricted normal invariants $\tilde{\mathcal{N}}_G(X)$ from (5.8) with equivariant K-theory under the standing assumption (5.1). We assume for convenience that $1 \in \mathrm{Iso}(Z)$.

Let $\widetilde{\mathrm{KO}}_G(X)$ be the reduced equivariant orthogonal K-theory. It is the subgroup of $\mathrm{KO}_G(X)$ consisting of differences $[\xi] - [\eta]$ with $\xi_x \cong \eta_x$ as $\mathrm{R}G_x$-modules for all x in the base space. The classifying space for $\widetilde{\mathrm{KO}}_G(X)$ is denoted $BO(G)$. It has connected H-fixed point sets for $H \subseteq G$, and is the G-connected cover of the classifying space for $\mathrm{KO}_G(X)$.

It is a consequence of the s-cobordism theorem that the structure set $\tilde{\mathscr{S}}_G(DU, SU)$ is trivial (for topological stable representations) so (5.12) gives

$$\tilde{\mathcal{N}}_G(DU/SU) = \sum\nolimits^{\oplus} L_{m(H)}(\mathrm{Z}[NH/H])$$

with H running over the conjugacy classes in $\mathrm{Iso}(U) = \mathrm{Iso}(Z)$ and $m(H) = \dim U^H$.

The G-signature defines a homeomorphism

$$\mathrm{sign}_G\colon L_k(\mathrm{Z}G) \to 4\mathrm{RO}_k(G)$$

with notation as in (3.4), and $\mathrm{RO}_{2l+1}(G) = 0$. There is a corresponding equivariant version when we define

$$\mathscr{RO}_m(G) = \sum\nolimits^{\oplus} \mathrm{RO}_{m(H)}(WH);\ \ m\colon \mathrm{Iso}(Z) \to \mathrm{Z}; (H) \in (\mathrm{Iso}(Z)).$$

The invariant sign $= \sum^{\oplus} \mathrm{sign}_{WH}$ defines an isomorphism

(6.1) $$\mathrm{sign}\colon \mathscr{L}_m(G) \otimes \mathrm{Z}[\tfrac{1}{2}] \xrightarrow{\ \cong\ } 4\mathscr{RO}_m(G) \otimes \mathrm{Z}[\tfrac{1}{2}]$$

With a vew towards the G-trivial case, analysed by D. Sullivan, cf. [24], one expects an isomorphism

$$\tilde{\mathcal{N}}_G(X) \otimes \mathrm{Z}[\tfrac{1}{2}] \xrightarrow{\ \cong\ } \sum\nolimits^{\oplus} \widetilde{\mathrm{KO}}_{WH}(X^H; \mathrm{Z}[\tfrac{1}{2}]).$$

Consider

$$X \mapsto \widetilde{\mathrm{KO}}_{WH}(X^H),\ \ WH = NH/H$$

as a functor on G-CW complexes. It is representable by a G-space which we denote $B(G; H)$, cf. [20]. Thus

$$\widetilde{KO}_{WH}(X^H) = [X, B(G; H)]^G \tag{6.2}$$

with $B(G; 1) = BO(G)$.

When G is abelian there is even a simple description of $B(G; H)$ by means of families of subgroups, based on the following

LEMMA 6.3. *Let $\mathscr{F}$ be a family of subgroups of G, closed under intersection. Let X and Y be G-spaces and write*

$$X_{\mathscr{F}} = \bigcup_{X^H}, H \notin \mathscr{F}.$$

Then

$$[X_{\mathscr{F}}, Y]^G \cong [X, E\mathscr{F} * Y]^G,$$

where $E\mathscr{F}$ is the "acyclic classifying space" of $\mathscr{F}$.

PROOF. Since $E\mathscr{F} = *E(G/K)$, $K \in \mathscr{F}$ it follows that $(E\mathscr{F})^\Gamma$ is empty when $\Gamma \notin \mathscr{F}$ and contractible if $\Gamma \in \mathscr{F}$. Thus $(E\mathscr{F})_{\mathscr{F}} = \varnothing$ and there is a map

$$\phi: [X, E\mathscr{F} * Y]^G \to [X_{\mathscr{F}}, Y]^G.$$

This is a bijection by equivariant obstruction theory because the Bredon cohomology groups

$$H_G^*(X, X_{\mathscr{F}}; \pi_*(E\mathscr{F} * Y))$$

vanishes identically.

We can apply (6.3) with $\mathscr{F}_\Gamma = \{H \subseteq G \mid H \not\supseteq \Gamma\}$. When G is abelian, $X_{\mathscr{F}} = X^\Gamma$ for $\mathscr{F} = \mathscr{F}_\Gamma$ so we can conclude that

$$B(G, H) = E\mathscr{F}_H * BO(G/H) \tag{6.4}$$

when G is abelian. For general G we don't know such a description.

Let G be any odd order group. We define an equivariant map

$$\sigma: F/\mathrm{PL}(G; Z^\infty) \to BO(G)_{\mathrm{odd}}$$

where $BO(G)_{\mathrm{odd}}$ classifies $\widetilde{KO}_G(X; Z[\tfrac{1}{2}]) = \widetilde{KO}_G(X) \otimes Z[\tfrac{1}{2}]$.

The construction is based upon the epimorphism (from [22])

$$\mathrm{KO}_G^*(X; Z[\tfrac{1}{2}]) \twoheadrightarrow \mathrm{Hom}_{\Omega_*^G}(\Omega_*^G(X), \mathrm{RO}(G) \otimes Z[\tfrac{1}{2}]) \tag{6.5}$$

with kernel $\mathrm{Ext}_{RO(G)}(\mathrm{KO}_*^G(X), \mathrm{RO}(G) \otimes Z[\tfrac{1}{2}])$

Here $\Omega_*^G(X)$ denotes the geometric oriented bordism groups of equivalence classes of G-maps $f: M^n \to X$, and all functors are considered $Z/4$-graded. We must construct a homomorphism

$$\sigma_G\colon \Omega_*^G(F/\mathrm{PL}(G;Z^\infty)) \to R0_*(G) \otimes \mathbf{Z}[\tfrac{1}{2}].$$

Let $\gamma\colon M \to X$ represent a cobordism class $\{M,\gamma\}$. There exists a G-manifold P with $\mathrm{sign}_G(\mathsf{P}) = 1$ and $M_{\mathsf{P}} = M \times \mathsf{P}$ topologically stable. Indeed we can take P to be a suitable projective space, cf. [22]. Consider the composition

$$\gamma_{\mathsf{P}}\colon M_{\mathsf{P}} \xrightarrow{\ \mathrm{pr}\ } M \xrightarrow{\ \gamma\ } F/\mathrm{PL}(G;Z^\infty).$$

It determines a normal map

$$\hat{M}_{\mathsf{P}} \xrightarrow{\ f\ } M_{\mathsf{P}},\ \hat{f}\colon T\hat{M}_{\mathsf{P}} \oplus V \longrightarrow TM_{\mathsf{P}} \oplus \zeta$$

and we let

(6.6)
$$\sigma_G(\{M,\gamma\}) = \mathrm{sign}_G(\hat{M}_{\mathsf{P}}) - \mathrm{sign}_G(M_{\mathsf{P}})$$
$$= \mathrm{sign}_G(\hat{M}_{\mathsf{P}}) - \mathrm{sign}_G(M).$$

This difference of G-signatures is independent of choice of P, since $\mathrm{sign}_G(\mathsf{P}) = 1$. We choose an element of $\mathrm{KO}_G^*(F/\mathrm{PL}(G;Z^\infty);\mathbf{Z}[\tfrac{1}{2}])$ which corresponds to the homomorphism σ_G under (6.5) and get the induced

(6.7)
$$\sigma_G\colon \tilde{\mathcal{N}}(X) \to \widetilde{\mathrm{KO}}_G(X;\mathbf{Z}[\tfrac{1}{2}]).$$

For finite G-CW complexes X, σ_G is determined modulo the group $\mathrm{Ext}_{\mathrm{RO}(G)}(\mathrm{KO}_*^G(X), \mathrm{RO}(G) \otimes \mathbf{Z}[\tfrac{1}{2}])$, which is finite. In the end we will show that $F/\mathrm{PL}(G;Z^\infty)$ can be expressed in terms of classifying spaces for K-theory, and we will be able to conclude that the Ext-term vanishes, so that σ_G in (6.7) is defined unambiguously after all. For the time being we shall be content with any choice of

$$\sigma_G \in \widetilde{\mathrm{KO}}_G(F/\mathrm{PL}(G;Z^\infty);\mathbf{Z}[\tfrac{1}{2}])$$

which corresponds to the homomorphism in (6.6).

For each $\Gamma \in I = \mathrm{Iso}(Z)$ we have the fixed set homomorphism

$$\mathrm{Fix}^\Gamma\colon \tilde{\mathcal{N}}_G(X) \to \tilde{\mathcal{N}}_{W\Gamma}(X^\Gamma).$$

It maps the triple (ξ, t, η) over X over X to the triple $(\xi^\Gamma, t^\Gamma, \eta^\Gamma)$ over X^Γ, with the induced action of $W\Gamma$. Then

$$\tilde{\mathcal{N}}_{W\Gamma}(X^\Gamma) = [X^\Gamma, F/\mathrm{PL}(W\Gamma;(Z^\Gamma)^\infty)]^{W\Gamma}$$

and (6.7) gives a map $\sigma_{W\Gamma}$ from $\tilde{\mathcal{N}}_{W\Gamma}(X^\Gamma)$ to $\widetilde{\mathrm{KO}}_{W\Gamma}(X^\Gamma;\mathbf{Z}[\tfrac{1}{2}])$. Set $\sigma = \sum^\oplus \sigma_{W\Gamma} \circ \mathrm{Fix}^\Gamma$

(6.8)
$$\sigma\colon \tilde{\mathcal{N}}_G(X) \to \textstyle\sum^\oplus \widetilde{\mathrm{KO}}_{W\Gamma}(X^\Gamma;\mathbf{Z}[\tfrac{1}{2}]),$$

with Γ running over the conjugacy classes in Iso (Z). On the classifying space level we get

(6.9) $$\sigma\colon F/\mathrm{PL}(G; Z^\infty)_{\mathrm{odd}} \to \prod B(G, \Gamma)_{\mathrm{odd}}, (\Gamma) \in (\mathrm{Iso}(Z)),$$

which we want to prove is a homotopy equivalence. Write

$$F_G(Z^\infty) = \operatorname{colim} F_G(S(Z^{\oplus k}))$$

$$F_G(Z_0^\infty) = \operatorname{colim} F_G(S(U_k))$$

where U_k is the complement of the fixed set in $Z^{\oplus k}$. We use similar notation for the automorphism groups $\widetilde{\mathrm{PL}}_G(\ )$ and $\mathrm{PL}_G(\ )$, and note from (1.8) and (2.10) the (split) fibrations

$$F_G(Z_0^\infty) \to F_G(Z^\infty) \overset{\longleftarrow}{\dashrightarrow} F$$

$$\widetilde{\mathrm{PL}}_G(Z_0^\infty) \to \mathrm{PL}_G(Z^\infty) \overset{\longleftarrow}{\dashrightarrow} \mathrm{PL}.$$

Furthermore, by (1.8) and (2.11), $\mathrm{PL}_G(Z^\infty) \simeq \widetilde{\mathrm{PL}}_G(Z_0^\infty)$ and $F_G(Z^\infty)/\mathrm{PL}_G(Z^\infty) \simeq F/\mathrm{PL}(G; Z^\infty)^G$, when the homogeneous space on the left is defined as the homotopy fibre of the the obvious mapping from $B\mathrm{PL}_G(Z^\infty)$ to $BF_G(Z^\infty)$.

The space $F_G(Z^\infty)$ has two components, corresponding to the units in the Burnside ring, and by [35]

$$SF_G(Z^\infty) \simeq \prod \Omega^\infty S_1^\infty(B(W\Gamma)_+), (\Gamma) \in (\mathrm{Iso}(Z))$$

where $\Omega^\infty S_1^\infty(B_+) \subset \Omega^\infty S^\infty(B_+)$ is the components of degree one.

With these notations we can reinterpret (2.7) and (2.12) in the following

PROPOSITION 6.10. (i) *There is a split fibration*

$$\tilde{F}_G(Z_0^\infty)/\widetilde{\mathrm{PL}}_G(Z_0^\infty) \xrightarrow{\ d\ } F_G(Z^\infty)/\mathrm{PL}_G(Z^\infty) \longrightarrow F/\mathrm{PL}$$

(ii) *The homotopy groups of the fiber is the stable structure set,*

$$\pi_k(\tilde{F}_G(Z_0^\infty)/\widetilde{\mathrm{PL}}_G(Z_0^\infty)) \cong \operatorname{colim} \tilde{\mathscr{S}}_G(D^k \times SU, \partial).$$

The colimit in 6.10 (ii) is over the suspension maps

$$\Sigma\colon \tilde{\mathscr{S}}_G(D^k \times SU, \partial) \to \tilde{\mathscr{S}}_G(D^k \times SW, \partial)$$

where $U \subset W \subset Z^\infty$ and $U^G = W^G = 0$. This map was defined in (2.13). It maps a structure $[t]$ represented by

$$t\colon (D^k \times SU, \partial) \to (D^k \times SU, \partial)$$

as follows. First extend t to

$$Dt\colon (D^k \times DU, S^{k-1} \times DU) \to (D^k \times DU, S^{k-1} \times DU)$$

If $U \oplus \chi = W$, then $\Sigma(t) = Dt \times D\chi \,|\, D^k \times SW$.

We close the section by defining what we call the *structure invariant*, and relate it to σ_G above. Consider the PL manifolds related to t:

$$X(t) = D^k \times SU \bigcup_{\partial t} D^k \times SU$$

$$DX(t) = D^k \times DU \bigcup_{\partial_1 Dt} D^k \times DU, \partial_1 Dt = Dt \,|\, S^{k-1} \times DU.$$

Then we have a G-map

$$T: DX(t) \to S^k \times DU, T = Dt \cup \mathrm{id}.$$

Multiply with a suitable G-manifold P with $\mathrm{sign}_G(\mathsf{P}) = 1$ and make

$$T_{\mathscr{P}}: DX(t) \times \mathscr{P} \to S^k \times DU$$

G-transverse to $S^k \times 0$. The transversal preimage is a submanifold $M(t) \subset DX(t) \times \mathsf{P}$ with normal bundle U. Its signature defines the structure invariant of $[t] \in \tilde{\mathscr{S}}_G(D^k \times SU, \partial)$,

(6.11) $$\tilde{s}_G([t]) := \mathrm{sign}_G(M(t)).$$

LEMMA 6.12. *The composition*

$\tilde{\mathscr{S}}_G(D^k \times SU, \partial) \to \pi_k(F_G(Z_0^\infty)/\mathrm{PL}_G(Z_0^\infty)) \xrightarrow{d_*} \pi_k(F_G(Z^\infty)/\mathrm{PL}_G(Z^\infty)) \xrightarrow{(\sigma_G)_*}$ $\mathrm{KO}_G^{-k}(\mathrm{pt}, \mathsf{Z}[\tfrac{1}{2}])$
is equal to $\tilde{s}_G \otimes \mathsf{Z}[\tfrac{1}{2}]$.

PROOF. Given $[t] \in \tilde{\mathscr{S}}_G(D^k \times SU, \partial)$ we form $Dt: D^k \times DU \to D^k \times DU$. Take Cartesian product with D^k (or D^l with $l \geqq k$) and pass to the interior to get the map

$$T: D^k \times (U \oplus \mathsf{R}^k) \to D^k \times (U \oplus \mathsf{R}^k).$$

This map is a PL-homeomorphism on the boundary, and can be deformed to commute with the projection onto D^k, since

$$\pi_k \widetilde{\mathrm{PL}}_G(U \oplus \mathsf{R}^k) \cong \pi_k \mathrm{PL}_G(U \oplus \mathsf{R}^k)$$

by Corollary 2.11.

Since T is a PL-homeomorphism on the boundary we can add $D^k \times (U \oplus \mathsf{R}^k)$ along ∂T to get $\hat{X}(T)$ and

$$\hat{T}: X(T) \to S^k \times (U \oplus \mathsf{R}^k)$$

($\hat{T} = T \times \mathrm{id}$). We multiply the domain with P and make $\hat{T}_{\mathsf{P}}$ G-transverse to $S^k \times 0$. Let $M(T)$ be a transverse preimage, so

$$(\sigma_G)_* \circ d_*[t] = \mathrm{sign}_G(M(T)).$$

On the other hand, we can already make the construction for

$$\operatorname{int}(Dt)\colon D^k \times U \to D^k \times U.$$

This gives a G-manifold

$$M(t) \subset \mathrm{P} \times (D^k \times U) \bigcup_{\hat{C}t} (D^k \times U)$$

with normal bundle U. Clearly $M(t)$ and $M(T)$ are G-bordant. Since

$$\tilde{s}_G([t]) = \operatorname{sign}_G M(t),$$

and $\operatorname{sign}_G(M(t)) = \operatorname{sign}_G M(T)$, this proves the claim.

We can compose the structure invariant $\tilde{s}_G$ with the fixed point map

$$\operatorname{Fix}^\Gamma\colon \tilde{\mathscr{S}}_G(D^k \times SU, \partial) \to \tilde{\mathscr{S}}_{W\Gamma}(D^k \times SU^\Gamma, \partial),$$

for each $(\Gamma) \in (\operatorname{Iso}(SU))$. The resulting invariants $\tilde{s}_{W\Gamma}$ adds up to give

$$\tilde{s} = \sum\nolimits^{\oplus} \tilde{s}_{W\Gamma} \circ \operatorname{Fix}^\Gamma\colon \tilde{\mathscr{S}}_G(D^k \times SU, \partial) \to \sum_{(\Gamma)}\nolimits^{\oplus} \mathrm{KO}^{-k}_{W\Gamma}(\mathrm{pt}; \mathrm{Z}[\tfrac{1}{2}]), \tag{6.13}$$

similarly to the definition of σ in (6.8). Now (6.12) implies.

PROPOSITION 6.14. *The following statements are equivalent*
(i) $\sigma \otimes \mathrm{Z}$ *is an isomorphism for all G-CW complexes X.*
(ii) $\tilde{s} \otimes \mathrm{Z}[\frac{1}{2}]$ *is an isomorphism for sufficiently large Z-restricted representations U with $U^G = 0$.*

§7 The inductive setting.

It is our goal to evaluate the surgery exact sequence (5.12) completely when X is the sphere of a stable representation. The surgery sequence (5.12) was set up under the connectivity assumption (5.1) about the Stiefel spaces, and in sect. 6 the set of normal invariants was analysed under the same assumption. As indicated already we aim to prove (5.1) by using the surgery exact sequence, generalizing the relatively free case treated in sect. 3. Thus we must set up an inductive procedure. This is the purpose of the present section.

The variable we induct over will be the group order. Thus we fix a group G (of odd order) and make the basic assumption (5.1) for groups of smaller order:

INDUCTIVE ASUMPTION 7.1. Let $\Gamma \subsetneqq G$. For topological stable $\mathrm{R}\Gamma$-modules $V \subset T$ with $\operatorname{Iso}(V) = \operatorname{Iso}(T)$, $\mathrm{PL}_\Gamma(T)/\mathrm{PL}_\Gamma(V)$ is $(\dim V^\Gamma - 1)$-connected.

With this assumption we have G-transversality in the special case of a map

$f: M \to \xi$ which is already G-transverse on M^G, and we have an exact sequence (cf. (5.13)):

$$(7.2) \qquad \tilde{\mathcal{N}}_G(D^{2k+1} \times SU/\partial) \xrightarrow{\lambda} \sum^{\oplus} L_{2k+m(\Gamma)}(\mathbf{Z}[W\Gamma]) \xrightarrow{\alpha} \tilde{\mathcal{S}}_G(D^{2k} \times SU, \partial) \xrightarrow{\eta} \tilde{\mathcal{N}}_G(D^{2k} \times SU, \partial) \to 0$$

whenever $U \oplus \mathbf{R}^{2k}$ is topologically stable and $U^G = 0$. Moreover

$$(7.3) \qquad \mathcal{N}_G(D^{2k} \times SU/\partial) = [D^{2k} \times SU/\partial, F/\mathrm{PL}]^G$$

where $F/\mathrm{PL} = F/\mathrm{PL}(G; (U \oplus \mathbf{R})^\infty)$. This short-hand notation will be used for the rest of the section.

We cannot define the mapping σ in (6.9) since we lack G-transversality in general. However, we have the following partial version. Let $\mathcal{P}$ denote the family of all proper subgroups of G and $E\mathcal{P}$ its acyclic classifying space. It is a G-space with

$$(E\mathcal{P})^G = \emptyset, (E\mathcal{P})^\Gamma \simeq * \text{ for } \Gamma \in \mathcal{P}.$$

Therefore the map

$$\pi: F/\mathrm{PL} \wedge E\mathcal{P}_+ \to F/\mathrm{PL}$$

has homotopy fiber Y with $Y^G = \Omega(F/\mathrm{PL}^G)$ and $Y^\Gamma \simeq *$ for $\Gamma \subsetneq G$. By obstruction theory

$$\pi_*: [X, F/\mathrm{PL} \wedge E\mathcal{P}_+]^G \xrightarrow{\cong} [X, F/\mathrm{PL}]^G$$

is an isomorphism if X^G is a single point, e.g. for $X = D^{2k} \times SU/\partial$. The σ_G of (6.6) extends to give

$$\sigma_G: \Omega^G_*(F/\mathrm{PL} \wedge E\mathcal{P}_+, \infty) \to \mathrm{KO}^G_*(\mathrm{pt}; \mathbf{Z}[\tfrac{1}{2}]).$$

Thus we obtain a G-map

$$\sigma: F/\mathrm{PL} \wedge E\mathcal{P}_+ \to \prod B(G;\Gamma), \ (\Gamma) \in (\mathrm{Iso}\,(U - 0))$$

We add a second inductive assumption:

INDUCTIVE ASSUMPTION (7.4). For $|\Gamma| < |G|$ and $|\Gamma|$ odd,

$$\sigma: \tilde{\mathcal{N}}_\Gamma(X) \otimes \mathbf{Z}[\tfrac{1}{2}] \xrightarrow{\cong} \sum^{\oplus} \widetilde{\mathrm{KO}}_{WK}(X^K; \mathbf{Z}[\tfrac{1}{2}]); \ (K) \in (\mathrm{Iso}\,(SU)).$$

is an isomorphism for each Γ – CW complex X.

PROPOSITION 7.5. *There is a G-homotopy equivalence away from* 2

$$\sigma: F/\mathrm{PL}_{\mathrm{odd}} \wedge E\mathcal{P}_+ \xrightarrow{\simeq} \prod B(B;\Gamma)_{\mathrm{odd}} \wedge E\mathcal{P}_+, \ (\Gamma) \in (\mathrm{Iso}\,(U))$$

PROOF. By the equivariant Whitehead theorem it suffices to show that

$$\sigma^H: F/\mathrm{PL}(G; U^\infty)^H \to \prod B(G;\Gamma)^H$$

is a homotopy equivalence for each proper subgroup H of G. But

$$\mathrm{Res}^H(F/\mathrm{PL}(G;U^\infty)) = F/\mathrm{PL}(H;U^\infty)$$

$$\mathrm{Res}^H(BG;\Gamma)) = \begin{cases} B(H;\Gamma) & \text{if } (\Gamma) \subseteq (H) \\ * & \text{if } (\Gamma) \nsubseteq (H) \end{cases}$$

and

$$\mathrm{Res}^H(\sigma): F/\mathrm{PL}(H;U^\infty) \to \prod B(H,\Gamma)$$

is a $\mathsf{Z}[\frac{1}{2}]$-local homotopy equivalence by (7.4). The result follows.

With (7.5) we can calculate the normal invariant term in (7.2) away from 2 to be

(7.6) $$\tilde{\mathcal{N}}_G(D^k \times SU, \partial) \otimes \mathsf{Z}[\tfrac{1}{2}] \cong \sum\nolimits^{\oplus} \widetilde{\mathrm{KO}}^{-k}_{W\Gamma}(SU^\Gamma; \mathsf{Z}[\tfrac{1}{2}]).$$

The L-term is also given by K-theory, since

(7.7) $$L_{2k+m(\Gamma)}(\mathsf{Z}[W\Gamma]) \otimes \mathsf{Z}[\tfrac{1}{2}] = \mathrm{KO}^{-2k-m(\Gamma)}_{W\Gamma}(\mathrm{pt}; \mathsf{Z}[\tfrac{1}{2}]).$$

In the key sect. 8 below we calculate λ, determine (7.2) and show (7.1) for $\Gamma = G$. This requires a suitable invariant of $\tilde{\mathcal{S}}_G(D^{2k} \times SU, \partial)$. We would like to use the structure invariant $\tilde{s}_G$ of (6.11) but this would involve us in a circular argument since its definition requires stable G-transversality, hence (7.1) for $\Gamma = G$; and this is what we attempt to prove.

Instead we use an extension of the ρ-invariant from sect. 3. First we need some preliminary results.

PROPOSITION 7.8. *For odd numbers k, $\pi_k(\mathrm{BPL}_G) \otimes \mathsf{Q} = 0$.*

PROOF. Since F_G has finite homotopy groups it is enough to show that

$$\pi_k(F_G/\mathrm{PL}_G) \otimes \mathsf{Q} = 0 \ \ (k \text{ odd}).$$

We know this to be true in the G-trivial case, so by (6.10 (i)), it suffices to consider $\pi_k(\tilde{F}_G/\widetilde{\mathrm{PL}}_G)$, or what is the same, $\tilde{\mathcal{S}}(D^k \times SU, \partial)$ for topological stable U with $U^G = 0$. Since $L_{\mathrm{odd}}(\mathsf{Z}[W\Gamma]) = 0$, (7.2) gives the exact sequence

(7.9) $$0 \to \tilde{\mathcal{S}}_G(D^k \times SU, \partial) \otimes \mathsf{Q} \to \sum\nolimits^{\oplus} \mathrm{KO}^{-k}_{W\Gamma}(SU^\Gamma; \mathsf{Q}) \xrightarrow{\lambda} \sum\nolimits^{\oplus} L_{k+m(\Gamma)}(\mathsf{Z}[W\Gamma]) \otimes \mathsf{Q}.$$

We use (7.9) to show that

$$\mathrm{Res}: \tilde{\mathcal{S}}_G(D^k \times SU, \partial) \otimes \mathsf{Q} \to \sum\nolimits^{\oplus} \tilde{\mathcal{S}}_\Gamma(D^k \times SU, \partial) \otimes \mathsf{Q},$$

into the sum over conjugacy classes of proper subgroups, is injective. The obvious induction will then complete the argument.

The group $KO_G^{-k}(SU; \mathbb{Q})$ is a factor in $K_G^{-1}(SU; \mathbb{Q})$ wich can be calculated from the exact sequence of the pair (DU, SU),

$$0 \to K_G^{-1}(SU) \xrightarrow{\delta_*} K_G(DU, SU) \xrightarrow{r} K_G(DU) \to \dots$$

When we identify $K_G(DU, SU) = RG$ via the Thom isomorphism, and $K_G(DU) = RG$ by retracting DU to its center, then r is identified with multiplication by the Euler class $\lambda_{-1}(U) \in RG$, (when we use the standard Thom class).

Since RG maps injectively into $\sum^{\oplus} RC$, with C running over the cyclic subgroups, the same is the case for $K_G^{-1}(SU)$. Hence there is nothing to prove unless G is cyclic.

If G is cyclic of order m then the kernel of

$$\text{Res}: RG \otimes \mathbb{Q} \to \sum^{\oplus} RH \otimes \mathbb{Q}, \ H \subsetneqq G$$

is isomorphic to the field $\mathbb{Q}(\zeta_m)$, $\zeta_m = e^{2\pi i/m}$. The isomorphism can be specified as the evaluation of characters at a chosen generator. But $e(U)(g) = 0$ if and only if $U^g \neq 0$, so

$$\text{Res}: K_G^{-1}(SU) \otimes \mathbb{Q} \to \sum^{\oplus} K_H^{-1}(SU) \otimes \mathbb{Q}, \ H \subsetneqq G$$

is injective as claimed. The same argument applies to the other summands of λ in (7.9).

Let $\Omega_k^G(X)$ denote the bordism group of G-maps $f: M \to X$ where M is a (locally linear) G-oriented PL G-manifold of dimension k. For every family $\mathscr{F}$ of subgroups of G, closed under subconjugation, $\Omega_k^G(E\mathscr{F})$ is the bordism groups of PL G-manifolds with isotropy groups in $\mathscr{F}$. We have by (7.1) sufficient transversality to show

$$(7.10) \qquad \Omega_k^G(E\mathscr{F}) = \varinjlim [S^{V+k}, E\mathscr{F}_+ \wedge \text{MSPL}_{|V|}(V \oplus \mathbb{R}^\infty)]$$

when $G \notin \mathscr{F}$ (cf. [22], [26]).

COROLLARY 7.11. *Under the assumptions of* (7.1) *and* (7.4) *the bordism groups* $\Omega_{2k+1}^G(E\mathscr{F}) \otimes \mathbb{Q} = 0$ *when* $G \notin \mathscr{F}$.

PROOF. Use (7.8), induction over subgroups, the Connor-Floyd neighboring sequences [37] and G-obstruction theory.

Let Δ_ξ be the $K_G(\ ; \mathbb{Z}[\frac{1}{2}])$ Thom class for the oriented G-bundle associated with the index operator, as above. The associated Euler class is $e(\xi)$. If $\xi = \chi$ is a faithful character of a cyclic group G then

$$e(\chi) = (1 - \chi)/(1 + \chi) \in R(G) \otimes \mathbb{Z}[\tfrac{1}{2}];$$

this determines the character $e(U)(g)$ in general.

We can now generalize (3.7) to get

$$\tilde{\rho}_G: \tilde{\mathscr{S}}_G(D^k \times SU, \partial) \to \mathrm{RO}_k(G) \otimes \mathbb{Q}\,(=\mathrm{KO}_G^{-k}(\mathrm{pt}; \mathbb{Q}))$$

For $[t] \in \tilde{\mathscr{S}}_G(D^{2k} \times SU, \partial)$ form the closed G-manifold

$$X(t) = D^{2k} \times SU \bigcup_{\partial t} D^{2k} \times SU.$$

It has odd dimension and $X(t)^G = \varnothing$, so by (7.11) a suitable number of copies of $X(t)$ bounds, say

$$r \cdot X(t) = \partial Y_r.$$

We can even choose Y_r to have the same isotropy groups as $X(t)$. We do so, and define

$$\tilde{\rho}_G[t] = \frac{1}{r} \operatorname{sign}_G(Y_r) \cdot e(U). \tag{7.12}$$

LEMMA 7.13. *The right hand side in* (7.12) *is independent of choice of* Y_r *as long as* $\operatorname{Iso}(Y_r) = \operatorname{Iso}(SU)$.

PROOF. The indeterminacy of Y_r amounts to the addition of a closed PL G-manifold M_r with $\operatorname{Iso}(M\,) \subseteq \operatorname{Iso}(SU)$. Thus we must show that

$$e(U) \cdot \operatorname{sign}_G(M) = 0.$$

for each such M.

Let $g \in G$ and denote by $\mathrm{R}(G)_g$ the localization of $\mathrm{R}(G)$ at the ideal of characters which vanish at g cf. [4]. If $U^g = 0$ then $e(U)(g) \neq 0$ so $e(U)$ becomes a unit in $R(G)_g$, and we must show that $\operatorname{sign}_g(M) = 0$. Since $\mathrm{R}(G)_g \cong \mathrm{R}(\langle g \rangle)_g$, we can assume G is cyclic, generated by g.

Both the representation ring and the G-bordism groups are functors over the Burnside ring $A(G)$ (via restriction and induction, cf. [28], [37]), and can be localized at the prime ideals of $A(G) \otimes \mathbb{Q}$. Let $q(G)$ be the prime ideal of $A(G) \otimes \mathbb{Q}$ consisting of virtual G-sets $X - Y$ with $\#(X^G) = \#(Y^G)$. We localize $\mathrm{R}(G)$ and the bordism ring of manifolds without stationary points, $\Omega_*^G(E\mathscr{P})$, at $q(G)$ and get:

$$\mathrm{R}(G)_{q(G)} = \mathrm{R}(G)_g = \mathbb{Q}(\zeta_{|G|})$$

$$\Omega_*^G(E\mathscr{P})_{q(G)} = 0.$$

The first equation follows from [36]; the second is contained in [28].

Compose with

$$\mathrm{Fix}^\Gamma: \tilde{\mathscr{S}}_G(D^k \times SU, \partial) \to \tilde{\mathscr{S}}_{W\Gamma}(D^k \times SU^\Gamma, \partial)$$

to get an invariant $\tilde{\rho}_{W\Gamma}$ for each $\Gamma \in \mathrm{Iso}(SU)$. We set

(7.14) $$\tilde{\rho} = \sum\nolimits^{\oplus} \tilde{\rho}_{W\Gamma} \circ \mathrm{Fix}^{\Gamma} : \tilde{\mathscr{S}}_G(D^k \times SU, \partial) \to \sum\nolimits^{\oplus} \mathrm{RO}_k(W\Gamma) \otimes \mathbf{Q}$$

with summation over the conjugacy classes in $\mathrm{Iso}(SU)$.

§8 The structure set for spheres, away from 2.

In this section we work under the inductive assumptions (7.1) and (7.4) about the connectivity of the PL Stiefel spaces and the structure of normal invariants. Our main result is that

$$\Sigma : \tilde{\mathscr{S}}_G(D^k \times SU, \partial) \otimes \mathbf{Z}[\tfrac{1}{2}] \to \tilde{\mathscr{S}}_G(D^k \times SW, \partial) \otimes \mathbf{Z}[\tfrac{1}{2}], \; k > 0,$$

is an isomorphism for each pair of $\mathbf{R}G$-modules $U \subseteq W$ with $U^G = W^G = 0$ for which $\mathbf{R}^k \oplus U$ and $\mathbf{R}^k \oplus W$ are topological stable. As a corollary we obtain

$$\pi_{k-1}(\mathrm{PL}_G(T)/\mathrm{PL}_G(V)) \otimes \mathbf{Z}[\tfrac{1}{2}] = 0 \text{ for } k \leqq \dim V^G$$

for stable representations $V \subseteq T$ with $\mathrm{Iso}(V) = \mathrm{Iso}(T)$ giving the (odd-local) inductive step for proving (5.1).

We fist consider the equivariant surgery obstruction

$$\lambda : \tilde{\mathscr{N}}_G(D^{k+1} \times SU, \partial) \to \mathscr{L}_{k+m}(G) = \mathscr{L}_{k+m}(G, D^{k+1} \times SU)$$

with $m(\Gamma) = \dim U^{\Gamma}$, $(\Gamma) \in \mathrm{Iso}(SU)$). Under the identification

(8.1) $$\begin{aligned} \mathscr{L}_{k+m}(G) \otimes \mathbf{Z}[\tfrac{1}{2}] &= \sum\nolimits^{\oplus} L_{k+m(\Gamma)}(\mathbf{Z}[W\Gamma]) \otimes \mathbf{Z}[\tfrac{1}{2}] \\ &= \sum\nolimits^{\oplus} \mathrm{RO}_{k+m(\Gamma)}(W\Gamma) \otimes \mathbf{Z}[\tfrac{1}{2}] \end{aligned}$$

the surgery obstruction becomes

$$\lambda(f, \hat{f} = \sum \mathrm{sign}_{W\Gamma}(M^{\Gamma})$$

for each U-restricted normal map $f : (M, \partial M) \to (D^{k+1} \times SU, \partial)$ with $k \equiv 0 \pmod 2$.

Here and below all sums are over the conjugacy classes in $\mathrm{Iso}(SU)$.

THEOREM 8.2. *There is a commutative diagram*

$$\begin{array}{ccc} \tilde{\mathscr{N}}_G(D^{2k+1} \times SU/\partial) \otimes \mathbf{Z}[\tfrac{1}{2}] & \xrightarrow{\lambda} & \sum^{\oplus} L_{2k+m(\Gamma)}(\mathbf{Z}[W\Gamma]) \otimes \mathbf{Z}[\tfrac{1}{2}] \\ \cong \downarrow \sigma & & \cong \downarrow \\ \sum^{\oplus} \mathrm{KO}_{W\Gamma}^{-2k-1}(SU^{\Gamma}, \mathbf{Z}[\tfrac{1}{2}]) & \xrightarrow{\delta^*} & \sum^{\oplus} \mathrm{KO}_{W\Gamma}^{-2k}(DU^{\Gamma}, SU^{\Gamma}; \mathbf{Z}[\tfrac{1}{2}]). \end{array}$$

where σ is the isomorphism from (7.6).

PROOF. The theorem will follow directly from Proposition 8.8 and Lemma 8.9 below.

Assumption (7.1) implies that Thom spaces of oriented PL G-bundles over spaces Y with $Y^G = \emptyset$ are K_G-oriented away from 2 (cf. [26, sect. 6]). Applied to normal bundles of PL G-manifolds this leads to K^G_*-homology orientations for oriented G-manifolds.

Let Y be a PL G-manifold, possibly with boundary, but without stationary points. Choose an embedding of Y in a large $\mathrm{R}G$-module, say

$$(Y, \partial Y) \subset (DV, SV),$$

with PL G-normal bundle ν. The resulting map

$$DV/SV \xrightarrow{\text{collapse}} T(\nu) \xrightarrow{\text{diag}} T(\nu) \wedge Y/\partial Y$$

gives equivariant S-duality. Specifically, slant product with the class of $K^G_{|V|}(DV, SV)$ which corresponds to $1 \in K^G_0(\mathrm{pt})$ under Bott-periodicity, gives an isomorphism

$$\tilde{\mathrm{K}}_G^{i+m}(T(\nu); \mathrm{Z}[\tfrac{1}{2}]) \xrightarrow{\cong} \mathrm{K}^G_{n-i}(Y, \partial Y; \mathrm{Z}[\tfrac{1}{2}]). \tag{8.3}$$

Here $m = \dim \nu$ and $n = \dim Y$. Let $[Y] \in \mathrm{K}^G_n(Y, \partial Y; \mathrm{Z}[\tfrac{1}{2}])$ be the class which corresponds to the Thom class $\Delta_\nu \in \mathrm{K}^m_G(T\nu; \mathrm{Z}[\tfrac{1}{2}])$. Then cap product with Y induces Poincaré duality

$$\cap [Y]: \mathrm{K}^i_G(Y; \mathrm{Z}[\tfrac{1}{2}]) \xrightarrow{\cong} \mathrm{K}^G_{n-i}(Y, \partial Y; \mathrm{Z}[\tfrac{1}{2}]). \tag{8.4}$$

Indeed, the map in (8.4) is equal to the map in (8.3) composed with Thom isomorphism.

For maps $f: M \to Y$ between G-manifolds (without stationary points), we define the Gysin homomorphism $f_!$, so that the diagram below commutes

$$\begin{array}{ccc} \mathrm{K}^i_G(M; \mathrm{Z}[\tfrac{1}{2}]) & \xrightarrow{f_!} & \mathrm{K}^{i-r}_G(Y; \mathrm{Z}[\tfrac{1}{2}]) \\ \downarrow \cap [M] & & \downarrow \cap [Y] \\ \mathrm{K}^G_{n-i}(M, \partial M; \mathrm{Z}[\tfrac{1}{2}]) & \xrightarrow{f_*} & \mathrm{K}^G_{n-i}(Y; \partial Y; \mathrm{Z}[\tfrac{1}{2}]) \end{array}$$

($r = \dim M - \dim Y$).

When $Y = \mathrm{pt}$, $\varphi: M \to \mathrm{pt}$ gives the topological index homomorphism

$$\varphi_!: \mathrm{K}_G(M^n; \mathrm{Z}[\tfrac{1}{2}]) \to \mathrm{R}G \otimes \mathrm{Z}[\tfrac{1}{2}].$$

With our choice of Thom class Δ the G-signature theorem becomes (cf. [26]):

THEOREM 8.5. *For* PL *G-manifolds* (*without stationary points*) $\mathrm{sign}_G(M) = \varphi_!(1)$.

Of course, (8.5) is also true if $M^G \neq \emptyset$, but this requires the connectivity results for Stiefel spaces we want to prove, or a different argument altogether.

Let $\langle , \rangle: K_G^i(M) \otimes K_j^G(M) \to K_{j-i}^G(\mathrm{pt})$ be the Kronecker pairing. Then

$$\varphi_!(x) = \langle x, [M] \rangle$$

We have formulated the above in $K_G^*(\ ; \mathbf{Z}[\frac{1}{2}])$ but could as well have used $\mathrm{KO}_G^*(\ ; \mathbf{Z}[\frac{1}{2}])$. In fact, the Thom class Δ_ξ lies in $\widetilde{\mathrm{KO}}_G^m(T\xi; \mathbf{Z}[\frac{1}{2}])$ when $m = \dim \xi$. Therefore we also have

$$[M] \in \mathrm{KO}_n^G(M; \mathbf{Z}[\tfrac{1}{2}]) \subset \mathrm{K}_n^G(M; \mathbf{Z}[\tfrac{1}{2}]).$$

In our calculations below it will be convenient to express the basic surjection of [22], and used in sect. 6 above,

$$\mu: \mathrm{KO}_G^*(X; \mathbf{Z}[\tfrac{1}{2}]) \to \mathrm{Hom}_{\Omega_*^G}(\Omega_*^G(X), \mathrm{RO}_*(G))$$

in terms of the Gysin homomorphism. The formula is

$$\mu(\xi)(\{M, f\}) = \varphi_! f^*(\xi) = \langle f^*(\xi), [M] \rangle \tag{8.6}$$

where $\{M, f\} \in \Omega_*^G(X)$ and $\xi \in \mathrm{KO}_G^*(X; \mathbf{Z}[\frac{1}{2}])$. This is a consequence of (8.5), and is left for the reader. Let

$$f: M \to Y, \hat{f}: TM \oplus U \to TY \oplus \zeta. \quad (\partial Y = \emptyset).$$

be a reduced normal map with Y stable, i.e. $10 < 2 \dim Y^H < \dim Y^K$, and $Y^G = \emptyset$. We get an element $\eta(f, f) \in [Y/\partial, F/\mathrm{PL}]^G$ and can apply σ, cf. (5.11), (7.6).

Proposition 8.7. *In the situation above*

$$\sigma_{W\Gamma} \circ \mathrm{Fix}^\Gamma(\eta(f, \hat{f})) = (f^\Gamma)_!(1) - 1$$

where $1 \in \mathrm{KO}_{W\Gamma}(Y^\Gamma; \mathbf{Z}[\frac{1}{2}])$ *is the trivial line bundle.*

Proof. It suffices to do the component corresponding to $\Gamma = 1$. By definition of σ_G,

$$\varphi_!(\sigma_G(\eta(f, \hat{f}))) = \mathrm{sign}_G(M) - \mathrm{sign}_G(Y).$$

On the other hand, (8.5) gives

$$\varphi_!(f_!(1) - 1) = \mathrm{sign}_G(M) - \mathrm{sign}_G(Y),$$

so $\sigma_G(\eta(f, \hat{f}))$ and $f_!(1) - 1$ define the same homomorphism from $\Omega_*^G(Y)$ to $\mathrm{KO}_*^G \otimes \mathbf{Z}[\frac{1}{2}]$. Hence the two elements agree in $\mathrm{KO}_G^*(Y; \mathbf{Z}[\frac{1}{2}])$, modulo the subgroup

$$\mathrm{Ext}_{\mathrm{KO}_*^G}(\mathrm{KO}_*^G(Y), \mathrm{KO}_*^G \otimes \mathbf{Z}[\tfrac{1}{2}]),$$

(cf. (6.5)). In the universal case $Y = F/\mathrm{PL} \wedge (E\mathscr{P})_+$ ($\mathscr{P}$ = family of all proper subgroups) the ext-term vanishes by (7.4), which identifies Y with a suitable classifying space, and by calculations from [44].

The surgery invariant $\lambda(f, \hat{f})$ of a G-normal map $f:(M, \partial M) \to (X, \partial X)$ is a difference of G-signatures because G has odd order:

$$\lambda(f, \hat{f}) = \sum^{\oplus} (\operatorname{sign}_{W\Gamma}(M^{\Gamma}) - \operatorname{sign}_{W\Gamma}(X^{\Gamma})).$$

PROPOSITION 8.8. *Suppose X is a stable* PL *G-manifold without stationary points. There is a commutative diagram*

$$\begin{array}{ccc} [X/\partial X, F/\mathrm{PL}]^G & \xrightarrow{\ \sigma\ } & \sum^{\oplus} \mathrm{KO}_{W\Gamma}(X, \partial X; \mathbf{Z}[\tfrac{1}{2}]) \\ & {\scriptstyle\lambda} \searrow \quad \swarrow {\scriptstyle \sum^{\oplus} \langle -, [X^{\Gamma}] \rangle} & \\ & \sum^{\oplus} (R(W\Gamma) \otimes \mathbf{Z}[\tfrac{1}{2}]). & \end{array}$$

PROOF. Let $f:(M, \partial M) \to (X, \partial X)$ be a normal map and $\eta(f, f) \in [X/\partial X, F/\mathrm{PL}]^G$ its normal invarint. Consider the induced normal map $(F, \hat{F})$ over the double $X \cup_{\partial} X$,

$$F: M \cup_{\partial f} X \to X \cup_{\partial} X, F = f \cup \mathrm{id}.$$

Then $\eta(F, \hat{F}) = j^* \eta(f, \hat{f})$ with

$$j^*: [X/\partial X, F/\mathrm{PL}]^G \to [X \cup_{\partial} X, F/\mathrm{PL}]^G.$$

induced from collapsing the second summand X. Let $\varphi: M \cup_{\partial f} X \to \mathrm{pt}$, and note that

$$\varphi_! F_!(1) = \operatorname{sign}_G(M \cup_{\partial f} X) = \operatorname{sign}_G(M, \partial M) - \operatorname{sign}_G(X, \partial X).$$

The last equation follows from Novikov's additivity lemma for signatures, cf. [7].

The top component σ_G of $\sigma: [X \cup_{\partial} X, F/\mathrm{PL}]^G \to \mathrm{KO}_G(X \cup_{\partial} X; \mathbf{Z}[\tfrac{1}{2}])$ maps $j^*(\eta(f, \hat{f})) = \eta(F, \hat{F})$ to $F_!(1) - 1$ by (8.7). But $\langle 1, [X \cup_{\partial} X] \rangle = 0$, because $X \cup_{\partial} X$ is a boundary. We have left to check commutativity in

$$\begin{array}{ccc} \mathrm{KO}_G(X, \partial X; \mathbf{Z}[\tfrac{1}{2}]) & \xrightarrow{\ j^*\ } & \mathrm{KO}_G(X \cup_{\partial} X; \mathbf{Z}[\tfrac{1}{2}]) \\ & {\scriptstyle \langle , [X] \rangle} \searrow \quad \swarrow {\scriptstyle \langle , [X \cup_{\partial} X] \rangle} & \\ & RG \otimes \mathbf{Z}[\tfrac{1}{2}] & \end{array}$$

This follows because

$$\mathrm{KO}_m^G(X \cup_{\partial} X; \mathbf{Z}[\tfrac{1}{2}]) \xrightarrow{\ j_*\ } \mathrm{KO}_m^G(X, \partial X; \mathbf{Z}[\tfrac{1}{2}])$$

maps $[X \cup_{\partial} X]$ to $[X]$.

LEMMA 8.9. *Let $Y = D^{k+1} \times SU$. There is an exact diagram*

$$\begin{array}{ccccc}
KO_G(Y,\partial Y;Z[\tfrac{1}{2}]) & \xrightarrow{\langle ,[Y]\rangle} & KO_G^{-|U|-k}(\mathrm{pt};Z[\tfrac{1}{2}]) & \xrightarrow{e(U)} & KO_G^{-k}(\mathrm{pt};Z[\tfrac{1}{2}]) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
KO_G^{-k-1}(SU;Z[\tfrac{1}{2}]) & \xrightarrow{\delta^*} & KO_G^{-k}(DU,SU;Z[\tfrac{1}{2}]) & \longrightarrow & KO_G^{-k}(DU;Z[\tfrac{1}{2}])
\end{array}$$

where $e(U)$ is the K-theory Euler class.

PROOF. Consider the commutative diagram:

$$\begin{array}{ccccc}
KO_G^{-k}(DU,SU;Z[\tfrac{1}{2}]) & \xleftarrow{\delta^*} & KO_G^{-k-1}(SU;Z[\tfrac{1}{2}]) & \xrightarrow[\cong]{\cup\Delta} & KO_G(D^{k+1}\times SU,\partial;Z[\tfrac{1}{2}]) \\
 & \searrow \langle -,[DU]\rangle & \downarrow \langle -,[SU]\rangle & \swarrow \langle -,[Y]\rangle & \\
 & & KO^G_{k+|U|}(\mathrm{pt};Z[\tfrac{1}{2}]) & &
\end{array}$$

Since $\langle -,[DU]\rangle$ is equal to the Thom isomorphism $KO_G^{-k}(DU,SU;Z[\tfrac{1}{2}]) \cong KO_G^{-k-|U|}(\mathrm{pt};Z[\tfrac{1}{2}])$, the result follows from the exact sequence in K-theory for the pair (DU, SU).

Multiplication with $e(U^\Gamma) \in RO_{m(\Gamma)}(W\Gamma) \otimes Z[\tfrac{1}{2}] = KO^{-m(\Gamma)}(\mathrm{pt};Z[\tfrac{1}{2}])$ defines a homomorophism

$$e(U^\Gamma): RO_{2k+m(\Gamma)}(W\Gamma) \otimes Z[\tfrac{1}{2}] \to RO_{2k}(W\Gamma) \otimes Z[\tfrac{1}{2}]$$

Its image is denoted $\langle e(U^\Gamma)\rangle$ and its kernel Ker $e(U^\Gamma)$, so

$$RO_{2k+m(\Gamma)}(W\Gamma)/\mathrm{Ker}\, e(U^\Gamma) = \langle e(U^\Gamma)\rangle.$$

By '8.8) and (8.9) the surgery exact sequence (5.12) applied to $D^k \times SU$ breaks up into exact sequences:

$$0 \to \sum_{(\Gamma)}^{\oplus} \langle e(U^\Gamma)\rangle \xrightarrow{\alpha} \tilde{\mathscr{S}}_G(D^{2k} \times SU, \partial) \otimes Z[\tfrac{1}{2}] \xrightarrow{\eta}$$

(8.10)
$$\sum_{(\Gamma)}^{\oplus} KO_{W\Gamma}^{-2k}(SU^\Gamma, Z[\tfrac{1}{2}]) \to 0$$

$$\tilde{\mathscr{S}}_G(D^{2k+1} \times SU, \partial) \otimes Z[\tfrac{1}{2}] = 0.$$

We want to solve the extension in (8.10). To this end we use the $\tilde{\rho}$-invariant from $\tilde{\mathscr{S}}_G(D^{2k} \times SU, \partial)$ to $\sum^{\oplus} RO(W\Gamma) \otimes Q$, cf. (7.14), and a second invariant, $\tilde{\eta}$, which we now define. Let $RO_*(G, U)$ be the kernel of the restriction map from $RO_*(G)$ to $\varprojlim RO_*(\Gamma)$, $\Gamma \in \mathrm{Iso}(SU)$. There is an exact sequence

$$0 \to RO_{2k}(G; U) \otimes Z[\tfrac{1}{2}] \to RO_{2k}(G) \otimes Z[\tfrac{1}{2}] \to$$
$$\mathrm{Free}\,(RO_{2k}(G)/\langle e(U)\rangle) \otimes Z[\tfrac{1}{2}] \to 0$$

where Free $A = A/\mathrm{Tor}\ A$ is the free quotient of the abelian group in question. Indeed, the restriction

$$\mathrm{Res}\colon \mathrm{RO}_{2k}(G)/\langle e(U)\rangle \to \varprojlim \mathrm{RO}_{2k}(\Gamma)$$

defines a rational isomorphism, and hence a monomorphism of the free part. (Localize at prime ideals, cf. [36]. Since

$$\mathrm{KO}_G^{-2k}(SU;\mathrm{Z}[\tfrac{1}{2}]) \cong (\mathrm{RO}_{2k}(G)/\langle e(U)\rangle)\otimes \mathrm{Z}[\tfrac{1}{2}]$$

the torsion free part of the normal invariant defines

(8.11) $\tilde{\eta}\colon \tilde{\mathscr{S}}_G(D^{2k}\times SU,\partial)\otimes \mathrm{Z}[\tfrac{1}{2}] \to \sum^{\oplus} \mathrm{Free}(\mathrm{RO}_{2k}(W\Gamma)/e(U^\Gamma))\otimes \mathrm{Z}[\tfrac{1}{2}]$.

Note that

$$\tilde{\eta} = \sum\nolimits^{\oplus} \eta_{W\Gamma}\circ \mathrm{Fix}^\Gamma.$$

The $\tilde{\rho}$-invariant can be calculted on the subgroup Image (α) in (8.10). Indeed, there is the following straightforward generalization of (3.8):

LEMMA 8.12. $\tilde{\rho}\circ\alpha(\sum^{\oplus} x_\Gamma) = \sum^{\oplus} x_\Gamma e(U^\Gamma)$.

Consider the suspension

$$\Sigma\colon \tilde{\mathscr{S}}_G(SU\times D^{2k},\partial) \to \tilde{\mathscr{S}}_G(SW\times D^{2k},\partial)$$

with $W = U\oplus\chi$ and $\chi^G = 0$. There is an obvious commutative diagram

$$\begin{array}{ccc}
\tilde{\mathscr{S}}_G(D^{2k}\times SU,\partial) & \xrightarrow{\eta} & \sum \mathrm{KO}_{W\Gamma}^{-2k}(SU^\Gamma;\mathrm{Z}[\tfrac{1}{2}]) \\
\downarrow \Sigma & & \uparrow \mathrm{Res} \\
\mathscr{S}_G(D^{2k}\times SW,\partial) & \xrightarrow{\eta} & \sum \mathrm{KO}_{W\Gamma}^{-2k}(SW^\Gamma;\mathrm{Z}[\tfrac{1}{2}]).
\end{array} \tag{8.13}$$

In particular, if Iso(SU) = Iso(SW) then

$$\tilde{\eta}\circ\Sigma = \tilde{\eta},$$

since Res in (8.13) is then an isomorphism of the free part.

Here is th key result, generalizing Lemma 3.9:

LEMMA 8.14. *Let $W = U\oplus\chi$ with $\chi^G = 0$, and let $x\in \mathrm{RO}_{2k+m}(G)$. For $g\in G$,*

$$\begin{aligned}
&\text{(i)}\ \ \tilde{\rho}_G(\Sigma\circ\alpha_G(x))(g) = e(U)(g)\cdot x(g) \quad \textit{if } e(\chi)(g) \neq 0\\
&\qquad\qquad\qquad\qquad\quad\ = 0 \qquad\qquad\qquad\ \ \textit{if } e(\chi)(g) = 0\\
&\text{(ii)}\ \tilde{\eta}_G(\Sigma\circ\alpha_G(x))(g) = e(U)(g)x(g) \quad\ \ \textit{if } e(\chi)(g) = 0.
\end{aligned}$$

PROOF. For the first equation we use (8.12) and

$$\tilde{\rho}_G(\Sigma(\alpha_G(x)) = \frac{1}{r}\tilde{\rho}_G(\alpha_G(x))\cdot \mathrm{sign}_G(M_r)\cdot e(\chi),$$

where M_r is a G-manifold with the same isotropy groups as $S(\chi)$ and $\partial M_r = r \cdot S(\chi)$, cf. the proof of Lemma 3.8. Form $\hat{M}_r = M_r \cup r \cdot D(\chi)$. It has r stationary points and $\text{sign}_G(M_r) = sign_G(\hat{M}_r)$. The G-signature theorem applies to show

$$\text{sign}_G(M_r)(g) = r \cdot e(\chi)(g)^{-1} \in R(G)_g.$$

Now (i) follows.

The proof of (ii) is more involved. Let C be the cyclic group generated by g. Since $e(\chi)(g) = 0$, $\chi^C \neq 0$ and $SW = SU * S\chi$ has a C-fixed point x_0. The normal invariant takes values in $\mathrm{KO}_G^{-2k}(SW; \mathbf{Z}[\frac{1}{2}])$ and we are interested in its image under the composition

$$\mathrm{KO}_G^{-2k}(SW; \mathbf{Z}[\tfrac{1}{2}]) \xrightarrow{\text{Res}} \mathrm{KO}_C^{-2k}(SW; \mathbf{Z}[\tfrac{1}{2}]) \xrightarrow{\cong} \mathrm{KO}_C^{-2k}(x_0; \mathbf{Z}[\tfrac{1}{2}]).$$

Since the whole surgery sequence is natural under restrictions to subgroups we can replace G by C for the rest of the argument. Note $C \neq G$ since $\chi^G = 0$, so one has C-transversality by (7.1) and [26].

Let $f: D^{2k} \times SW \to D^{2k} \times SW$ represent an element $[f] \in \tilde{\mathcal{S}}_C(D^{2k} \times SW, \partial)$. Then $\tilde{\eta}([f]) \in \mathrm{KO}_C^{-2k}(x_0; \mathbf{Z}[\frac{1}{2}])$ can be calculated geometrically as follows. Choose a closed C-manifold P with $\text{sign}_C(\mathsf{P}) = 1$ such that

$$f(\mathsf{P}): \mathsf{P} \times D^{2k} \times SW \to D^{2k} \times SW \xrightarrow{f} D^{2k} \times ST$$

can be made transverse to $D^{2k} \times x_0$. Let $X \subset \mathsf{P} \times D^{2k} \times SW$ be a transverse pre-image of $D^{2k} \times x_0$; then

$$\tilde{\eta}_C[f] = \text{sign}_C(X) \in \mathrm{RC} \otimes \mathbf{Z}[\tfrac{1}{2}]. \tag{8.15}$$

The C-cobordism class of X depends on P but its index does not, since we use only manifolds P with $\text{sign}_C(\mathsf{P}) = 1$.

Let $h: SE \to D^{2k} \times SU$ represent $[h] \in \tilde{\mathcal{S}}_G(D^{2k} \times SU, \partial)$; actually $SE = D^{2k} \times SU$, but we choose to write SE to separate the range and domain of h notationally. Let

$$Dh: DE \to D^{2k} \times DU, \ DE = D^{2k} \times DU$$

be the construction from (2.14). It restricts to h on SE and to a PL-homeomorphism on $S^{2k-1} \times DU$.

Choose a suitable P as above, so that

$$Dh_{\mathsf{P}}: \mathsf{P} \times DE \xrightarrow{\text{pr}} DE \xrightarrow{Dh} D^{2k} \times DU$$

can be made C-transverse to $D^{2k} \times 0$. Let $X \subset \mathsf{P} \times DE$ be a transverse preimage. Then

$$\tilde{\eta}_C(\Sigma([h])) = \text{sign}_C(X).$$

On the other hand, suppose the structure $[SE, h]$ is in the image of

$$\alpha_C: L_{2k+m}(\mathbf{Z}C) \to \tilde{\mathscr{S}}_C(D^{2k} \times SU, \partial), \quad m = \dim U.$$

Each element x of the *L-group can be represented by a C*-map,

$$F: (N; \partial_+ N, \partial_- N) \to (D^{2k+1} \times SU; D^{2k}_+ \times SU, D^{2k}_- \times SU)$$

by the equivariant version of Wall's realization theorem (cf. [38, Ch. 9] and [13]). Then

(a) $(\partial_+ N, \partial_+ F) = (SE, h)$,

(b) $\partial_- F$ is a PL-homeomorphism,

(c) $\mathrm{sign}_G(N) = x$.

Consider

$$N_1 = N \cup D^{2k}_- \times DU, \quad F_1 = F \cup \mathrm{id}.$$

Its boundary is $S^{2k-1} * SU$ and $F_1: N_1 \to D^{2k} \times DU$ is a PL-equivalence in a neighbourhood of $D^{2k} \times 0$. We glue N_1 to $D(E)$ to obtain

$$F_2: N_1 \cup_\partial D(E) \to S^{2k} * SU.$$

Then take cartesian product with P and make the resulting map $(F_2)_{\mathsf{P}}$ from $\mathsf{P} \times (N_1 \cup_\partial D(E))$ to $S^{2k} * SU$ transverse to $S^{2k} \times 0$. The inverse image is C-cobordant to $X \cup_\partial D^{2k}$ and it has trivial normal bundle U in $N_1 \cup_\partial D(E)$. By the C-signature theorem,

$$\mathrm{sign}_g(X \cup_\partial D^{2k}) = e(U)(g) \cdot \mathrm{sign}_g(N_1 \cup_\partial DE).$$

where g generates C. But

$$\mathrm{sign}_g(N) = \mathrm{sign}_g(N_1 \cup_\partial DE)$$

is the element $x(g)$ in the formula we want to prove, since the identification of $L_{2k+|U|}(\mathbf{Z}C)_{\mathrm{odd}}$ with $\mathrm{KO}_U^{-2k-|U|}(\mathrm{pt})_{\mathrm{odd}}$ is via the signature. Finally, $\mathrm{sign}_g(X) = \mathrm{sign}_g(X \cup D^{2k})$.

REMARK 8.16. The formula (i) in Lemma 8.14 implies that $\tilde{\rho}_G: \tilde{\mathscr{S}}_G(D^{2k} \times ST, \partial) \to \mathrm{R}(G) \otimes \mathbf{Q}$ is not in general $\mathbf{Z}[\frac{1}{2}]$-integral. Indeed for $G = \mathbf{Z}/p^2$, let ψ be a faithful character, and take $U = n \cdot \psi$ and $\chi = \psi^{(p)}$, where $\chi(g) = \psi(g^p)$.

Suppose $2k + m \equiv 0 \pmod 4$ so in (8.14), $x \in \mathrm{RO}(G) \otimes \mathbf{Z}[\frac{1}{2}]$. For $x = 1$, (8.14) (i) gives

$$\tilde{\rho}_G(\Sigma \alpha_G(1))(T) = (\zeta - 1/\zeta + 1)^n, \quad \zeta = e^{\pi i/p2}$$

$$\tilde{\rho}_G(\Sigma \zeta_p(1))(T^p) = 0$$

for a suitable generator $T \in \mathbf{Z}/p^2$. Let Φ be the p'th cyclotomic polynomial; $\Phi(x) = 1 + x + \ldots + x^{p-1}$. Then

$$\tilde{\rho}_G(\Sigma\alpha_G(1)) = (\psi - 1/\psi + 1)^n\cdot\left(1 - \frac{1}{p}\Phi(\chi)\right)$$

and this expression is not in $R(Z/p^2)\otimes Z[\frac{1}{2}]$.

THEOREM 8.17. *Suppose $U \subseteq W$ are RG-modules with $U^G = W^G = 0$. Let k be a positive integer and assume $V \oplus R^{2k}$ and $W \oplus R^{2k}$ are both topological stable. Then*

(i) $\Sigma: \tilde{\mathscr{S}}_G(D^{2k}\times SU,\partial)\otimes Z[\frac{1}{2}] \to \tilde{\mathscr{S}}_G(D^{2k}\times SW,\partial)\otimes Z[\frac{1}{2}]$ *is injective*
(ii) $\tilde{\eta}\circ\Sigma$ *surjects onto* $\sum^{\oplus} KO_{W\Gamma}^{-2k}(SW^{\Gamma}; Z[\frac{1}{2}])$, $(\Gamma)\in(\mathrm{Iso}(SU))$
(iii) Σ *is an isomorphism if* $\mathrm{Iso}(W) = \mathrm{Iso}(U)$.

PROOF. It is direct from (8.13) and (8.14) that Σ is injective: if $\Sigma(y) = 0$ then y must be in the image of α by (8.13), say $y = \alpha(x)$, and $\Sigma\circ\alpha(x) = 0$ implies that $e(U^\Gamma)(g)x^\Gamma(g) = 0$ for all $g\in W\Gamma$. Thus $e(U^\Gamma)\cdot x_\Gamma = 0$ and hence $x_\Gamma = 0$ in $\langle e(U^\Gamma)\rangle$ for all Γ.

The surjectivity of Σ is harder. Suppose $1\in \mathrm{Iso}(SU)$ and consider the composition

$$\begin{array}{ccccc} RO_{2k+|U|}(G)/\mathrm{Ker}\, e(U) & \xrightarrow{\alpha_G} & \tilde{\mathscr{S}}_G(D^{2k}\times SU,\partial)\otimes Z[\frac{1}{2}] & & \\ & & \downarrow \Sigma & & \\ & & \mathscr{S}_G(D^{2k}\times SW,\partial)\otimes Z[\frac{1}{2}] & \xrightarrow{\eta_G} & KO_G^{-2k}(SW; Z[\frac{1}{2}]) \end{array}$$

It maps into $\langle e(U)\rangle/\langle e(W)\rangle$ by (8.13). We show it maps onto. Consider first the map into the free quotient. Since

$$\mathrm{Res}\colon \mathrm{Free}\, RO_*(G)/\langle e(W)\rangle \to \textstyle\sum^{\oplus}\{R(C)\mid C \text{ cyclic}, W^C \neq 0\}$$

is injective it follows from 8.14 (ii) that

$$\eta_G\circ\Sigma\circ\alpha_G(x) = e(U)\cdot x$$

in Free $\langle e(U)\rangle/\langle e(W)\rangle$. Hence Free $(\eta_G\circ\Sigma\circ\alpha_G)$ is onto.

Second, if $W = U\oplus\chi$ the subgroup $RO_*(G;\chi)\subset RO_*(G)$ projects onto the torsion subgroup of $RO_*(G)/\langle e(\chi)\rangle$, and $e(U)RO_*(G;\chi)$ projects onto Tor $(\langle e(U)\rangle/\langle e(W)\rangle)$.

For $x\in RO_{2k+|U|}(G;\chi)/\mathrm{Ker}\, e(U)$, (8.14(i)) gives

$$\tilde{\rho}_G(\Sigma\circ\alpha_G(x)) = x\cdot e(U).$$

The subgroup

$$X = \Sigma\circ\alpha_G(RO_{2k+|U|}(G;\chi)/\mathrm{Ker}\, e(U))$$

contains the image of

$$\alpha_G\colon RO_{2k+|W|}(G)/\text{Ker } e(W) \to \tilde{\mathscr{S}}_G(D^{2k} \times SW, \partial) \otimes \mathbf{Z}[\tfrac{1}{2}]$$

by (8.14) and because $\tilde{\rho}_G$ is monic on each of these subgroups. Since

$$\begin{array}{ccc} 0 \to \text{Tor}(X/\text{Im }\alpha_G) & \xrightarrow{\eta_G} & \text{Tor}(\langle e(U)\rangle/\langle e(W)\rangle) \\ \downarrow \tilde{\rho}_G & & \\ \text{Tor}(\langle e(U)\rangle/\langle e(W)\rangle) & & \\ \downarrow & & \\ 0 & & \end{array}$$

is exact, and since all groups are finite both maps must be isomorphisms. Thus $\tilde{\eta}_G \circ \Sigma$ is surjective. The same works for the other components, proving (ii). Finally (iii) follows from (8.10) and (8.14). □

COROLLARY 8.18. *The structure set* $\tilde{\mathscr{S}}_G(D^{2k} \times SU, \partial) \otimes \mathbf{Z}[\frac{1}{2}]$ *is torsion free whenever* $U \oplus \mathbf{R}^{2k}$ *is topological stable.*

PROOF. Using induction over $|G|$ and (8.10), we may assume that the torsion subgroup of $\tilde{\mathscr{S}}_G(D^{2k} \times SU, \partial)$ maps trivially under $\eta \circ \text{Fix}^\Gamma$ to $KO_{W\Gamma}^{-2k}(SU, \mathbf{Z}[\frac{1}{2}])$ for $\Gamma \neq 1$. Thus we may assume $1 \in \text{Iso}(SU)$, and the torsion group injects into the component $KO_G^{-2k}(SU; \mathbf{Z}[\frac{1}{2}])$.

Choose an $\mathbf{R}G$-module W with $W \oplus \mathbf{R}^{2k}$ stable, $U \subset W$, and such that W contains each irreducible representation. By (8.17(i))

$$\Sigma\colon \tilde{\mathscr{S}}_G(D^{2k} \times SU, \partial) \otimes \mathbf{Z}[\tfrac{1}{2}] \to \tilde{\mathscr{S}}_G(D^{2k} \times SW, \partial) \otimes \mathbf{Z}[\tfrac{1}{2}]$$

is injective. If G is not cyclic, then $W^g \neq 0$ for any $g \in G$, so $e(W) = 0$. Thus $KO_G^{-2k}(SW, \mathbf{Z}[\frac{1}{2}])$ is torsion free, and the same must be the case for the structure sets.

If G is cyclic, choose $W = U \oplus \chi$ with χ a (large) free representation. Again Σ is an isomorphism, and by (8.14(ii)) the composition

$$\tilde{\mathscr{S}}_G(D^{2k} \times S\chi, \partial) \otimes \mathbf{Z}[\tfrac{1}{2}] \xrightarrow{\Sigma_U} \tilde{\mathscr{S}}_G(D^{2k} \times SW, \partial) \otimes \mathbf{Z}[\tfrac{1}{2}] \xrightarrow{\eta_G} KO_G^{-2k}(SW; \mathbf{Z}[\tfrac{1}{2}])$$

is surjective. Since Σ_U is injective and $\tilde{\mathscr{S}}_G(D^{2k} \times S\chi, \partial) \otimes \mathbf{Z}[\frac{1}{2}]$ is torison free by (3.11), the result follows.

COROLLARY 8.19. *Let* $V \subseteq T$ *be a pair of topological stable* $\mathbf{R}G$*-modules with* $\text{Iso}(V) = \text{Iso}(T)$. *Under the assumption of* (7.1) *and* (7.4),

$$\pi_k(PL_G(T)/PL_G(V)) \otimes \mathbf{Z}[\tfrac{1}{2}] = 0 \textit{ for } k \leq \dim V^G - 1.$$

PROOF. This follows from (1.4), (2.6) and (8.17), just as in the relatively free case treated in (3.12).

§9 The 2-local structure set.

This section examines $\tilde{\mathscr{S}}_G(D^k \times SU, \partial) \otimes Z_{(2)}$ when $U^G = 0$, and $|G|$ is odd under the assumptions (7.1) and (7.4). The main result is that suspension is an isomorphism in the usual stability range.

The space F/PL is a G-infinite loop space, [45]. This implies there are homomorphisms

$$\mathrm{Ind}_H^G\colon [D^k \times SU/\partial, F/\mathrm{PL}]^H \to [D^k \times SU/\partial, F/\mathrm{PL}]^G$$

is surjective. Since Σ_U is injective and $\tilde{\mathscr{S}}_G(D^{2k} \times S\chi, \partial) \otimes Z[\frac{1}{2}]$ is torison free by (3.11), the result follows.

COROLLARY 8.19. *Let $V \subseteq T$ be a pair of topological stable RG-modules with* $\mathrm{Iso}(V) = \mathrm{Iso}(T)$. *Under the assumption of* (7.1) *and* (7.4),

$$\pi_k(\mathrm{PL}_G(T)/\mathrm{PL}_G(V)) \otimes Z[\tfrac{1}{2}] = 0 \textit{ for } k \le \dim V^G - 1.$$

PROOF. This follows from (1.4), (2.6) and (8.17), just as in the relatively free case treated in (3.12).

§9 The 2-local structure set.

This section examines $\tilde{\mathscr{S}}_G(D^k \times SU, \partial) \otimes Z_{(2)}$ when $U^G = 0$, and $|G|$ is odd under the assumptions (7.1) and (7.4). The main result is that suspension is an isomorphism in the usual stability range.

The space F/PL is a G-infinite loop space, [45]. This implis there are homomorphisms

$$\mathrm{Ind}_H^G\colon [D^k \times SU/\partial, F/\mathrm{PL}]^H \to [D^k \times SU/\partial, F/\mathrm{PL}]^G$$

which together with the obvious restriction maps Res_G^H and the conjugation maps C_g make $H \to [D^k \times SU/\partial, F/\mathrm{PL}]$ into a Mackey functor. From [28] we then have:

THEOREM 9.1. *There are isomorphisms*

(i) $\mathrm{Res}\colon [D^k \times SU/\partial, F/\mathrm{PL}]_{(2)}^G \xrightarrow{\cong} \varprojlim_H [D^k \times SU/\partial, F/\mathrm{PL}]_{(2)}^H$

(ii) $\mathrm{Ind}\colon \varinjlim_H [D^k \times SU/\partial, F/\mathrm{PL}]_{(2)}^H \xrightarrow{\cong} [D^k \times SU/\partial, F/\mathrm{PL}]_{(2)}^G$

where $H \in \mathrm{Iso}(SU)$.

Let $V = \mathrm{R}^k \oplus U$. Then $SV/S^{k-1} \simeq_G D^k \times SU/\partial$, and if $SU^H \neq \varnothing$ for some subgroup H,

$$SV/S^{k-1} \simeq_H SV \vee S^k.$$

Consequently

$$\text{(9.2)} \qquad [D^k \times SU/\partial, F/\mathrm{PL}]^H \cong [SV, F/\mathrm{PL}]^H \oplus \pi_k(F/\mathrm{PL}^H).$$

On the other hand,

LEMMA 9.3. *For topological stable representations V with $V^G \neq 0$, $\tilde{\mathscr{S}}_G(SV) = *$.*

PROOF. Let $f\colon M^n \to SV$ represent an element af $\tilde{\mathscr{S}}_G(SV)$. By definition, the local linear structure of M and SV agrees. We remove disks around two stationary points $x, y \in M^G$. The resulting manifold $M_0 = M^n - \mathring{D}^n(x) - \mathring{D}^n(y)$ is G-simple homotopy equivalent to $SV_0 \times I$; $V_0 \oplus \mathrm{R} = \mathrm{V}$. From the equivariant s-cobordism theorem, M_0 is PL-homeomorphic to $SV_0 \times I$. Thus $M = DV_0 \cup_h DV_0$ is the twisted double along an equivariant PL-homeomorphism of SV_0. Since h extends to DV_0 via coning, M will be PL-homeomorphic to $SV = DV_0 \cup_h DV_0$. Finally, the G-homotopy classes of G-homotopy equivalences of SV are enumerated by the units $A(G)^\times$ in the Burnside ring. But G has odd order, so $A(G)^\times = \{\pm 1\}$. Hence each G-homotopy automorphism of SV is homotopic to a diffeomorphism, and represents the trivial structure.

Let $H \in \mathrm{Iso}\,(SU)$. Consider the commutative diagram

$$\begin{array}{ccc} \tilde{\mathscr{S}}_H(D^k \times SU, \partial) & \longrightarrow & \tilde{\mathscr{S}}_H(SV) \\ \big\downarrow \eta & & \big\downarrow \eta \\ [D^k \times SU/\partial, F/\mathrm{PL}]^H & \longrightarrow & [SV, F/\mathrm{PL}]^H \\ & \lambda \searrow \qquad \swarrow \lambda & \\ & \mathscr{L}_{m+k-1}(H). & \end{array}$$

The lower horizontal map is induced from the projection $p\colon SV \to D^k \times SU/\partial$, the upper one maps a structure $f\colon (M, \partial) \to (D^k \times SU, \partial)$ into the structure

$$f \cup \mathrm{id}\colon M \cup_{\partial f} S^{k-1} \times DU \to SV,$$

and the maps η and λ are from the surgery exact sequence. The previous lemma asserts the triviality of $\tilde{\mathscr{S}}_G(SV)$, so using (9.1) and (9.2) we get:

COROLLARY 9.5. *The 2-local surgery exact sequence for $D^k \times SU$ takes the form*

$$0 \to \mathscr{L}_{k+m}(G)_{(2)}/\varinjlim_H \mathscr{L}_{k+m}(H)_{(2)} \xrightarrow{\ \alpha\ } \tilde{\mathscr{S}}_G(D^k \times SU, \partial)_{(2)}$$

$$\xrightarrow{\ \eta\ } \varprojlim_H \pi_k(F/\mathrm{PL}^H)_{(2)} \to 0$$

Here m is the function on $(\mathrm{Iso}\,(SU))$ *which maps the conjugacy class* (Γ) *to* $\dim U^\Gamma$.

The domain of α in (9.5) is torsion free when $(k+m)$ is even, since the Arf-element is divided out. Indeed $\tilde{L}_{2k}(\mathsf{Z}[WH]) = L_{2k}(\mathsf{Z}[WH])/L_{2k}(\mathsf{Z})$ is torsion free, detected by the equivariant signature,

$$\text{sign}: L_{2k}(\mathsf{Z}[WH])/L_{2k}(\mathsf{Z}) \rightarrowtail \text{RO}_{2k}(WH)/\text{RO}_{2k}(1)$$

by results from [36], [39]. Combined with (8.18) we get

THEOREM 9.6. *Under the assumptions* (7.1) *and* (7.4), $\tilde{\mathscr{S}}_G(D^{2k+1} \times SU, \partial) = 0$ *and* $\tilde{\mathscr{S}}_G(D^{2k} \times SU, \partial)$ *is torsion free.*

LEMMA 9.7. *The quotient* $\mathsf{R}(G)_{(2)}/\varinjlim_{H} \mathsf{R}(H)_{(2)}$ *is torsion free, and* $e(U)$ *defines a unit.*

PROOF. Since $\mathsf{R}(G)_{(2)} = \varinjlim \mathsf{R}(C)_{(2)}$ with C running over the cyclic subgroups it is enough to do the case where G is cyclic. Then

$$\mathsf{R}(G)_{(2)} \cong \prod \mathsf{Z}_{(2)}[\zeta_r], \quad r \,\|\, G|$$

where ζ_r is a primitive r'th root of 1. Therefore,

$$\mathsf{R}(G)_{(2)}/\varinjlim \mathsf{R}(H)_{(2)} = \prod \mathsf{Z}_{(2)}[\zeta_s],$$

where s runs over the divisors of $|G|$ which do not divide any $|H|$. The image of $e(U)$ in the factor $\mathsf{Z}_{(2)}[\zeta_s]$ is equal to the character value $e(U)(g)$, $g \in G$ any element of order r. But $e(U)(g) = 0$ *if and only if* $U^g \neq 0$. This happens only when g belongs to some isotropy group H. Thus the projection of $e(U)$ into $Z_{(2)}[\zeta_s]$ is a product of terms of the form $(\zeta_s^i - 1)/(\zeta_s^i + 1)$ with $\zeta_s^i \neq 1$. Such elements are units.

PROPOSITION 9.8. *With the assumptions of* (8.17), *the suspension*

$$\Sigma: \tilde{\mathscr{S}}_G(D^{2k} \times SU, \partial)_{(2)} \rightarrow \tilde{\mathscr{S}}_G(D^{2k} \times SW, \partial)_{(2)}$$

is an isomorphism.

PROOF. The normal invariants $\lim \pi_{2k}(F/\text{PL}^H)$ are the same for the two structures, and the triangle

$$\begin{array}{ccc} \tilde{\mathscr{S}}_G(D^{2k} \times SW, \partial)_{(2)} & \searrow^{\eta} & \\ \big\downarrow \Sigma & & \lim \pi_{2k}(F/\text{PL}^H)_{(2)} \\ \tilde{\mathscr{S}}_G(D^{2k} \times SU, \partial)_{(2)} & \nearrow_{\eta} & \end{array}$$

is commutative by the 2-local analogue of (8.13). Thus it suffices to see that the two images of α in (9.5), Im α^U and Im α^W, correspond under suspension.

Let $W = \chi \oplus U$. Since W and U have the same isotropy groups; (8.14) gives:

$$\tilde{\rho}_G(\Sigma \circ \alpha_G^U(x_G)) = e(U) \cdot \text{sign}_G(x_G),$$

with a similar formula for the other components of $\tilde{\rho}$.

We show Σ is surjective. For each $y \in \tilde{L}_{2k+|W|}(\mathbf{Z}G)_{(2)}$ we can find $x \in \tilde{L}_{2k+|U|}(\mathbf{Z}G)_{(2)}$ with $\operatorname{sign}_G(y) \circ e(\chi) = \operatorname{sign}_G(x)$. Then

$$\tilde{\rho}_G(\Sigma \alpha_G^U(x)) = \tilde{\rho}_G(\alpha_G^W(y))$$

by (8.12).

Since

$$\tilde{\rho}_G \circ \alpha_G : L_{2k+|W|}(\mathbf{Z}G)_{(2)}/\varinjlim L_{2k+|W|}(\mathbf{Z}H)_{(2)} \to \mathrm{RO}_{2k}(G)_{(2)}$$

is injective, (9.7), $\Sigma\, \alpha_G^U(x) = \alpha_G^W(y)$. Hence Σ is surjective. The proof of injectivity is similar.

The argument used in (8.19) works equally well for the 2-local case, so altogether we have

COROLLARY 9.10. Under the assumptions of (8.19), $\mathrm{PL}_G(T)/\mathrm{PL}_G(V)$ is $(\dim V^G - 1)$-connected.

§10. The final inductive step.

In the previous two sections we have worked under the two inductive assumptions (7.1) and (7.4). In (9.10) we concluded the inductive step for (7.1), giving the right connectivity for the stable PL Stiefel spaces. In particular we have stable G-transversality in the PL category, and consequently from sect. 5,

$$\mathcal{N}_G(X) = [X, F/\mathrm{PL}]^G.$$

In sect 6 we defined the equivariant Sullivan mapping

$$\sigma : \mathcal{N}_G(X) \to \sum\nolimits^{\oplus} \widetilde{\mathrm{KO}}_{WH}(X^H; \mathbf{Z}[\tfrac{1}{2}]).$$

Our final inductive step is to prove:

THEOREM 10.1. *Away from 2, σ is an isomorphism.*

The proof of (10.1) follows immediately from (6.14) and the following result, about the structure invariant form (6.11) and (6.13).

THEOREM 10.2. *Let U be an $\mathbf{R}G$-module with $U^G = 0$ and suppose $U \oplus \mathbf{R}^{2k}$ is topological stable. Then the structure invariant*

$$\tilde{s}; \tilde{\mathcal{S}}_G(D^{2k} \times SU, \partial) \otimes \mathbf{Z}[\tfrac{1}{2}] \to \sum_{(H)}\nolimits^{\oplus} R0_{2k}(WH) \otimes \mathbf{Z}[\tfrac{1}{2}],$$

$(H) \in (\mathrm{Iso}\,(SU))$, is an isomorphism.

We first relate $\tilde{s}$ to the invariants $\tilde{\rho}$ and $\tilde{\eta}$ of sect. 8. Each of the invariants is a sum of components, one for each conjugacy class in Iso(SU). For convenience we suppose SU has a free part (i.e. $1 \in \operatorname{Iso}(SU)$) and consider only the top components:

$$\tilde{s}_G: \tilde{\mathscr{S}}_G(D^{2k} \times SU, \partial) \to \mathrm{RO}_{2k}(G)$$

$$\tilde{\rho}_G: \tilde{\mathscr{S}}_G(D^{2k} \times SU, \partial) \to \mathrm{RO}_{2k}(G; U) \otimes \mathbf{Q}$$

$$\tilde{\eta}_G: \tilde{\mathscr{S}}_G(D^{2k} \times SU, \partial) \to \operatorname{Free}(\mathrm{RO}_{2k}(G)/\langle e(U)\rangle).$$

LEMMA 10.3. *For $[t] \in \tilde{\mathscr{S}}_G(D^{2k} \times SU, \partial)$ and $g \in G$ the following character relations hold:*

(i) $\tilde{s}_G([t])(g) = \tilde{\rho}_G([t])(g)$ *if* $U^g = 0$
(ii) $\tilde{s}_G([t])(g) = \tilde{\eta}_G([t])(g)$ *if* $U_g \neq 0$.

PROOF. Recall (from sect. 2) that each $[t] \in \tilde{\mathscr{S}}_G(D^{2k} \times SU, \partial)$ is represented by the restriction of a G-map

$$Dt: D^{2k} \times DU \to D^{2k} \times DU$$

with $t_0 = Dt \mid S^{k-1} \times DU$ a PL-homeomorphism. Let

$$DX(t) = D^{2k} \times DU \cup_{t_0} D^{2k} \times DU$$

and extend Dt to G-map

$$T: DX(t) \to S^{2k} \times DU.$$

For suitable P (with $\operatorname{sign}_G \mathsf{P} = 1$) we get a transversal diagram

$$\begin{array}{ccc} \mathsf{P} \times DX(t) & \xrightarrow{T_{\mathsf{P}}} & S^{2k} \times DU \\ \cup & \pitchfork & \cup \\ M & \longrightarrow & S^{2k} \end{array} \quad , \ T_{\mathsf{P}} = T \circ \mathrm{proj}.$$

Then $\tilde{s}_G([t]) = \operatorname{sign}_G(M)$, and $\tilde{\eta}_G([t])(g) = \operatorname{sign}_G(M)(g)$ whenever $U^g = 0$, cf. the proof of 8.14 (ii).

Second, let $g \in G$ be an element with $U^g = 0$, and hence $e(U)(g) \neq 0$. We have

$$\tilde{\rho}_G([t])(g) = \frac{1}{r} \operatorname{sign}_G(Y_r(t))(g) \cdot e(U)(g)$$

where

$$\partial Y_r(t) = r \cdot \partial DX(t), \quad \operatorname{Iso}(Y_r(t)) = \operatorname{Iso}(SU).$$

Form the closed manifold $\hat{Y}_r(t) = Y_r(t) \cup_\partial r \cdot DXt)$ and note that

$$\operatorname{sign}_G(\hat{Y}_r(t)) = \operatorname{sign}_G(Y_r(t)).$$

The left hand side can be calculated from the PL G-signature theorem (8.5) and the mappings

$$r \cdot M \xrightarrow{\psi} \mathsf{P} \times \hat{Y}_r(t) \xrightarrow{\varphi} \mathrm{pt}.$$

We localize at $g \in G$ and consider $K_G(X)_g$, cf. [4]. Since $\psi^* \psi_!(x) = e(U) \cdot x$,

$$\psi_!: K_G(r \cdot M)_g \to K_G(\mathsf{P} \times \hat{Y}_r(t))_g$$

maps $e(U)^{-1} \cdot 1$ into 1. Hence

$$\varphi_! \psi_!(e(U)^{-1} \cdot 1) = \varphi_!(1) = \mathrm{sign}_g(\mathsf{P} \times \hat{Y}_r(t)) = \mathrm{sign}_g(\hat{Y}_r(t)).$$

On the other hand, $\varphi_! \circ \psi_! = (\varphi \circ \psi)_!$, and

$$(\varphi \circ \psi)_!(e(U)^{-1})(g) = e(U)(g)^{-1} \cdot (\varphi \circ \psi)_!(1)(g) = e(U)(g)^{-1} \cdot \mathrm{sign}_g(r \cdot M).$$

This gives the required formula:

$$r \cdot \mathrm{sign}_g(M) \cdot e(U)(g)^{-1} = \mathrm{sign}_g(Y_r(t)).$$

PROOF OF (10.2). The structure set $\tilde{\mathscr{S}}_G(D^{2k} \times SU, \partial)$ is torsion free and detected by the inariants $\tilde{\rho}$ and $\tilde{\eta}$. Hence $\tilde{s}$ is injective. We have the diagram

$$\begin{array}{ccccccccc}
0 \to \sum^{\oplus} \langle e(U^H) \rangle \otimes \mathsf{Z}[\tfrac{1}{2}] & \to & \tilde{\mathscr{S}}_G(D^{2k} \times SU, \partial) \otimes \mathsf{Z}[\tfrac{1}{2}] & \to & \sum^{\oplus} \mathrm{KO}_{WH}^{-2k}(SU^H; \mathsf{Z}[\tfrac{1}{2}]) \to 0 \\
\| & & \downarrow \tilde{s} & & \downarrow \bar{s} \\
0 \to \sum^{\oplus} \langle e(U^H) \rangle \otimes \mathsf{Z}[\tfrac{1}{2}] & \to & \sum^{\oplus} RO(WH) \otimes \mathsf{Z}[\tfrac{1}{2}] & \to & \sum^{\oplus} \mathrm{RO}(WH)/\langle e(U^H) \rangle \otimes \mathsf{Z}[\tfrac{1}{2}] -
\end{array}$$

with $\bar{s}$ induced from $\tilde{s}$. The range and domain of $\bar{s}$ are abstractly isomorphic, and since $\bar{s}$ is injective, it maps the torsion subgroups isomorphically. The composite

$$\tilde{s}: \tilde{\mathscr{S}}_G(D^{2k} \times SU, \partial) \otimes \mathsf{Z}[\tfrac{1}{2}] \to \sum^{\oplus} \mathrm{Free}(\mathrm{RO}_{2k}(WH)/\langle e(U^H) \rangle) \otimes \mathsf{Z}[\tfrac{1}{2}]$$

is equal to $\tilde{\eta}$, so it is onto by (10.3) and sect. 2. Hence $\bar{s}$ must be surjective.

We end the paper by summarizing our results on the homotopy groups of $\tilde{F}_G/\widetilde{\mathrm{PL}}$ and F_G/PL_G when $|G|$ is odd.

Let U be a fixed $\mathsf{R}G$-module with $U \oplus \mathsf{R}^{2k}$ topological stable and $U^G = 0$. Let m: (Iso(SU)) $\to \mathsf{Z}$ be the dimension function, $m(H) = \dim U^H$. We have the associated (simple) equivariant L-groups

$$\mathscr{L}_m(G; SU) = \sum_{(H)}^{\oplus} L_{m(H)}(\mathsf{Z}(WH]), \quad (H) \in (\mathrm{Iso}(SU))$$

$$\mathscr{L}_m(G; U) = \sum_{(H)}^{\oplus} L_{m(H)}(\mathsf{Z}[WH]), \quad (H) \in (\mathrm{Iso}(U)).$$

With the notation of (6.10) we have

THEOREM 10.5. (i) $\pi_k(\tilde{F}_G(U_0^\infty)/\widetilde{\mathrm{PL}}_G(U_0^\infty)) = \mathscr{L}_{k+m}(G; SU)$.
(ii) $\pi_k(F_G(U^\infty)/\mathrm{PL}_G(U^\infty)) = \mathscr{L}_{k+m}(G; U)$.

PROOF. We prove (i). The space $\tilde{F}(U_0^\infty)/\widetilde{\mathrm{PL}}(U_0^\infty)$ is a G-infinite loop space, so the homotopy groups of its H-fixed sets become a Mackey functor on Iso (SU). In particular, we have the decomposition (cf. [28]):

$$\pi_k(\tilde{F}_G(U_0^\infty)/\widetilde{\mathrm{PL}}_G(U_0^\infty))_{(2)} = \sum_{(H)}^{\oplus} \pi_k(\tilde{F}_G(U_0^\infty)/\widetilde{\mathrm{PL}}_G(U_0^\infty))_{q(H,2)},$$

$$= \sum_{(H)}^{\oplus} \pi_k(\tilde{F}_k(U_0^\infty)/\mathrm{PL}_H(U_0^\infty))^{WH}_{q(H,2)}$$

associated with the maximal ideals in the Burnside ring of G-sets with isotropy groups in Iso (SU). There is a similar decomposition

$$\mathscr{L}_m(G; SU)_{(2)} \cong \sum^{\oplus} \mathscr{L}_m(G; SU)_{q(H,2)}$$

$$\sum^{\oplus} \mathscr{L}_m(H; SU)^{WH}_{q(H,2)}.$$

Thus it suffices to consider the 'top' component associated with the maximal ideal $q(G, 2)$. This component may alternatively be described as the kernel of the restriction map to all $H \in$ Iso (SU). Our suspension result (9.8) together with (6.10 (ii)) gives

$$\pi_k(\tilde{F}_G(U_0^\infty)/\widetilde{\mathrm{PL}}_G(U_0^\infty)) = \tilde{\mathscr{S}}_G(D^k \times SU, \partial)$$

and by (9.5),

$$\tilde{\mathscr{S}}_G(D^k \times SU, \partial)_{q(G,2)} \cong (\mathscr{L}_{k+m}(G; SU))/\varinjlim \mathscr{L}_{k+m}(H; SU)_{q(G,2)}$$

$$= \mathscr{L}_{k+m}(G; SU)_{q(G,2)}.$$

This proves the 2-local part of (i). Localized away from 2, (i) is a consequence of (10.2). Finally, (ii) follows from (i) and (6.10).

REMARK 10.6. The structure invariant describes the structure set localized away from 2, and even at 2 in degrees congruent to 0 (mod 4). Indeed,

$$\tilde{s}: \tilde{\mathscr{S}}_G(D^{4k} \times SU, \partial) \xrightarrow{\cong} 4\mathscr{R}\mathrm{O}_m(G)$$

where $\mathscr{R}\mathrm{O}_m(G) = \sum_{(H)}^{\oplus} (1 + (-1)^{m(H)}\psi^{-1})\mathrm{R}(WH)$.

REFERENCES

1. D. Anderson, W. C. Hsiang, *The functors K_{-i} and pseudo-isotopies of polyhedra*, Ann. of Math. 105 (1977), 201–223.
2. D. Anderson, W. C. Hsiang, *Extending combinatorial piecewise linear structures on stratified spaces II*, Trans. Amer. Math. Soc. 260 (19080), 223–253.
3. P. L. Antonelli, D. Burghelea, P. J. Kahn, *The concordance homotopy groups of geometric automorphism groups*, Lecture notes in Math. 215, Springer-Verlag.
4. M. F. Atiyah, G. B. Segal, *The index of elliptic operators II*, Annals of Math. 87 (1968), 485–530.
5. M. F. Atiyah, G. B. Segal, *Equivariant K-theory and completion*, J. Differential Geom. 3 1969), 1–18.
6. M. F. Atiyah, I. M. Singer, *The index of elliptic operators I*, Annals of Math. 87 (1968), 484–531.
7. M. F. Atiyah, I. M. Singer, *The index of elliptic operators III*, Annals of Math. 87 (1968), 546–604.
8. G. Bredon, *Equivariant Cohomology Theories*, Lecture notes in Math. 34 (1967), Springer Verlag.
9. G. Brumfiel, I. Madsen, R. J. Milgram, *PL characteristic classes and cobordism*, Annals of Math. 97 (1972), 82–159.
10. G. Brumfield, J. Morgan, *Homotopy theoretic consequences of N. Levitts obstruction theory to transversality for spherical fibrations*, Pacific J. Math. 67 (1976), 1–100.
11. D. Burghelea, R. Lashof, *The homotopy type of the space of diffeomorphisms I, II*, Trans Amer. Math. Soc. 469 (1974), 1–36, 37–50.
12. G. de Rham, *Torsion et type simple d'homotopie*, vol. 34 (1967), Springer Verlag.
13. K. Dovermann, M. Rothenberg, *An equivariant surgery sequence and equivariant diffeomorphism and homeomorphism classification.* Mem. Amer. Math. Soc. vol. 71, No. 379 (1988).
14. A. Dress, *Induction and structure theorems for orthogonal representations of finite groups*, Annals of Math. 102 (1975), 291–325.
15. K. Dovermann, T. Petrie, *G-surgery II*, Mem. Amer. Math. Soc. 260 (1982).
16. H. Hauschild, *Äquivariante homotopie I*, Arch. Math. 29 (1977), 158–165.
17. A. Haefliger, V. Poenaru, *La classification des immersions combinatoires*, Publ. Math. I.H.E.S. 23 (1964), 75–91.
18. A. Haefliger, C. T. C. Wall, *PL bundles in the stable range*, Topology 4 (1965), 209–214.
19. R. Lashof, M. Rothenberg, *G-smoothing theory*, Proc. Symp. Pure Math. 32 (1978), 211–266, Amer. Math. Soc. Providence.
20. L. G. Lewis, J. P. May, J. E. McClure, S. Warner, *Equivariant cohomology theories* (to appear).
21. S. Lopez de Medrano, *Involutions on Manifolds*, Ergebnisse der Mathematik und ihre Grenzgebiete, Band 59 (1971), Springer-Verlag.
22. I. Madsen, *Geometric equivariant bordism and K-theory*, Topology 25 (1986), 217–227.
23. I. Madsen, *Equivariant transversality and the space of manifold structures*, Indian J. Math. vol. 25 (1983).
24. I. Madsen, R. J. Milgram, *The Classifying Spaces for Surgery and Cobordism of Manifolds*, Annals of Math. Studies, nr. 92 (1979), Princeton University Press.
25. I. Madsen, M. Raussen, *Smooth and locally linear G homotopy representations*, Algebraic Topology, Göttingen 1984, Proceedings, Lecture notes in Math. 1172, Springer , 130–156.
26. I. Madsen, M. Rothenberg, *On the classification of G-spheres I: Equivariant transversality*, Acta Math., 160 (1988) 65–104.
27. I. Madsen, M. Rothenberg, *On the classification of* G-spheres III: preprint, Aarhus University, 1985.
28. I. Madsen, J.-A. Svensson, *Induction in unstable equivariant homotopy theory and non-invariance of Whitehead torsion*, Contemp. Math. 37 (1985), 99–113.
29. C. Morlets, *Topologie des Variétés semi-lineares*, Annales Ecole Normal 1 (1968), 313–394.
30. M. Rothenberg, *Torsion invariants and finite transformation groups*, Proc. Symp. Pure Math. vol. 33 (1978), 267–312, Amer. Math. Soc. Providence.
31. C. Rourke, B. Sanderson, *Block bundles I*, Ann. of Math. 87 (1968), 1–28.
32. C. Rourke, B. Sanderson, *Block bundles II*, Ann. of Math. 87 (1968), 255–277.
33. C. Rourke, B. Sanderson, *Block bundles III*, Ann. of Math. 87 (1968), 431–483.

34. C. Rourke, B. Sanderson, *Δ-sets I, II*, Quart. J. Math. Oxford 22 (1971), 321-338, 465–485.
35. G. B. Segal, *Equivariant Stable homotopy theory*, Actes Congrés intern Math. Tome 2 (1970), 59–63.
36. G. B. Segal, *The representating ring of a compact Lie group*, Publ. Math. I.H.E.S. 34 (1968), 113–128.
37. T. tom Dieck, *Transformation groups and representation theory*, Lecture notes in Math. 766 (1979), Springer-Verlag.
38. C. T. C. Wall, *Surgery on Compact Manifolds*, Academic Press, London 1970.
39. C. T. C. Wall, *On the classification of hermitian forms VI: Group rings*, Annals of Math. 103 (1976), 1–80.
40. A. G. Wasserman, *Equivariant differential topology*, Topology 8 (1969, 127–150.
41. D. Burghelea, R. Lashof, M. Rothenberg, *Groups of automorphisms of manifolds*, Lecture nots in Math. 473 (1975), Springer Verlag.
42. W. Lück, I. Madsen, *Equivalent L-theory I, II*, preprints, Aarhus University, 1987, Math. Z. (to appear).
43. M. Nagata, *The equivariant homotopy type of the classifying space for normal maps*, Thesis, University of Chicago, 1987.
44. L. H. Hodgkin, V. P. Snaith, *Topics in K-theory*. Lecture notes in Math. 496, Springer (1975).
45. K. Shimakawa, *Infinite loop G-spaces associated to monoidal G-gradad categories*, Publ. RIMS, Kyoto University (1989).

MATEMATISK INSTITUT
NY MUNKEGADE
DK-8000 ÅRHUSC
DENMARK

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CHICAGO
CHICAGO, IL 60637
U.S.A.