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HERMITIAN NATURAL TENSORS

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Abstract.

We define natural tensors for almost hermitian manifolds, study the space of ∞-jets of almost hermitian structures (g, F) on a disk in $\mathbb{R}^{2n}$, obtain the classification theorem for regular hermitian natural tensors and determine all homogeneous regular hermitian natural connections on almost hermitian manifolds.

§0. Introduction

In [7] Epstein introduces the concept of natural tensor fields on Riemannian manifolds. This concept is in the line of the invariants introduced by Gilkey in [8] and elucidated in [2]. In later papers ([9], [6]) the analogous concept of hermitian invariant is defined for hermitian manifolds and an approach to a good definition for almost hermitian manifolds is given in [11]. These hermitian invariants are also used in [13]. In this paper we give the notion of hermitian natural tensor for almost hermitian manifolds. We do it following the scheme of [7]. After giving some definitions in §2, we study, in §3, the ∞-jet of an almost hermitian structure (g, F): we get the compatibility conditions between the ∞-jets of g and F necessary for (g, F) to be an almost-hermitian structure, then we show that a hermitian natural tensor depends only on the ∞-jet of (g, F) and give the set of these ∞-jets in a more convenient form. In §4 we obtain the classification theorem for regular hermitian natural tensors by using the real representation of $U(n)$ and the Iwahori's version ([12]) of the Weyl's theorem for $U(n)$. Here we can notice that when we restrict our attention to hermitian manifolds, the space of hermitian invariants in [6] is the complexified of the space of regular hermitian natural functions (tensors of type (0,0)) in our definition. This fact follows from 4.4, the Theorem of §3 in [6] and the well-known expression relating the torsion and the curvature of the canonical hermitian connection with the Riemannian

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curvature and the covariant derivatives of the Kaehler form (cfr. [13]). Finally, in §5 we define the natural hermitian connections and determine all the homogeneous regular hermitian natural connections and those such that $Dg = 0$ and $DJ = 0$.

It is also possible to study the C^∞ case, but both the result (a C^∞ hermitian natural tensor of type (p, q) which is homogeneous of weigth w and nonzero is a regular hermitian natural tensor) and the method of proof follow closely those used by Epstein, [7], in the Riemannian case.

The homogeneous regular almost complex (without depending on a metric) natural tensors have been studied in [1]. The results there are very different from ours; for example, there is no homogeneous regular almost complex natural connection.

In a later paper ([4]), we have considered hermitian natural differential operators.

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§1. Notation.

In order to simplify the formulas below we shall adopt the following conventions of notation:

1.1. Capital latin letters I, J, K, L will mean finite sequences of numbers, i.e. $I = (i_1, \dots, i_r)$ with $1 \leqq i_s \leqq 2n, 1 \leqq s \leqq r$ for some integers r and n, and $|I|$ will denote the number of elements of the sequence ($|I| = r$ in the above example).

Given $x = (x^1, \dots, x^{2n}) \in \mathbb{R}^{2n}$, the expresion x^I will mean

$$x^I = x^{i_1} \dots x^{i_r}$$

For example, the Taylor expansion of a function $f\colon \mathbb{R}^{2n} \to \mathbb{R}$

$$f(x) \cong f(0) + \sum_{r \geqq 1} \sum_{k=1}^{r} \sum_{i_k = 1}^{2n} f_{i_1 \dots i_r} x^{i_1} \dots x^{i_r}$$

will be written as

$$f(x) \cong f(0) + \sum_{|I| \geqq 1} f_I x^I.$$

1.2. J will be employed also to denote an almost-complex structure on a manifold, though this should not be misleading.

1.3. Let (M, g, J) be an almost-hermitian manifold of real dimension $2n$. If A_I^{ij} is a number depending on indices i, j, I, with respect to a J-orthonormal basis

$\{e_1,\ldots,e_n,e_{n+1}=Je_1,\ldots,e_{2n+2}=Je_{2n}\}$ of T_pM, then

A_I^{i*j} will denote A_I^{i+nj} if $1\leqq i\leqq n$ and $-A_I^{i-nj}$ if $n+1\leqq i\leqq 2n$.

In a similar way should be understood the * in the indices of A^i_{jI}, A_{ijI}.

1.4. If $\sigma\in S_r, u<r, l=(i_1,\ldots i_u)$, and A_I^{ij} is a number as in 1.3, we define $A^{ij}_{I\sigma(a,b)}=A^{ij}_{i_{\sigma(a)}\ldots i_{\sigma(b)}}$, where $r\geqq b>a\geqq 1$, and $b-a-1=u\leqq r$.

1.5. Greek letters $\alpha,\beta,\ldots$ will denote multi-indices

$$\alpha=(\alpha_1,\ldots,\alpha_{2n}),\qquad \mathrm{B}=(\beta_1,\ldots,\beta_{2n}),\ \alpha_j,\beta_j\in\mathbf{Z}^+, 1\leqq \mathrm{j}\leqq 2n,$$

$$|\alpha|=\alpha_1+\ldots+\alpha_{2n},\text{ and }\frac{\partial^{|\alpha|}}{\partial x^\alpha}=\frac{\partial^{\alpha_1+\ldots+\alpha_n}}{(\partial x^1)^{\alpha_1}\ldots(\partial x^{2n})^{\alpha_{2n}}}$$

1.6. We shall use the Einstein convention of sumation for repeated indices, even if one of them is affected by *.

1.7. If $\{\Omega_\alpha\}_{\alpha\in\mathsf{A}}$ is a family of 2-contravariant tensors and $A=(\alpha_1,\ldots,\alpha_s)$ is a finite sequence of elements of A, we shall denote

$$\Omega_A^I=\Omega_{\alpha_1}^{i_1i_2}\ldots\Omega_{\alpha_s}^{i_{2s-1}i_{2s}}$$

1.8. If g is a 2-covariant tensor, $I=(i_1,\ldots,i_m)$, $\sigma\in S_m$, $J=(j_1,\ldots,j_q)$, $0\leqq r\leqq s\leqq m$, $s-r=q$, we shall denote

$$g_{I\sigma(r,s)J}=g_{i_{\sigma(r)}j_1}\cdots g_{i_{\sigma(s)}j_q}.$$

1.9. If ω^i are 1-forms and e_j vectors in $T_pM, I=(i_1,\ldots,i_r)$, $J=(j_1,\ldots,j_s)$, we shall denote

$$\omega^I=\omega^{i_1}\otimes\ldots\otimes\omega^{i_r},\quad e_J=e_{j_1}\otimes\ldots\otimes e_{j_s}.$$

§2. The definition of a hermitian natural tensor.

DEFINITION 2.1. A hermitian natural tensor is a map t which associates to each almost-hermitian manifold (M,g,J) a section $t_{(M,g,J)}$ of the tensor algebra over M which is natural in the following sense: 'if $\psi:(M,g,J)\to(M',g',J')$ is a holomorphic $(J'\cdot\psi_*=\psi_*\cdot J)$ isometry of M onto an open set of M', then $\psi_*t_{(M,g,J)}=t_{(M',g',J')}|_{\psi}(M)$'.

t is said of type (p,q) if $t_{(M,g,J)}(m)$ is a p-times contravariant and q-times covariant tensor, at each point m of M.

t is said to be homogeneous of weight w if

$$t_{(M,c^2g,J)}=c^wt_{(M,g,J)},\ (c\in\mathsf{R},c>0).$$

To introduce the regularity condition H. Donnelly, [6], uses a system of holomorphic coordinates on a hermitian manifold; thus, in a general almost hermitian manifold, we need the following:

DEFINITION 2.2. Let (M, g, J) be an almost hermitian manifold of real dimension $2n$ and x a point of M. A coordinate system $(x^1, \ldots, x^{2n})$ centred at x will be called a J-coordinate system if $(\partial/\partial x^{n+j})(x) = J(\partial/\partial x^i)(x)$. A J-coordinate system which is normal with respect to the Levi-Civita connection and the vectors $\partial/\partial x^j$ are orthonormal at the origin x will be called a J-normal coordinte system.

DEFINITION 2.3. Let t be a hermitian natural tensor of type (p, q). We shall say that t is regular if for each almost hermitian manifold (M, g, J), any point x in M and every J-coordinate system centered at x, the components $t_K^I(I = (i_1, \ldots, i_p), K = (j_1, \ldots, j_q))$ of t are universal polynomials in the variables

$$g_{ij}, g^{kl}, F_{rs}, \frac{\partial^{|\alpha|}}{\partial x^\alpha} g_{ij}, \frac{\partial^{|\beta|}}{\partial x^\beta} F_{rs},$$

where α, β are multi-indices and F is defined, as usually, by $F(X, Y) = g(JX, Y)$.

Remark. The expression 'universal polynomial' means that this polynomial expression allows to compute t_K^I in all J-coordinate systems.

As in the riemannian case ([2], [7], [8]) a hermitian natural tensor is clearly locally defined and the following is equivalent to Definition 2.1.

DEFINITION 2.4. A hermitian natural tensor is a map which associates to each disk $D(r)$ in $\mathbb{R}^{2n}$ with centre at 0 and radius r and each almost hermitian structure (g, J) on $D(r)$ a C^∞ tensor field $t_{(D(r), g, J)}$ on $D(r)$ such that if $\psi: (D(r), g, J) \to (D(s), g', J')$ is a holomorphic isometry onto an open set of $D(s)$, then $\psi_* t_{(D(r), g, J)} = t_{(D(s), g', J')}|_{\psi(D(r))}$.

§3. The set of ∞-jets of g and F.

In the following we use Definition 2.4.

THEOREM 3.1. *$t_{(D(r), g, J)}(0)$ depends only on the ∞-jets of g and F (or J) at 0.*

PROOF. Let (g_1, F_1) and (g_2, F_2) be two almost hermitian structures on a disk D such that g_1 and g_2 have the same ∞-jet at 0 and F_1 and F_2 have the same ∞-jet at 0. Let $V_1 = \{x \text{ in } D: x_1 \geq 2n|x_j|, \text{ for } 2 \leq i \leq 2n\}$ and $V_2 = \{x \text{ in } D: -x_1 \geq 2n|x_i|, \text{ for } 2 \leq i \leq 2n\}$. Then $V_1 \cap V_2 = \{0\}$. Take an orthonormal basis $\{e_1 \ldots, e_n, e_{n+1}, \ldots, e_{2n}\}$ of $T_0 D$ (with respect to $g_1(0) = g_2(0)$) such that $e_{n+i} = J_1(0)e_i$ (recall that $J_1(0) = J_2(0)$). Take an extension of $\{e_k\}$ to a g_i-orthonormal frame $\{E_k^i\}$, such that $E_{n+j}^i = J_i E_j^i$, $i = 1,2$ (here we don't use Einstein convention). Then we have C^∞ functions a_k^{ij} defined on an open disk $D(s) \subset D$ by $E_k^i = a_k^{ij}\, \partial/\partial x^j$. The standard procedure to get a local orthonormal frame of the form $E_1, \ldots, E_n$, $JE_1, \ldots, JE_n$ gives us the a_k^{ij} as C^∞-functions in the variables

g_{jk}, g^{lm} and F_{rs}. Then, because of the coincidence of ∞-jets at 0, the functions a_k^{1j} and a_k^{2j} have the same ∞-jet at 0. By Withney's extension theorem (applied as in [7]), there are C^∞ functions a_k^j on a disk $D(r) \subset D$ such that $\det(a_k^j) \neq 0$ on $D(r)$ and $a_k^j|_{D(r) \cap V_i} = a_k^{ij}|_{D(r) \cap V_i}$, $i = 1,2$. Then we define $E_k = a_k^j \, \partial/\partial x^j$ on $D(r)$; $JE_i = E_{n+i}$, $JE_{n+i} = -E_i$ for $1 \leqq i \leqq n$; $g(E_i, E_j) = \delta_{ij}$ for $1 \leqq i,j \leqq 2n$, and $F(X, Y) = g(JX, Y)$, for tangent vectors fields X, Y on $D(r)$. This defines an almost-hermitian structure (g, F) on $D(r)$ which coincides with (g_i, F_i) on $D(r) \cap V_i$. Thus, if t is a hermitian natural tensor, we have

$$t_{(D,g_1,F_1)}(0) = t_{(D(r),g_1,F_1)}(0) = t_{(D(r),g_1,F_1)|D(r) \cap V_1}(0) = t_{(D(r),g,F)}(0). \quad Q.E.D.$$

We can compute $t_{(D(r),g,J)}(0)$ in any J-coordinate system and, in partiular, in a J-normal coordinate system centered at 0. By using these coordinates we have, for $g_{ij}(x)$ and $F_{ij}(x)$ the Taylor series:

$$g_{ij}(x) \cong \delta_{ij} + \sum_{|I| > 1} g_{ijI} x^I, \tag{3.1}$$

$$F_{ij}(x) \cong \delta_{i^*j} + \sum_{|I| \geqq 1} F_{ijI} x^I. \tag{3.2}$$

In [7] it is proved that the coefficients g_{ijI} in (3.1) satisfy the following conditions:

(3.G.1) They are symmetric in the first two indices,

(3.G.2) They are symmetric in the last r indices.

(3.G.3) $\sum_{\sigma \in S_{r+1}} g_{iI\sigma(1,r+1)} = 0$. In particular, $g_{iji_1} = 0$.

In order to obtain the conditions satisfied by F_{ijI} we first consider the Taylor series of the entries g^{ij} of the matrix (g^{ij}) inverse of (g_{ij}):

$$g^{ij}(x) \cong \delta^{ij} + \sum_{|I| > 1} g_I^{ij} x^I \tag{3.3}$$

From the conditions $g^{ij} g_{jk} = \delta_k^i$, it follows that g_I^{ij} are functions of g_{ijI} given by the recurrent formulae

(3.G.4.r) $r!\, g_I^{ij} + r!\, g_{ijI} + \sum_{\sigma \in S_r} \sum_{s=2}^{r-2} g_{ikI\sigma(1,s)} g_{I\sigma(s+1,r)}^{kj} = 0$ for all $r \geqq 2$.

In particular, for $r = 2$ and 3,

(3.G.4.2) $g_{i_1 i_2}^{ij} + g_{iji_1 i_2} = 0$.

(3.G.4.3) $g_{i_1 i_2 i_3}^{ij} + g_{iji_1 i_2 i_3} = 0$.

If (g, F) defines an almost hermitian structure on $D(r)$, the tensor J defined by $g(JX, Y) = F(X, Y)$ verifies $J^2 = -\mathrm{id}$, which is equivalent to $g(JX, JY) = g(X, Y)$, since F is skewsymmetric. From $g(JX, Y) = F(X, Y)$ we have $J_i^k g_{kj} = F_{ij}$, then $J_i^k = F_{ij}g^{jk}$, and $J^2 = -\mathrm{id}$ is so equivalent to $F_{ik}g^{kl}F_{lm}g^{mj} = -\delta_i^j$, which using the series (3.2) and (3.3) gives

$$-\delta_i^j + \sum_{|I| \geqq 1} A_{ijI}x^I + \sum_{|I|,|J| \geqq 1} B_{ijIJ}x^I x^J + \sum_{|I|,|J|,|K| \geqq 1} C_{ijIJK}x^I x^J x^K$$

$$+ \sum_{|I|,|J|,|K|,|L| \geqq 1} D_{ijIJKL}x^I x^J x^K x^L = 0,$$

where

$A_{ijI} = F_{i^*jI} - F_{ij^*I} - g_I^{ij} - g_I^{i^*j^*}$,
$B_{ijIJ} = F_{i^*kI}g_J^{kj} + F_{ikI}g_J^{k^*j} + g_I^{i^*k}F_{kjJ} - F_{ikI}g_J^{kj^*} + F_{ikI}F_{kjJ} + g_I^{i^*k}g_J^{k^*j}$,
$C_{ijIJK} = F_{ikI}F_{klJ}g_K^{lj} + F_{ikI}g_J^{kl}F_{ljK} + g_I^{i^*k}F_{klJ}g_K^{lj} + F_{ikI}g_J^{kl}g_K^{l^*j}$,
$D_{ijIJKL} = F_{ikI}g_J^{kl}F_{lmK}g_L^{mj}$,
and $g_I^{ij} = 0$ if $|I| = 1$.

From this polynomial equation, the skewsymmetry of F and the equality of cross derivatives, we obtain the conditions that the coefficients F_{ijI} have to satisfy:

(3.F.1) They are skewsymmetric in the first two indices;
(3.F.2) They are symmetric in the last r indices;
(3.F.3) They verify the equalities:

$$\text{(3.F.3.r)}\quad r!\,A_{ijI} + \sum_{\sigma\in S_r}\left\{\sum_{s=1}^{r-1} B_{ijI\sigma(1,s)J\sigma(s+1,r)} + \sum_{s=1}^{r-3}\sum_{t=s+1}^{r-1} C_{ijI\sigma(1,s)J\sigma(s+1,t)K\sigma(t+1,r)} + \sum_{s=1}^{r-5}\sum_{t=s+2}^{r-3}\sum_{u=t+1}^{r-2} D_{ijI\sigma(1,s)J\sigma(s+1,t)K\sigma(t+1,u)L\sigma(u+1,r)}\right\} = 0$$

In particular, for $r = 1, 2$ we have

$$\text{(3.F.3.1)}\quad F_{i^*ji_1} - F_{ij^*i_1} = 0.$$

$$\text{(3.F.3.2)}\quad -2g_{i_1i_2}^{ij} - 2g_{i_1i_2}^{i^*j^*} + F_{iki_1}F_{kji_2} + F_{iki_2}F_{kji_1} + 2F_{i^*ji_1i_2} - 2F_{ij^*i_1i_2} = 0.$$

As it is pointed out in [7], each coefficient g_{ijI}, $|I| = r$, defines a $(0, r+2)$-tensor g_r at 0 by the formula

$$g_r(e_i, e_j, e_{i_1}, \ldots, e_{i_r}) = g_{iji_1\ldots i_r}, \text{ where } e_k = \partial/\partial x^k(0).$$

Similarly, since different J-normal coordinate systems are related by an element of $U(n)$, each F_{ijI}, $|I| = r$, defines a $(0, r+2)$-tensor f_r at 0 by

$$f_r(e_i, e_j, e_{i_1}, \ldots, e_{i_r}) = F_{iji_1\ldots i_r}, \tag{3.4}$$

and for these tensors we have

$$F_{i^*ji_1\ldots i_r} = f_r(Je_i, e_j, e_{i_1}, \ldots, e_{i_r}).$$

Let V denote the tangent space at 0. Let M be the set of sequences (g_r, f_r) in $\prod_{r\geq 1}\left(\overset{r+2}{\otimes} V^* \times \overset{r+2}{\otimes} V^*\right)$ satisfying the conditions (3.G.1) to (3.G.3) and (3.F.1) to (3.F.3). From (3.1) and (3.2) M can be considered as the set of ∞-jets at 0 of the almost-hermitian structures (g, F) on $D(s)$. For any (g, F), $j(g, F)$ will denote the corresponding element of M. Then we have the following.

DEFINITION 3.2. For every hermitian natural tensor of type (p, q) we define a map $\mathscr{T}$ from M to $\otimes^p V \otimes \otimes^q V^*$ as $\mathscr{T}(j(g, F)) = t_{(g,F)}(0)$.

DEFINITION 3.3. Given A in $U(n)$, we define the action of A on the structures (g, F) on $D(s)$ as follows: let $(x^1, \ldots, x^{2n})$ be a J-normal coordinate system centred at 0. Let $(y^1, \ldots, y^{2n})$ be the J-normal coordinate system at 0 such that $\partial/\partial y^j(0) = A^{-1}((\partial/\partial x^j)(0))$. Then we define $A_* g$, on a neighborhood of 0, by

$$(A_* g)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)(p) = g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)(p).$$

Indeed, $A_* g$ is essentially the same tensor g as viewed in the new J-normal coordinate system. Similar definition is given for $A_* F$, and $A_*(g, F) = (A_* g, A_* F)$, which is also an almost hermitian structure.

Notice that this definition is necessary in order that the action of A be defined on the tangent space of each point of a neighbourhood of 0, and not only on V.

REMARK. The above definitions yield

$$j(A_*(g, F))(0) = A_* j(g, F),$$

where the action of A on a tensor β in $\otimes^r V \otimes \otimes^s V^*$ is defined as usually by

$$(A_*\beta)(\omega^1, \ldots, \omega^r, X_1, \ldots, X_s) = \beta(A^*\omega^1, \ldots, A^*\omega^r, A^{-1}X_1, \ldots, A^{-1}X_s).$$

The following proposition shows the invariance of $\mathscr{T}$.

PROPOSITION 3.4. *If t is a hermitian natural tensor of type (p, q) then $\mathscr{T}$ is invariant by $U(n)$, i.e., for any A in $U(n)$, $A_*\mathscr{T}(j(g, F)) = \mathscr{T}(A_* j(g, F))$.*

PROOF. Since t is natural,

$A_* t_{(g,F)} = t_{A_*(g,F)}$, and then,

$A_*\mathscr{T}(j(g, F)) = A_* t_{(g,F)} = t_{A_*(g,F)} = \mathscr{T}(j(A_*(g, F))) = \mathscr{T}(A_* j(g, F)).$

The next theorem is an extension to the almost hermitian situation of a well known result in riemannian geometry. A practical interest of it is in the study of the linear dependence of a system of generators of a certain space of homogeneous regular natural hermitian tensors, as we do in 5.2 and 5.3 Without it, it would be necessary to compute the tensors on concrete examples (see, for example, [11]).

THEOREM 3.5. *If* $(g_r, f_r)_{r \geqq 1}$ *is in* M *then there is an almost hermitian structure* (g, F) *on a neighborhood of* 0 *whose Taylor series expansion gives us the elements* $(g_r, f_r)_{r \geqq 1}$.

PROOF. Let $(g_r, f_r)_{r \geqq 1} \in$ M and take $V \equiv \mathrm{R}^{2n} \equiv \mathrm{C}^n$. Then we have the coefficients $g_{ijI}, F_{ijI} \in \mathrm{R}$. Let $g'_{ij} = g'_{ji}$ and $F'_{ij} = -F'_{ji}$ be C^∞-functions with derivatives at the origin given by

$$\frac{\partial^{|I|} g'_{ij}}{\partial x^I} = r!\, g_{ijI}, \quad \frac{\partial^{|I|} F'_{ij}}{\partial x^I} = r!\, F_{ijI},$$

and satisfying $g'_{ij}(0) = \delta_{ij}$, $F'_{ij}(0) = \delta_{i*j}$. Then, the pair (g', F') defines an almost hermitian structure at the origin, i.e.:

$$F'_{ik} g'^{kl} F'_{lm} g'^{mj}(0) = -\delta^j_i,$$

and, in a neighborhood of 0,

$$F'_{ik} g'^{kl} F'_{lm} g'^{mj} = -\lambda^j_i$$

where $\lambda = (\lambda^j_i)$ is a matrix of positive determinant which represents a g′-symmetric endomorphism λ. In fact, $\lambda = -J'^2$, where J' is the tensor defined by $g'(J'X, Y) = F'(X, Y)$ and, since F'_{ij} is skewsymmetric, $g'(J'^2X, Y) = -g'(J'X, J'Y) = g'(X, J'^2Y)$, i.e., J'^2 is a g′-symmetric endomorphism. Then, there is a matrix $\mu = \mu^j_i$ of positive determinant which represents a g'-symmetric endomorphism μ, such that $\mu^4 = \lambda$. Moreover, since $-\lambda(0) = -\mathrm{id}$ (id = identity matrix), we have $\mu(0) = \mathrm{id}$. Now, we define

$$F_{ij} = F'_{ik}\mu^k_j \qquad g_{ij} = g'_{ik}\mu^k_j. \tag{3.5}$$

Then, we have $F_{ij}(0) = F'_{ij}(0), g_{ij}(0) = g'_{ij}(0)$. Let

$$\mu^j_i \cong \delta^j_i + \sum_{|I| \geqq 1} \mu^j_{iI} x^I$$

be the Taylor series of the μ^j_i. The equations (3.F.1) to (3.F.3) are equivalent to

$$\frac{\partial^{|I|}}{\partial x^I}(-\lambda^j_i)(0) = -\frac{\partial^{|I|}}{\partial x^I}(\mu^k_i \mu^l_k \mu^m_l \mu^j_m)(0) \text{ for every } I = (i_1, \ldots, i_r),$$

and these are equivalent to

$$\left(\delta_i^k + \sum_{|I|\geqq 1} \mu_{iI}^k x^I\right)\left(\delta_k^l + \sum_{|J|\geqq 1} \mu_{kJ}^l x^J\right)$$

$$\left(\delta_l^m + \sum_{|K|\geqq 1} \mu_{lK}^m x^K\right)\left(\delta_m^j + \sum_{|L|\geqq 1} \mu_{mL}^j x^L\right) = \delta_i^j,$$

whence we get the following conditions:

$$4\mu_{ii_1}^j = 0; 2\cdot 4\mu_{ii_1i_2}^j + \beta_2 = 0; \ldots; r!4\mu_{ii_1\ldots i_r}^j + \beta_r = 0; \ldots$$

where β_r is a polynomial in the μ_{kI}^l, $1 \leqq |I| \leqq r-1$. That implies $\mu_{kI}^l = 0$ for every I, i.e., $(\partial^{|I|}\mu_i^j/\partial x^I)(0) = 0$. Then, from (3.5), we have

$$\frac{\partial^{|I|}F_{ij}}{\partial x^I}(0) = \frac{\partial^{|I|}F'_{ij}}{\partial x^I}(0) \text{ and } \frac{\partial^{|I|}g_{ij}}{\partial x^I}(0) = \frac{\partial^{|I|}g'_{ij}}{\partial x^I}(0),$$

and $F_{ik}g^{kl}F_{lm}g^{mj} = -\delta_i^j$ in a neighborhood of 0. Thus (g, F) defines an almost hermitian structure on a disk centred at 0, whose Taylor series has the coefficients g_{ijI} and F_{ijI}. However the coordinates may not be normal, but it is shown in [7] that if we change to normal coordinates for g, then the ∞-jet of the change of coordinates is the identity at 0, so that in a J-normal coordinate system the ∞-jet of (g, F) is the sequence $(g_r, f_r)_{r\geqq 1}$.

The next theorem provides a new description of the set M.

THEOREM 3.6. *Let V be the tangent space at 0 endowed with an almost hermitian structure (g, J) and let $W = \{f \in V^* \wedge V^*: f(JX, JY) = -f(X, Y)\}$. Then there is a $U(n)$-invariant bijection from* M *to the vector subspace $\prod_{r\geqq 1} (Y_r \times (W \otimes \odot^r V^*))$ of the vector space $\prod_{r\geqq 1} (\otimes^{r+2} V^* \times \otimes^{r+2} V^*)$, where $\odot$ means symmetric tensor product and Y_r is an irreducible* GL (V)*-submodule with Young diagram having r squares in the first row and 2 squares in the second row, except that if $r = 1$, $Y_r = \{0\}$.*

PROOF. It is shown in [7] that the set of sequences $(g_r)_{r\geqq 2}$ (g_r belongs to $\otimes^{r+2} V^*$) is in 0(2n)-invariant bijection with the vector space $\prod_{r\geqq 2} Y_r$. On the other hand, the map from $V^* \wedge V^* \otimes \odot^r V^*$ to $W \otimes \odot^r V^*$, denoted by $\tilde{}$, defined by

(3.6) $2\tilde{f}_r(X, Y, Z_1, \ldots, Z_r) = f_r(X, Y, Z_1, \ldots, Z_r) - f_r(JX, JY, Z_1, \ldots, Z_r)$

is $U(n)$-invariant. The equation (3.F.3.r) is equivalent to

$$f_r(e_i, e_j, e_{j_1}, \ldots, e_{j_r}) + f_r(Je_i, Je_j, e_{j_1}, \ldots, e_{j_r}) = F_{ijI} + F_{i^*j^*I} =$$

$$= A_{ij^*I} - F_{i^*j^*I} + g_I^{ij^*} - g_I^{i^*j} + F_{i^*j^*I} =$$

$$= -\frac{1}{r!}\sum_{\sigma\in S_r}\left\{\sum_{s=1}^{r-1} B_{ij^*I\sigma(1,s)J\sigma(s+1,r)} + \sum_{s=1}^{r-3}\sum_{t=s+1}^{r-1} C_{ij^*I\sigma(1,s)J\sigma(s+1,t)K\sigma(t+1,r)}\right.$$

$$\left. + \sum_{s=1}^{r-5}\sum_{t=s+2}^{r-3}\sum_{u=t+1}^{r-2} D_{ij^*I\sigma(1,s)J\sigma(s+1,t)K\sigma(t+1,u)L\sigma(u+1,r)}\right\} + g_I^{ij^*} - g_I^{i^*j}.$$

Then the sequences $(g_r, f_r)_{r\geqq 1}$ satisfy (3.F.3) if and only if

$$f_r(X, Y, Z_1, \ldots, Z_r) + f_r(JX, JY, Z_1, \ldots, Z_r) = \alpha_r(X, Y, Z_1, \ldots, Z_r) \tag{3.7}$$

for every r, where α_r is a tensor given by

$$\alpha_r(e_i, e_j, e_{j_1}, \ldots, e_{j_r}) =$$

$$= -\frac{1}{r!}\sum_{\sigma\in S_r}\left\{\sum_{s=1}^{r-1} B_{ij^*I\sigma(1,s)J\sigma(s+1,r)} + \sum_{s=1}^{r-3}\sum_{t=s+1}^{r-1} C_{ij^*I\sigma(1,s)J\sigma(s+1,t)K\sigma(t+1,r)}\right.$$

$$\left. + \sum_{s=1}^{r-5}\sum_{t=s+2}^{r-3}\sum_{u=t+1}^{r-2} D_{ij^*I\sigma(1,s)J\sigma(s+1,t)K\sigma(t+1,u)L\sigma(u+1,r)}\right\} + g_I^{ij^*} - g_I^{i^*j}.$$

Observe that from (3.G.4.r) and the definitions of B, C, D it follows that the $B_{ij^*I\sigma(1,s)J\sigma(s+1,r)}$, in the above expressions of α_r are functions of f_s, g_{r-s}, g_s, f_{r-s}, with $1 \leqq s \leqq r-1$; the $C_{ij^*I\sigma(1,s)J\sigma(s+1,t)K\sigma(t+1,r)}$ are functions of f_s, f_{t-s}, g_{r-t}, f_{r-t}, g_{t-s}, f_{r-t}, g_s, with $1 \leqq s \leqq r-3$, $s+1 \leqq t \leqq r+1$; the $D_{ij^*I\sigma(1,s)J\sigma(s+1,t)K\sigma(t+1,u)L\sigma(u+1,r)}$ are functions of f_s, g_{t-s}, f_{u-t}, g_{r-u}, with $1 \leqq s \leqq r-5$, $s+2 \leqq t \leqq r-3$, $t+1 \leqq u \leqq r-2$; and $g_I^{ij^*} - g_I^{i^*j}$ depends only on g_s, $2 \leqq s \leqq r$. Then α_r depends only on $g_2, \ldots, g_r$, $f_1, \ldots, f_{r-1}$.

If we restrict the map $\tilde{}$ to M we have, from (3.6) and (3.7), the $2\,\tilde{}f_r = 2f_r - \alpha_r$. Then on M there exists an inverse map of $\tilde{}$ given by $2f_r = 2\,\tilde{}f_r + \alpha_r$ and this completes the proof.

As a consequence of this theorem,

$$\prod_{r\geqq 1} (Y_r \times (W \otimes \odot^r V^*))$$

can also be considered as the space of ∞-jets of the almost hermitian structures (g, F) on $D(s)$. Moreover, if t is a natural tensor, the map $\mathscr{T}$ of the definition 3.2 can be viewed as a map from the above space to $\otimes^p V \otimes \otimes^q V^*$ given by

$$\mathscr{T}((\tilde{}g_r, \tilde{}f_r)_{r\geqq 1}) = \mathscr{T}((g_r, f_r)_{r\geqq 1}). \tag{3.8}$$

We also have the following reformulation of the proposition 3.4.

PROPOSITION 3.7. *If t is a hermitian natural tensor of type (p, q), then $\mathscr{T}$, considered as in* (3.8), *is $U(n)$-invariant.*

PROOF. Let A be in $U(n)$. Since $\tilde{}$ is $U(n)$-invariant, we have

$A_*(\tilde{}g_r, \tilde{}f_r) = A_*\tilde{}(g_r, f_r) = \tilde{}A_*(g_r, f_r)$.

By the Proposition 3.4

$$A_*\mathscr{T}(g_r, f_r) = \mathscr{T}A_*(g_r, f_r).$$

Then, using (3.8), we have

$$\mathscr{T}A_*(\tilde{}g_r, \tilde{}f_r) = \mathscr{T}(\tilde{}^{-1}A_*(\tilde{}g_r, \tilde{}f_r)) = \mathscr{T}(\tilde{}^{-1}\tilde{}A_*(g_r, f_r)) = \mathscr{T}A_*(g_r, f_r) =$$
$$= A_*\mathscr{T}(g_r, f_r) = A_*\mathscr{T}(\tilde{}g_r, \tilde{}f_r).$$

§4. The classification theorem for regular hermitian natural tensors.

LEMMA 4.1 (*Weyl's Theorem for a real representation of* $U(n)$, [12]; *see also* [14]). *Let* $V \equiv \mathrm{R}^{2n} \equiv \mathrm{C}^n$ *endowed with the canonical hermitian structure* (g, F). *Then the* R-*module* $\mathrm{Hom}_{U(n)}(\otimes^r V^*, \mathrm{R})$ *of* $U(n)$-*invariant* R-*linear maps from* $\otimes^r V^*$ *to* R *vanishes for r odd, and when r is even,* $r = 2s$, *it is spanned by elementary contractions of the type*

$$\psi_\sigma^{\alpha_1 \ldots \alpha_s}(\omega^1 \otimes \ldots \otimes \omega^r) = \Omega_{\alpha_1}(\omega^{\sigma(1)}, \omega^{\sigma(2)}) \ldots \Omega_{\alpha_s}(\omega^{\sigma(2s-1)}, \omega^{\sigma(2s)}).$$

where σ *is a permutation of* $\{1, \ldots, r\}$, $1 \leqq \alpha_1, \ldots, \alpha_s \leqq 2$, $\Omega_1 = g^{\cdot\cdot}$ *and* $\Omega_2 = F^{\cdot\cdot}$, $g^{\cdot\cdot}$ *and* $F^{\cdot\cdot}$ *being the metric and the Kaehler form induced on* V^* *by* $g_{..}$ *and* $F_{..}$, *respectively, through the canonical isomorphism between* V *and* V^* *given by the metric* g.

LEMMA 4.2. *The* R-*module* $\mathrm{Hom}_{U(n)}(\otimes^r V^*, \otimes^p V \otimes \otimes^q V^*)$ *of* $U(n)$-*invariant* R-*linear homomorphisms is zero if* $r + p + q$ *is odd and, if* $r + p\ q = 2s$, *it is spanned by the elements of the form*

$$\psi_\sigma^A(\omega_{I_r} dx^{I_r}) = \Omega_A^I\, \omega_{I\sigma(1,r)} g_{I\sigma(r+p+1,r+p+q)J} \frac{\partial}{\partial x^{I\sigma(r+1,r+p)}} \otimes dx^J,$$

where $I_r = (i_1, \ldots, i_r)$, $I = (i_1, \ldots, i_{2s})$, $A = (\alpha_1, \ldots, \alpha_s)$, $J = (j_1, \ldots, j_q)$, σ *is a permutation of* $\{1, \ldots, 2s\}$, $1 \leqq \alpha_1, \ldots, \alpha_s \leqq 2$.

PROOF. Let α be in $E = \mathrm{Hom}(\otimes^{r+p} V^* \otimes \otimes^q V, \mathrm{R})$ and denote by $\underline{\alpha}$ its image in $B = \mathrm{Hom}(\otimes^r V^*, \otimes^p V \otimes \otimes^q V^*)$ under the canonical isomorphism. It is easy to see that α is $U(n)$-invariant if and only if $\underline{\alpha}$ is. On the other hand, the canonical isomorphism between E and $\mathrm{C} = \mathrm{Hom}(\otimes^{r+p+q} V^*, \mathrm{R})$ induced by the metric preserves the $U(n)$-invariant elements. Then we can obtain a system of generators of $B_{U(n)} = \mathrm{Hom}_{u(n)}(\otimes^r V^*, \otimes^p V \otimes \otimes^q V^*)$ by taking such a system in $\mathrm{C}_{u(n)}$ and their images by the above isomorphisms. The generators of $C_{U(n)}$ are given in lemma 4.1. They can be written as:

$$\Phi_\sigma^A = \Omega_A^I \frac{\partial}{\partial x^{I\sigma(1,2s)}},$$

with $2s = p + q + r$. Their images in B under the isomorphisms given above are

$$\psi_\sigma^A = \Omega_A^I g_{I\sigma(r+p+1,2s)J} \frac{\partial}{\partial x^{I\sigma(1,r+p)}} \otimes dx^J,$$

and their action on the r-covariant tensor $\omega = \omega_I dx^I$ gives just the required elements. Q.E.D.

THEOREM 4.3. *Let t be a regular hermitian natural tensor of type (p, q). Then t is a sum of tensors whose components have the form*

$$\underbrace{F^{\cdot\cdot}\ldots F^{\cdot\cdot}}_{2b}\underbrace{g_{..}\ldots g_{..}}_{2b}\Omega^{\cdot\cdot}_{\alpha_1}\ldots\Omega^{\cdot\cdot}_{\alpha_s}\prod_{r=1}^{N}\underbrace{(\underbrace{f_{r\ldots\ldots}}_{r+2}\cdots\underbrace{f_{r\ldots\ldots}}_{r+2})}_{h_r^a}\prod_{r=2}^{N}\underbrace{(\underbrace{g_{r\ldots\ldots}}_{r+2}\cdots\underbrace{g_{r\ldots\ldots}}_{r+2})}_{h_r^Y}\underbrace{g_{..}\cdots g_{..}}_{q},$$

where $0 \leqq 2b \leqq 2\left(\sum_{r=1}^{N} h_r^a\right)$, $2s = \kappa + p + q$, $\kappa = 3\sum_{r=2}^{N}(r+2)(h_r^a + h_r^Y)$; N, h_r^a *and* h_r^Y *are natural numbers, and the following contractions are taken:*

a) *The first index of each* $F^{\cdot\cdot}$ *with the first index of each of the first* $2b$ $g_{..}$*'s;*

b) *Take b of the* f_r*'s, then the second index of each* $F^{\cdot}$ *is contracted with one of the first two indices of these* f_r*'s;*

c) *The first indices of the last q* $g_{..}$*'s, the rest of indices of* f_r*'s, the second index of each of the first* $2b$ $g_{..}$*'s and all the indices of the* g_r*'s are contracted with* $\kappa + q$ *indices of the* $\Omega^{\cdot\cdot}_{\alpha_1}, \ldots, \Omega^{\cdot\cdot}_{\alpha_s}$. *There remain p upper and q lower non-contracted indices.*

PROOF. t is regular if and only if $\mathscr{T}: \mathsf{M} \to \otimes^p V \otimes \otimes^q V^*$ is a polynomial in a finite subsequene $(f_1, g_2, f_2, \ldots, g_N, f_N)$ of $(g_r, f_r)_{r \geqq 1}$. This holds if and only if $\mathscr{T}: \prod_{r \geqq 1} (Y_r \times (W \otimes \odot^r V^*)) \to \otimes^p V \otimes \otimes^q V^*$ is the composition of the projection π_N onto a finite product $\prod_{r=1}^{N} (Y_r \times (W \otimes \odot^r V^*))$ followed by a polynomial map P; i.e., $\mathscr{T} = P \cdot \pi_N$. Then, by (3.7), we have $A_*(P \cdot \pi_N(\tilde{}g_r, \tilde{}f_r)) = A_*\mathscr{T}(\tilde{}g_r, \tilde{}f_r) = \mathscr{T} A_*(\tilde{}g_r, \tilde{}f_r) = P \cdot \pi_N A_*(\tilde{}g_r, \tilde{}f_r) = P A_* \pi_N(\tilde{}g_r, \tilde{}f_r)$, and P is $U(n)$-invariant. We can write P as a sum of $U(n)$-invariant homogeneous polynomial of degrees $h_1^a, h_2^Y, h_2^a, \ldots, h_N^Y, h_N^a$ in the variables $f_1, g_2, f_2, \ldots, f_N, g_N$, respectively, where f_r is in $W \otimes \odot^r V^* = a_r$ and g_r is in Y_r. By a polarization process (see, for example, [2] or [5]) each homogeneous polynomial (wich we continue denoting by P) can be considered as induced by a $U(n)$-invariant multilinear map

$$a_1 \times \overset{(h_1^a)}{\ldots\ldots} \times a_1 \times Y_2 \times \overset{(h_2^Y)}{\ldots\ldots} \times Y_2 \times \ldots\ldots \times Y_N \times \overset{(h_N^Y)}{\ldots\ldots} \times Y_N \times a_N \times \overset{(h_N^a)}{\ldots\ldots} \times a_N \to \otimes^p V \otimes \otimes^q V^*,$$

and this is induced by a $U(n)$-invariant linear map

$$a_1 \otimes \overset{(h_1^a)}{\cdots\cdots} \otimes a_1 \otimes Y_2 \otimes \overset{(h_2^Y)}{\cdots\cdots} \otimes Y_2 \otimes \cdots\cdots \otimes Y_N \otimes \overset{(h_N^Y)}{\cdots\cdots} \otimes Y_N \otimes a_N \otimes \overset{(h_N^a)}{\cdots\cdots} \otimes a_N \to \otimes^p V \otimes \otimes^q V^*,$$

Since Y_r is a GL $(2n, \mathsf{R})$-direct summand of $\otimes^{r+2} V^*$ and a_r is a $U(n)$-direct summand of $\otimes^{r+2} V^*$, the last map can be viewed as induced by a $U(n)$-invariant linear map $\boldsymbol{P}$: $\otimes_\kappa V^* \to \otimes^p V \otimes \otimes^q V^*$. Then, if we also denote $\mathscr{T}$ the summand of $\mathscr{T} = P \cdot \pi_N$ which corresponds to the homogeneous polynomial P, from Lemma 4.2 we have

$$\mathscr{T}((g_r, f_r)_{r \geqq 1}) = \mathscr{T}((\tilde{\ } g_r, \tilde{\ } f_r)_{r \geqq 1}) = P((\tilde{\ } g_r, \tilde{\ } f_r)_{1 \leqq r \leqq N}) =$$

$$= \boldsymbol{P}(\tilde{\ } f_1 \otimes \overset{(h_1^a)}{\cdots\cdots} \otimes \tilde{\ } f_1 \otimes \tilde{\ } g_2 \otimes \overset{(h_2^Y)}{\cdots\cdots} \otimes \tilde{\ } g_2 \otimes \cdots\cdots \otimes \tilde{\ } g_N \otimes \overset{(h_N^Y)}{\cdots\cdots} \otimes \tilde{\ } g_N \otimes \tilde{\ } f_N \otimes \overset{(h_N^a)}{\cdots\cdots} \otimes \tilde{\ } f_N)$$

$$= \Omega_A^I \, \tilde{\ } f_{1I\sigma(1,3)} \cdots \tilde{\ } g_{2I\sigma(3h_1^a+1, 3h_1^a+4)} \cdots \tilde{\ } f_{NI\sigma(\kappa-n-1,\kappa)} \, \tilde{\ } g_{I\sigma(\kappa+p+1,\kappa+p+q)J}$$

$$\frac{\partial}{\partial x^{I\sigma(\kappa+1,\kappa+p)}} \otimes dx^J,$$

where $2s = \kappa + p + q$, $A = (\alpha_1, \ldots \alpha_{2s})$.

Now the theorem follows from this expression, the formulae giving the bijection $\tilde{\ }$ of theorem 3.6

$$(\tilde{\ } g_r)_{ijklI} = (g_r)_{ilkjI} - (g_r)_{ikljI} - (g_r)_{jkliI} - (g_r)_{jlkiI} \quad \text{(see [7])},$$

$$(\tilde{\ } f_r)_{ijI} = (1/2)((f_r)_{ijI} - (f_r)_{i^*j^*I}) \quad \text{(see (3.6))},$$

and the formulae (at the point 0)

$$(f_r)_{i^*j^*I} = (f_r)_{klI} J_i^k J_j^l = (f_r)_{klI} g_{iu} F^{uk} g_{vj} F^{vl}.$$

COROLLARY 4.4. *Let t be a regular hermitian natural tensor. Then t is of one of the following forms:*

a) *The metric $g_{..}$ with values in $\otimes^2 T^*M$ or its image $g^{..}$, by the canonical isomorphism given by the metric, with values in $\otimes^2 TM$.*

b) *The Riemann curvature tensor R and its covariant derivatives $\nabla^k R$.*

c) *The Kaehler form $F_{..}$ and its derivatives $\nabla^k F$.*

d) *The tensor product of tensors of type* a), b) *or* c).

e) *The tensors obtained by contractions of upper and lower indices in the above tensors.*

f) *All linear combinations of tensors of the above types.*

PROOF. It follows from Theorem 4.3 and the Taylor series expansions of a tensor in normal coordinates given in [10].

REMARK. It is interesting to remark that the tensors given by Corollary 4.4 include all those obtained from types a), b), c), d) or e) by permuting arguments, because those can also be obtained by contractions with $g_{..}$ and $g^{..}$.

COROLLARY 4.5. *Let t be a non identically zero homogeneous regular hermitian natural tensor of weight w and type (p, q). If we define the degree d of t as the number of derivatives of g and F appearing in the expression of t when we use a J-normal coordinate system, then* $w = -d - p + q$. *Consequently, w is even and* $q - p \geqq w$.

PROOF. The same arguments as in §5.3 of [7] show the following: let c be a constant and $\mathbf{g} = c^2 g$ (then $\mathbf{F} = \mathbf{g}(J \bullet, \bullet) = c^2 F$). If $(x^1, \ldots, x^{2n})$ is a J-normal coordinate system for (g, F) and $(y^1, \ldots, y^{2n})$ a J-normal coordinate system for $(\mathbf{g}, \mathbf{F})$, then, if $|I| = r$,

$$\mathbf{g}_{ijI} = c^{-r} g_{ijI}; \mathbf{F}_{ijI} = c^{-r} F_{ijI}; dy^i = c\, dx^i; \partial/\partial y^i = c^{-1}\, \partial/\partial x^i;$$
$$\mathbf{F}_{ij} = F_{ij}; \quad \mathbf{g}_{ij} = g_{ij}; \quad \mathbf{F}^{ij} = F^{ij}; \quad \mathbf{g}^{ij} = g^{ij}.$$

A monomial of those given in Theorem 4.3 has degree

$$d = h_1^a + 2h_2^a + \ldots + Nh_N^a + 2h_2^Y + \ldots + Nh_N^Y,$$

and weight

$$w = -h_1^a - 2h_2^a - \ldots - Nh_N^a - 2h_2^Y - \ldots - Nh_N^Y - p + q,$$

which proves that $w = -d - p + q$. On the other hand, Theorem 4.3 says that $\kappa + p + q$ is even if $t \neq 0$, and $\kappa + p + q + w = \kappa + 2q - d = 2q + 2h_1^a + \sum_{r=2}^{N} 2(h_r^a + h_r^Y)$ is even, then w is even.

Corollary 4.5 implies that the concepts of weight ([2] or [7]) and degree in the partial derivatives ([8, 9] or [11]) are quivalent.

§5. Hermitian natural connections.

Analogously to tensors, we can give the following definition:

DEFINITION 5.1. A hermitian natural connection is a map which associates to each almost hermitian manifold (M, g, J) a linear connection $D^{(M,g,J)}$ on TM such that if $f: (M, g, J) \to (M', g', J')$ is a holomorphic isometry of M onto an open set of M', then

$$D_X^{(M,g,J)} Y = D_{f_* X}^{(M',g',J')} f_* Y \text{ for every } X, Y \in \mathscr{X}(M).$$

We shall say that a hermitian natural connection D is regular if, for every point x in M and every J-coordinate system centred at x, the Christoffel symbols

$$\Gamma_{ij}^k = dx^k \left(D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right)$$

of D are universal polynomials in the variables

$$g_{ij}, g^{kl}, F_{rs}, \frac{\partial^{|\alpha|}}{\partial x^\alpha} g_{ij}, \frac{\partial^{|\beta|}}{\partial x^\beta} F_{rs},$$

The concept of weight of a homogeneous hermitian natural connection can also be defined as for hermitian natural tensors. Evidently, the Levi-Civita connection ∇ is regular hermitian natural and homogeneous of weight zero. All connection D as before has the form $\nabla + B$, where B is a $(1,2)$-tensor. Then, a hermitian natural connection which is homogeneous must be of weight zero.

THEOREM 5.2. *Let D be a homogeneous (of weight zero) regular hermitian natural connection. For hermitian manifolds of real dimension $2n \geqq 6$, the difference tensor B of the connections D and ∇ is in the vector space $\boldsymbol{B}$ with basis $\{B_1, \ldots, B_{24}\}$, where*

$B_1 = \nabla J,$	$B_2(X,Y) = B_1(Y,X),$	$g(B_3(X,Y),Z) = g(B_1(Z,X),Y),$
$B_4 = -J\cdot\nabla J,$	$B_5(X,Y) = B_4(Y,X),$	$g(B_6(X,Y),Z) = g(B_4(Z,X),Y),$
$B_7(X,Y) = -J\nabla_{JX}(J)Y,$	$B_8(X,Y) = B_7(Y,X),$	$g(B_9(X,Y),Z) = g(B_7(Z,X),Y),$
$B_{10}(X,Y) = \nabla_{JX}(J)Y,$	$B_{11}(X,Y) = B_{10}(Y,X),$	$g(B_{12}(X,Y),Z) = g(B_{10}(Z,X),Y),$
$B_{13} = g \otimes \delta J,$	$B_{14} = l \otimes \delta F,$	$B_{15} = \delta F \otimes l,$
$B_{16} = g \otimes J\delta J,$	$B_{17} = l \otimes \delta F\cdot J,$	$B_{18} = \delta F\cdot J \otimes l,$
$B_{19} = F \otimes \delta J,$	$B_{20} = J \otimes \delta F,$	$B_{21} = \delta F \otimes J,$
$B_{22} = F \otimes J\delta J,$	$B_{23} = J \otimes \delta F\cdot J,$	$B_{24} = \delta F\cdot J \otimes J,$

where l is the identity automorphism, δF is the coderivative of F and δJ is defined by $g(\delta J, X) = \delta F(X)$.

For dimension 4, we have the relations $B_1 - B_2 + B_3 - B_7 - B_9 + B_8 = 0$ and $-B_4 + B_5 - B_6 - B_{10} + B_{11} - B_{12} = 0$, then $\{B_1, \ldots, B_6, B_8, B_9, B_{11}, B_{12}, B_{13}, \ldots, B_{24}\}$ is a basis of $\boldsymbol{B}$. For dimension 2, $\boldsymbol{B} = \{0\}$.

PROOF. Obviously, B is a homogeneous hermitian natural tensor of weight zero and type (1,2). Them, from Corollary 4.5, B has degree $d = -1 + 2 = 1$. In a J-normal coordinate system $g_{ijk}(0) = 0$; thus the components of B are polynomials of degree one in $f_{1ijk} = F_{ijk} = \nabla_{\partial/\partial x^k}(F)(\partial/\partial x^i, \partial/\partial x^j)(0)$. Then, from Theorem 4.3, the members of a system of generators of $\boldsymbol{B}$ are obtained from $\Omega^{\cdot\cdot}\Omega^{\cdot\cdot}\Omega^{\cdot\cdot}F_{\ldots}g_{..}g_{..}$ or from $F^{\cdot\cdot}\Omega^{\cdot\cdot}\Omega^{\cdot\cdot}\Omega^{\cdot\cdot}F_{\ldots}g_{..}g_{..}g_{..}$ by contracting all the indices except one upper and two lower ones. The only possibilities are (up to a constant) those listed above. Then $\{B_i, i = 1, \ldots, 24\}$ is a system of generators of the vector space $\boldsymbol{B}$. Next we study their linear dependence.

First we consider the dimension $2n \geqq 6$. A linear combination of $B_1, \ldots, B_{24}$ gives (by applying it to the vectors $\partial/\partial x^1, \partial/\partial x^2, \partial/\partial x^3$ at 0) a linear combination of F_{321}, F_{132}, F_{123}, F_{23^*1}, F_{13^*2}, F_{12^*3}, $F_{23^*1^*}$, $F_{13^*2^*}$, $F_{12^*3^*}$, F_{231^*}, F_{132^*} and F_{13^*2}. From theorem 3.5 and the conditions (3.F.1), (3.F.2) and (3.F.3.1), given arbitrary values for these F_{ijk}, there is an almost hermitian structure (g, F) on $D(r)$ such that

$$\partial F_{ij}/\partial x^k(0) = \nabla_{\partial/\partial x^k}(F)(\partial/\partial x^i, \partial/\partial x^j)(0) = F_{ij}, \tag{5.0}$$

for i,j,k in the set $\{1,2,3,1^*,2^*,3^*\}$ (notice that $1^* = n+1$, $2^* = n+2$,

$3^* = n + 3$). Then a linear combination of these F_{ijk} is zero if and only if all the coefficients are zero; thus, the coefficients of $B_{11}, \ldots, B_{12}$ vanish. In order to get the coefficients $B_{13}, \ldots, B_{24}$ to be zero take a linear combination of these tensors and evaluate it on the vectors ∂/x^i, ∂/x^j and ∂/x^k at the point 0 and take $i = j$, $i = j^*, i = k$, $i = k^*$, $j = k$, and $j = k^*$, successively. (Note that, as above, given arbitrary values $\mathscr{F}_k$ to $\Sigma_i F_{kii}$, it follows from theorem 3.5, (3.F.1), (3.F.2) and (3.F.3.1) that there is an almost hermitian structure (g, F) on $D(r)$ such that the F_{ijk} satisfy (5.0) and $\Sigma_i F_{kii} = \mathscr{F}_k$).

In dimension 4 the only F_{ijk} which can take arbitrary values are F_{121}, F_{121^*}, F_{122}, F_{122^*}, F_{12^*1}, $F_{12^*1^*}$, F_{12^*2}, $F_{12^*2^*}$. So, given a linear combination $\Sigma \lambda_i \mathbf{B}_i$ and applying it to all possible arguments among $\partial/\partial x^1$, $\partial/\partial x^2$, $\partial/\partial x^3 = \partial/\partial x^{1^*}$, $\partial/\partial x^4 = \partial/\partial x^{2^*}$, and equalizing to zero we get

$$\lambda_1 = \lambda_3 = -\lambda_7 = \lambda_8 = -\lambda_2 = -\lambda_9, \tag{5.1}$$

$$\lambda_4 = \lambda_6 = \lambda_{10} = \lambda_{12} = -\lambda_5 = -\lambda_{11}. \tag{5.2}$$

In dimension 2 all B_i vanish, since $\nabla J = 0$.

COROLLARY 5.3. *Let $D = \nabla + B$ be a homogeneous hermitian natural connnection. Then:*

a) *If D is metric* (i.e. $Dg = 0$) *and the dimension is $2n \geqq 6$, B lies in the space with basis*

$$\mathscr{B} = \{B_1, B_2 - B_3, B_4, B_5 - B_6, B_7, B_8 - B_9, B_{10}, B_{11} - B_{12}, B_{13} - B_{14}, B_{16} + B_{17}, B_{19} - B_{20}, B_{21}, B_{22} + B_{23}, B_{24}\};$$

if the dimension is 4, *B* lies in the space with basis $\mathscr{B} - \{B_{10}\}$.

b) *If $DJ = 0$, B is of the form*

$$B = (1/2)B_4 + \lambda_2(B_2 + B_8) + \lambda_5(B_5 - B_{11}) + \lambda_6(B_6 - B_{12}) + \lambda_3(B_3 + B_9) + \lambda_{13}(B_{13} + B_{22}) + \lambda_{14}(B_{14} - B_{23}) + \lambda_{15}B_{15} + \lambda_{16}(B_{16} - B_{19}) + \lambda_{17}(B_{17} + B_{20}) + \lambda_{18}B_{18} + \lambda_{21}B_{21} + \lambda_{24}B_{24}, \quad \lambda_i \in \mathrm{R}.$$

c) *If $DJ = 0 = Dg$, B is of the form*

$$B = (1/2)B_4 + \zeta_1(B_5 - B_6 - B_{11} + B_{12}) + \zeta_2(B_2 + B_8 - B_3 - B_9) + \zeta_3(B_{13} + B_{22} - B_{14} + B_{23}) + \zeta_4(B_{16} - B_{19} + B_{17} + B_{20}) + \zeta_5 B_{21} + \zeta_6 B_{24}, \quad \zeta_i \in \mathrm{R}.$$

PROOF: c) follows from a) and b). First we prove a). $Dg = 0$ if and only if $g(B(X, Y), Y) = 0$, for any X, Y in $\mathscr{X}(M)$. Writing $B = \Sigma_i \lambda_i B_i$ and using the Theorem 5.2, $g(B(X, Y), Y) = 0$ if and only if

$$(\lambda_2 + \lambda_3)\nabla_Y(F)_{XY} + (\lambda_5 + \lambda_6)\nabla_Y(F)_{XJY} + (\lambda_8 + \lambda_9)\nabla_{JY}(F)_{XJY} + (\lambda_{11} + \lambda_{12})\nabla_{JY}(F)_{XY} + (\lambda_{13} + \lambda_{14})g(X, Y)\delta F(Y) + \lambda_{15}g(Y, Y)\delta F(X) + (-\lambda_{16} + \lambda_{17})g(X, Y)\delta F(JY) + \lambda_{18}g(X, Y)\delta F(JX) + (\lambda_{19} + \lambda_{20})F(X, Y)\delta F(Y) + (-\lambda_{22} + \lambda_{23})F(X, Y)\delta F(JY) = 0. \tag{5.3}$$

Taking $X = \partial/x^1$ and $Y = \partial/\partial x^2$ and evaluating (5.3) at the origin of a J-normal coordinate system, the same arguments as those in proof of Theorem 5.2 show that the formula (5.3) implies (for dimension $2n \geqq 4$) $\lambda_2 = -\lambda_3, \lambda_5 = -\lambda_6, \lambda_8 = -\lambda_9$ and $\lambda_{11} = -\lambda_{12}$. The same arguments as those in proof of 5.2 to get the coefficients of $B_{13}, \ldots, B_{24}$ to be zero give $\lambda_{13} = -\lambda_{14}, \lambda_{15} = 0, \lambda_{16} = \lambda_{17}, \lambda_{18} = 0, \lambda_{19} = -\lambda_{20}, \lambda_{22} = \lambda_{23}$. This proves a).

In order to prove b) we first note that $DJ = 0$ if and only if

(5.4) $0 = \nabla_X(J)Y + B(X, JY) - JB(X, Y) \equiv \alpha(X, Y)$, for any X, Y in $\mathscr{X}(M)$.

Taking $B = \Sigma_i \lambda_i B_i$ and computing $g(\alpha(\partial/\partial x^1, \partial/\partial x^2), \partial/\partial x^3)$ at the origin in a J-normal coordinate system, we obtain, following the same method as in a), in dimension $2n \geqq 6$, that

$$\lambda_1 = 0, 1 - 2\lambda_4 = 0, \lambda_7 = \lambda_{10} = 0, \lambda_2 = \lambda_8, \lambda_5 = \lambda_{11}, \lambda_3 = \lambda_9, \tag{5.5}$$

$$\lambda_6 = \lambda_{12}, \lambda_{13} = \lambda_{22}, \lambda_{16} = -\lambda_{19}, \lambda_{14} = -\lambda_{23}, \lambda_{17} = \lambda_{20},$$

which proves b). Since B_7 and B_{10} do not appear in the formula of b), that also holds for dimension 4.

The characteristic, second and Levi-Civita connections, and also the Weyl connection (so called in [15]), are examples of homogeneous regular hermitian natural connections. For Kaehler manifolds, the unique homogeneous regular hermitian natural connection is the Levi-Civita connection.

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