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SIMPLE SINGULARITIES OF FUNCTIONS ON SUPERMANIFOLDS

V. SERGANOVA and A. WEINTROB

Introduction.

The study of singular points of a smooth function on a manifold is related as is shown in works by Arnold and his school [A] with objects from different branches of mathematics: groups generated by reflections, Lie algebras, automorphic forms, etc.

Inspired by remarkable achievements of singularity theory we decided to consider functions on supermanifolds from this point of view.

In this paper we classify simple (with respect to the group of diffeomorphisms of the pre-image) germs of functions on a supermanifold. An attempt to find the relations of our classification with other branches of mathematics failed as yet, cf. [A] with [S].

However we have decided to publish the results of the classification in the hope to draw attention to this subject which, we are sure, deserves it.

The paper consists of two parts. The first one introduces the main notions: stable equivalence, local algebra, etc., and certain general results are proved: analogues of the Morse Lemma, theorems on finite definiteness, etc. The second part contains the classification of simple germs itself.

The main differences of the supercase as compared with the conventional one are perhaps the following. The fact that a singularity is isolated is not equivalent any more to its finite multiplicity. Moreover, not all the singular points of an even function in which the corank of the second differential equals p/q, where $q > 0$, are isolated. Perhaps that is why we cannot find for our classification relations similar to the even case (since to define the corresponding geometric objects: intersection forms, monodromy groups, etc. isolatedness is crucial). The absence for supermanifolds of adequate analogues of the usual topological notions like the degree of the map is the reason for our algebraic defition of Milnor's number as the dimension of the local superalgebra of the singularity. A certain justifica-

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tion for doing so is that Milnor's number coincides (as in the even, manifold case) with the dimension of the base of a versal deformation.

A few words about our future plans. For the time being we consider only the action of the diffeomorphism group Diff_0 on the space of even germs. Elsewhere we will classify the functions with respect to the *supergroup* of diffeomorphisms whose underlying is Diff_0. This will enable us to consider not only even functions and 2) give a correct definition of the modality of a singularity (as a minimal pair m/n, such that a neighborhood of the orbit of a singular germ may be covered by a finite number of m/n-parameter families of orbits). One more direction which might lead to the phenomena reflecting hidden as yet supersymmetry of a singularity is the study of its bifurcational diagram (an algebraic subsupervariety in the base of the versal deformation corresponding to the singular germs).

1. Beginning of the classification of germs of functions on supermanifolds.

1.1. *Definitions and notations.* Let $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ be a supermanifold. The *functions* on M are arbitrary sections of the structural sheaf $\mathcal{O}_M$. We will study the normal forms to which a function on M can be locally reduced. The majority of the classification work is performed for the case when M is complex and the functions are analytic. Nevertheless, all the results of Chapter 1 which are simple generalizations of the corresponding facts from the purely even theory [A]) hold in either of the three categories: analytic (over R or C), smooth (C^∞) or formal. All the singularities of finite multiplicity are by Theorem 2.1.2 of a polynomial form in some coordinate system which effaces the differences between these three categories but the problem of description of real forms still remains open. We do not study this problem here.

We will only be interested in the local behaviour of functions and therefore we will assume that M is a subsuperdomain in the (p, q)-dimensional linear superspace $\mathrm{C}^{p/q}$ (resp. $\mathrm{R}^{p/q}$) with even coordinates $x_1, x_2, \dots, x_p$ and odd ones $\xi_1, \dots, \xi_q$, i.e. M is an open subset of C^p (resp. R^p) with coordinate system $(x_1, \dots, x_p)$ and $\mathcal{O}_{\mathcal{M}} = \mathcal{O}_M \otimes \wedge^* V$, where $\mathcal{O}_M$ is a ring of analytic (infinitely differentiable) functions on M and $V = \langle \xi_1, \dots, \xi_q \rangle$ the q-dimensional vector space over C (resp. R).

We will denote by $\mathcal{O}_{\mathcal{M},m}$ the ring of germs of functions at $m \in M$. It is a local supercommutative superalgebra with the maximal ideal $\mathfrak{M}_m$ consisting of the functions whose restriction to M vanishes at m. A choice of a coordinate system in a neighborhood of m determines an isomorphism of $\mathcal{O}_{\mathcal{M},m}$ and the superalgebra $\mathcal{O}_{p/q}$ of germs of analytic (smooth) functions at the origin of $\mathrm{C}^{p/q}$ (resp. $\mathrm{R}^{p/q}$).

1.2. *Equivalence of germs.*

1.2.1. On the space $\mathcal{O}_{\mathcal{M},m}$, the group $G = \mathrm{Diff}_0$ of diffeomorphisms of the supermanifold M preserving m acts; we will identify G with the group $G_{p,q}$ of automorphisms of $\mathcal{O}_{p/q}$. Any such diffeomorphism (automorphism) may be considered as a change of variables

$$x_i \mapsto y_i = f_{0,i} + \sum f_{2,i}^{j_1 j_2} \xi_{j_1} \xi_{j_2} + \sum f_{4,i}^{j_1 j_2 j_3 j_4} \xi_{j_1} \xi_{j_2} \xi_{j_3} \xi_{j_4} + \dots$$

$$\xi_j \mapsto \eta_j = \sum f_{i,j}^{j_1} \xi_{j_1} + \sum f_{3,j}^{j_1 j_2 j_3} \xi_{j_1} \xi_{j_2} \xi_{j_3} + \dots,$$

where $i = 1, \dots, p, j = 1, \dots, q, f_{k,i}^{j_1 j_2 \dots j_k} \in \mathcal{O}_p$, and the matrices

$$\left(\frac{\partial f_{o,i}}{\partial x_k}\right), k = 1, \dots, p \text{ and } (f_{1,l}^j), l = 1, \dots, q$$

are invertible.

The germs belonging to one G-orbit are called *equivalent*. (In [A] this equivalence is called R-equivalence).

1.2.2. The problem on local normal forms of functions is to describe the orbits of the $G_{p,q}$-action in $\mathcal{O}_{p/q}$. The group and the space of orbits (however understood) are infinite dimensional. Therefore the complete solution of this problem is a rather hopeless task. But on the orbits there is a natural stratification and the first terms of this stratification are subject to complete investigation.

1.2.3. REMARK. $G_{p/q}$ is the underlying group of the supergroup $\hat{G}_{p,q}$ of the diffeomorphisms of M which also acts on $\mathcal{O}_{p/q}$. Therefore, from the supermanifold theory point of view it would be more correct to study the quotient space $\mathcal{O}_{p/q}/\hat{G}_{p/q}$; this will be done elsewhere. However, it is clear now why we confine ourselves to homogeneous germs: the $G_{p/q}$-equivalence for non-homogeneous elements of $\mathcal{O}_{p/q}$ is too weak and the corresponding classification is the description of pairs consisting of an even and an odd function.

1.3. *Non-singular germs.*

1.3.1. A point $m \in M$ is called a *singular point* of a function $f \in \mathcal{O}_{\mathcal{M}}$ if all the partial derivatives $\partial f/\partial x_i$ and $\partial f/\partial \xi_j$ vanish at m. A germ $f \in \mathcal{O}_{\mathcal{M},m}$ is called *singular* if m is singular for a representative of the germ.

In other words, "f is a singular germ" means that $f \equiv \text{const} \bmod \mathfrak{M}_m^2$.

Almost all germs are non-singular and the first classification result refers to this most general case.

1.3.2. PROPOSITION. *Let a function f be non-singular at $m \in M$. Then the germ of ρ at m is equivalent to* const $+ x_1$ *if f is an even function and ξ_1 if f is odd.*

PROOF. The implicit function theorem, see [L].

2. Morse's lemma and stable equivalence.

2.1. *Morse's lemma.*

2.1.1. By Proposition 1.3.2 we may now only consider singular germs of even and odd functions. Any even germ is of the form $f = a + f_2$, where $a \in \mathbf{C}$, and $f_2 \in \mathfrak{m}^2$. The $G_{p/q}$-action does not affect the constant a, therefore to describe all the orbits in the space of germs of the functions it suffices to study the orbits in the space of germs of functions that vanish at the origin together with their differential. In other words, we will consider the $G_{p/q}$-action in the space $\mathfrak{m}^2_{p/q}$ then the 2-jet of the germ f is a bilinear form on $T_m M$ called the *second differential* of f. The *rank of the second differential* of a singular germ f (or just the *rank* of f) is the rank at m) of the matrix

$$\begin{pmatrix} \dfrac{\partial^2 f}{\partial x_i x_j} & \dfrac{\partial^2 f}{\partial x_i \partial \xi_k} \\ \dfrac{\partial^2 f}{\partial \xi_l \partial x_j} & \dfrac{\partial^2 f}{\partial \xi_l \partial \xi_k} \end{pmatrix}.$$

The *corank*, $\operatorname{cork} f = p/q - \operatorname{rk} f$ is an important invariant of a singular germ.

Notice that for the non-singular germs the quadratic part of the Taylor series is not invariantly defined.

2.1.2. For germs homogeneous with respect to parity the following analogue of MORSE'S LEMMA holds.

Let f be a homogeneous germ of $\mathfrak{m}^2$ whose second differential is non-degenerate. Then f is equivalent to its quadratic part.

We will prove this statement in a trifle more general form called the *parametric Morse's lemma* (2.1.4).

2.1.3. Let $f_1(x, \xi)$ be a function on a supermanifold M_1 and $f_2(y, \eta)$ a function on a supermanifold M_2. The function $f_1(x, \xi) + f_2(y, \eta)$ on $M = M_1 \times M_2$ is called the *direct sum of* f_1 and f_2.

2.1.4. THEOREM. *Let $f \in \mathfrak{m}^2_{p/q}$ be homogeneous and $rkf = k/l$. Then f is equivalent to the direct sum of a non-degenerate quadratic form $b(x, \xi) \in \mathfrak{m}^2_{k/l}$ and a germ $g(y, \eta) \in \mathfrak{m}^3_{p-k/q-l}$ whose 2-jet is zero.*

PROOF. First consider the case when f is even. Let $\mathfrak{n}$ be the ideal of $\mathcal{O}_{p/q}$ generated by the odd variables. Reducing f modulo $\mathfrak{n}$ we get a function $\bar{f}$ in p even variables. Applying the conventional parametric Morse's lemma [A] reduce $\bar{f}$ to the form

$$\bar{f} = x_1^2 + \ldots + x_k^2 + \bar{g}(y_1, \ldots, y_{p-k}), \; \bar{g} \in \mathfrak{m}^3_{p-k}.$$

Complement the even coordinate functions x_i, y_j with odd ones $\xi_1, \ldots, \xi_l$, $\eta_1, \ldots, \eta_{q-l}$ so that the second differential in the resulting coordinate system is of the form $x_1^2 + \ldots + x_k^2 + \xi_1\xi_2 + \xi_3\xi_4 + \ldots$.

Then f takes in these coordinates the form

$$f = x_1^2 + \ldots + x_k^2 + x_1 f_1 + x_2 f_2 + \ldots + x_k f_k + \xi_1\xi_2 + \xi_3\xi_4 + \ldots + \xi_1\varphi_2 + \\ + \varphi_1\xi_2 + \ldots + g$$

where $f_i \in \mathfrak{n}^2$, $\varphi_j \in \mathfrak{n}$. Performing the change of variables

$$x_i \mapsto x_i - \tfrac{1}{2}f_i \;\; \xi_j \mapsto \xi_i - \varphi_i,$$

we reduce f to the form

$$f = x_1^2 + x_2^2 + \ldots + x_1\tilde{f}_1 + x_2\tilde{f}_2 + \ldots + \xi_1\xi_2 + \xi_3\xi_4 + \ldots + \xi_1\tilde{\varphi}_2 + \tilde{\varphi}_1\xi_2 + \\ + \xi_3\tilde{\varphi}_4 + \tilde{\varphi}_3\xi_4 + \ldots + \hat{g},$$

where now $\tilde{f}_i \in \mathfrak{n}^4$, $\tilde{\varphi}_j \in \mathfrak{n}^3$.
Repeating this procedure several times we get a representation in which $\tilde{f}_i$, $\tilde{\varphi}_j \in \mathfrak{n}^{q+1}$. But $\mathfrak{n}^{q+1} = 0$ therefore this is the desired representation of f.

For an odd germ the proof is still simpler. In fact, expressing f in a coordinate system in which its second differential is of the form $x_1\xi_1 + x_2\xi_2 + \ldots + x_p\xi_p$, we get

$$f = x_1\xi_1 + \ldots + x_p\xi_p + \xi_1 f_1 + \ldots + \xi_p f_p + g(y, \eta),$$

where $f_i \in \mathfrak{n}^2$, $g \in \mathfrak{n}^3$. The change $x_i \mapsto x_i - f_i$ reduces to a representation where $f_i \in \mathfrak{n}^4$. At the following step we get $f_i \in \mathfrak{n}^8$, and in several steps we get $f_i \in \mathfrak{n}^{q+1} = 0$ as required.

2.2. *Stable equivalence.*

2.2.1. The parametric Morse's lemma reduces the classification of singular germs to the description of germs with zero second differential, i.e. to the study of $G_{p,q}$-orbits in $\mathfrak{m}_{p/q}^3$. Besides, it shows that it is more reasonable to classify the germs not by the number of even or odd variables on which it depends but by the corank of the second differential. More exactly these considerations are formulated in the following definitions.

2.2.2. The germs $f_1 \in \mathcal{O}_{p/q}$ and $f_2 \in \mathcal{O}_{m/n}$ are called *stably equivalent* if there exist non-degenerate quadratic forms b_1 and b_2 in k_1/l_1 and k_2/l_2 variables respectively such that $p + k_1 = m + k_2$ and $q + l_1 = h + l_2$ and the germs $f_1 \oplus b_1$ and $f_2 \oplus b_2$ are equivalent.

This definition immediately implies that the corank is an invariant of stable equivalence. Moreover for homogeneous functions depending on the same number of even and odd variables stable equivalence is equivalent to the usual equivalence. Therefore, the following definition is correct.

2.2.3. A *singularity* (of parity $\varepsilon \in \mathbf{Z}_2$) is a class of stable equivalence of germs.

Before we prove the above statement let us give one more definition and a useful lemma.

2.2.4. The *gradient ideal* I_f of a germ $f \in \mathcal{O}_{p/q}$ is the ideal generated by the partial derivatives $\partial f/\partial x_i$, $\partial f/\partial \xi_j$.

2.2.4.1. REMARK. The gradient ideal is an invariant of a germ which is the tangent superspace to the orbit of f under the action of the supergroup of all diffeomorphisms preserving $m \in \mathcal{M}$.

2.2.4.2. EXAMPLE. The gradient ideal of a non-singular germ coincides with the whole of $\mathcal{O}_{p/q}$. Conversely, if $I_f = \mathcal{O}_{p/q}$ then f is a non-singular germ.

2.2.4.3. EXAMPLE. If the quadratic differential of a singular homogeneous germ f is non-degenerate then $I_f = \mathfrak{m}$. The converse is also true.

2.2.5. PROPOSITION. *Let f and φ be homogeneous germs of the same parity such that $\varphi \in \mathfrak{m} I_f$ and $I_\varphi \subset \mathfrak{m} I_f$. Then f and $f + \varphi$ are equivalent.*

PROOF. Is based on the homotopy method whose essence is the following.

Instead of one diffeomorphism F sending $f + \varphi$ into f we will seak a family of diffeomorphisms F_t such that

$$F_t^*(f + t\varphi) = f,\ F_o = \mathrm{id},\ F_t(0) = 0.$$

Differentiating the first identity with respect to t we get

$$(*) \qquad V(F_t(x,\xi),t)(f + t\varphi) + \varphi(F(x,\xi)) = 0,$$

where $V(F_t(x,\xi),t) := \dfrac{d}{dt} F_t(x,\xi)$ is the vector field corresponding to the family of diffeomorphisms F_t. The map $(x,\xi,t) \mapsto (F(x,\xi),t)$ is a diffeomorphism and therefore solving $(*)$ reduces to seeking a vector field $V(x,\xi,t)$ on $M \times \mathsf{R}$ such that $V \cdot (f + t\varphi) = -\varphi$. If, besides, $V(0,0,t) = 0$ then a solution of the initial value problem

$$\frac{d}{dt} F_t(x,\xi) = V(F_t(x,\xi),t),\ F_0(x,\xi) = (x,\xi)$$

exists for sufficiently small x for $t \in [0,1]$. The existence is guaranteed by Shander's theorem [Sh] and the extendability to the segment follows from the fact that $F(0,0,t) \equiv 0$ is a solution for $(x,\xi) = 0$.

Let $\mathcal{O}$ be the superalgebra of germs of functions in $(x\xi)$ and $\mathfrak{m}$ its maximal ideal. Let $\hat{\mathcal{O}} = \mathcal{O} \hat{\otimes} \mathcal{O}_{[0,1]}$ be the superalgebra of families of germs that depend smoothly (analytically) on a parameter $t \in [0,1]$. We seek germs $a_i \in [\mathfrak{m} \cdot \hat{\mathcal{O}}]_{\bar{0}}$ and

$\alpha_j \in [\mathfrak{m} \cdot \tilde{\mathcal{O}}]_{\bar{1}}$ such that

$$\sum a_i \frac{\partial}{\partial x_i}(f + t\varphi) + \sum \alpha_j \frac{\partial}{\partial \xi_j}(f + t\varphi) = -\varphi$$

For $t = 0$ such germs can be found since $\varphi \in \mathfrak{m} I_f$. Let us show that $I_\varphi \subset \mathfrak{m} I_f$ implies $I_f \subset \tilde{I}_{f+t\varphi}$, where $\tilde{I}_{f+t\varphi}$ is the ideal of $\tilde{\mathcal{O}}$ generated by the $\frac{\partial}{\partial x_i}(f + t\varphi)$, $\frac{\partial}{\partial \xi_j}(f + t\varphi)$. In this case $\varphi \in \mathfrak{m} \tilde{I}_{f+t\varphi}$ and the coefficients a_i, α_j in the decomposition

$$\varphi = \sum a_i \frac{\partial}{\partial x_i}(f + t\varphi) + \sum \alpha_j \frac{\partial}{\partial \xi_j}(f + t\varphi)$$

give all that is needed.

The inclusion $I_f \subset \tilde{I}_{f+t\varphi}$ is proved by the following trick (a variant of Nakayama's lemma). Making use of the fact that $(\partial/\partial x_i)\varphi$ and $(\partial/\partial \xi_j)\varphi$ belong to $\mathfrak{m} I_f$ we find $b_{ki}, c_{lj}, \in [\mathfrak{m}\tilde{\mathcal{O}}]_{\bar{0}}$ and $\beta_{kj}, \gamma_{li} \in [\mathfrak{m}\tilde{\mathcal{O}}]_{\bar{1}}$ such that

$$\frac{\partial}{\partial x_k} f = \frac{\partial}{\partial x_k}(f + t\varphi) - t\frac{\partial}{\partial x_k}\varphi = \frac{\partial}{\partial x_k}(f + t\varphi) - t\sum b_{ki}\frac{\partial}{\partial x_i} f - t\sum_j \beta_{kj}\frac{\partial}{\partial \xi_j} f,$$

$$\frac{\partial}{\partial \xi_l} f = \frac{\partial}{\partial \xi_l}(f + t\varphi) - t\frac{\partial}{\partial \xi_l}\varphi = \frac{\partial}{\partial \xi_l}(f + t) - t\sum \gamma_{li}\frac{\partial}{\partial x_i} f - t\sum c_{lj}\frac{\partial}{\partial \xi_j} f.$$

Transporting to the left-hand side the terms with partial derivatives of f we get a system of linear equations

$$(1 + T)\operatorname{grad} f = \operatorname{grad}(f + t\varphi),$$

where $T = \begin{pmatrix} (b_{ki})(\beta_{kj}) \\ (\gamma_{li})(c_{lj}) \end{pmatrix}$. All the matrix elements of T belong to $\mathfrak{m}\tilde{\mathcal{O}}$ therefore $1 + T$ is invertible and

$$\operatorname{grad} f = (1 + T)^{-1}\operatorname{grad}(f + t\varphi).$$

Therefore $I_f \subset \tilde{I}_f + t\varphi$ and we are done.

2.2.5.1. Corollary. *If f and φ are homogeneous germs of the same parity, $f \in \mathfrak{m}^3$ and $\varphi \in I_f^2$ then $f + \varphi$ is equivalent to f.*

Proof. The conditions of the Proposition hold since $I_f \subset \mathfrak{m}^2$ and partial derivation sends I_f into $\mathfrak{m}$ and I_f^2 into $\mathfrak{m} I_f$.

2.2.5.2. Corollary: *Morse's lemma.*

In fact, let f be a non-degenerate quadratic form (either even or odd) then $I_f = \mathfrak{m}$ and $\varphi \in \mathfrak{m}^3$ immediately implies $I_\varphi \subset \mathfrak{m} I_f$.

One more corollary of Proposition 2.2.5 is the theorem on finite determinacy 3.2.1.

2.2.6. THEOREM. *Let homogeneous functions $f_1, f_2 \in \mathcal{O}_{p/q}$ be stably equivalent. Then they are equivalent.*

PROOF. Apply to the functions f_1 and f_2 the parametric Morse's lemma and reduce them to the form $f_1^0 \oplus b_1$ and $f_2^0 \oplus b_2$, where b_1 and b_2 are non-degenerate quadratic forms and f_1^0 and f_2^0 are germs with zero 2-jet. The stable equivalence of f_1 and f_2 means that there exist non-degenerate quadratic forms b'_1 and b'_2 of the same rank since f_1 and f_2 depend on the same number of variables) for which the germs $f_1^0 \oplus b_1 \oplus b'_1$ and $f_2^0 \oplus b_2 \oplus b'_2$ are equivalent. Then the forms $b_1 \oplus b'_1$ and $b_2 \oplus b'_2$ are isomorphic and therefore it suffices to prove the statement for stably equivalent functions with zero 2-jet.

Thus, let now f_1 and f_2 be functions of variables $(x_1, \ldots, x_q, \xi_1, \ldots, \xi_q)$ belonging to $\mathfrak{m}^3$ and b_1 and b_2 non-degenerate quadratic forms in variables $(y_1, \ldots, y_2, \eta_1, \ldots, \eta_2)$ such that the germ of $f_1 + b_1$ is equivalent to the germ of $f_2 + b_2$. Performing a linear change of variables (y, η), we may assume that the forms b_1 and b_2 are reduced to the canonical forms

$$b(y, \eta) = y_1^2 + \ldots + y_k^2 + \eta_1\eta_2 + \ldots + \eta_{2l-1}\eta_{2l}$$

for even functions and

$$b(y, \eta) = y_1\eta_1 + \ldots + y_k\eta_k$$

for odd ones.

Consider a diffeomorphism $F(x, \xi, y, \eta)_+ \in G$, sending $f_1 \oplus b$ into $f_2 \oplus b$. The linear part of F preserves b therefore the Jacobi matrix J of F is of the form

$$\begin{array}{c} \\ (x,\xi) \\ (y,\eta) \end{array} \begin{array}{c} (x,\xi) \quad (y,\eta) \\ \begin{pmatrix} A & B \\ O & C \end{pmatrix} \end{array}.$$

Therefore, the invertibility of J implies that of the matrix A which is the Jacobi matrix for the map F': $\mathbf{C}^{p'/q} \to \mathbf{C}^{p/q}$ which is the composition of an embedding $\mathbf{C}^{p/q} \to \mathbf{C}^{p+r/q+s}$ and the projection $\mathbf{C}^{p+r/q+s} \to \mathbf{C}^{p/q}$. By the implicit function theorem F' is a difeomorphism and therefore replacing $f_1(x, \xi)$ by an equivalent germ $f'_1 = f_1(F'(x, \xi))$ we get a pair of equivalent germs $f'_1 + b_1$ and $f_2 + b_2$ and the change of variables sending $f_2 + b_2$ into $f'_1 + b_1$ is of the form

$$x_i \mapsto x_i + \sum y_k a_k(x, \xi) + \sum \eta_l \Psi_l(x, \xi) + \ldots$$
$$\xi_j \mapsto \xi_j + \sum y_k \beta_{kj}(x, \xi) + \sum \eta_l d_{kl}(x, \xi) + \ldots$$

$$y_k \mapsto y_k + c(x, \xi) + \ldots$$

$$\eta_k \mapsto \eta_l + \ldots \gamma(x, \xi) + \ldots$$

where dots stands for the terms that belong to the square of $I' = \langle y, \eta \rangle$. (To make the linear part of this change of variables the identity in y, η this diffeomorphism should be multiplied by a linear transformation preserving b).

Now let us show that $h = f_1' - f_2$ belongs to the square of the gradient ideal $I_{f_2} \subset \mathcal{O}_{p/q}$, By Corollary 1.5.4.1 this would mean that f_2 and f_1' are equivalent. In fact, for even functions we get modulo I'^2:

$$f_1' + y_1^2 + \ldots + b_1 = f_2 + y_3^2 + \ldots + h =$$

$$= f_2(x + ya + \eta d) + \sum (y_k + c_k(x_1 \xi) + \ldots)^2 + \sum (\eta_l + \gamma(x, \xi) + \ldots) \cdot$$

$$\cdot (\eta_{l+1} + \gamma(x, \xi) + \ldots) =$$

$$= f_2 + y_i a_i \frac{\partial f_2}{\partial y_i} + \sum \eta_l d_l \frac{\partial f_2}{\partial \eta_l} + \ldots + \sum y_k^2 + 2 y_k c_k + c^2 k +$$

$$+ \sum (\eta_l \eta_{l+1} + \eta_l \gamma_{l+1} - \eta_{l+1} \gamma_l + \ldots + \gamma_l \gamma_{l+1} + \ldots).$$

Equating the coefficients of y_k, η_l and the terms we get

$$c_k = -\frac{1}{2} a_k \frac{\partial f_2}{\partial y_k}, \quad \gamma_{2l+1} = d_{2l} \frac{\partial f_2}{\partial \eta_{2l+1}},$$

$$\gamma_{2l} = -\alpha_{2l} \frac{\partial f_2}{\partial \eta_{2l}}, \quad h = \sum_k c_k^2 + \sum_l \gamma_l \gamma_k,$$

yielding $h \in I^2 f_2$ and by Proposition 2.2.2, the functions f_i' and f_2 are equivalent. But f_1 and f_1' are also equivalent therefore so are f_1 and f_2. For even functions the end of the proof is similar.

3. Local algebra and deformations.

3.1. *Germs of finite multiplicity.* One of the main algebraic invariants of a singularity is its local algebra.

3.1.1. *The local superalgebra of a singular germ* f is $\mathcal{O}_f := \mathcal{O}/I_f$. If $\mathcal{O}_f$ is of finite dimensions then the singular point is said to be of *finite multiplicity* and $\mu = \mu_0/\mu_1 = \dim \mathcal{O}_f$ is called its *multiplicity* (or *Milnor's number*).

3.1.2. REMARK. Clearly, if of two germs f_1 and f_2 at least one is of finite multiplicity then $\mathcal{O}_{f_1} \oplus \mathcal{O}_{f_2} = \mathcal{O}_{f_1} \otimes \mathcal{O}_{f_2}$. If f_2 is a non-singular quadratic form then $\mathcal{O}_{f_2} = \mathbf{C}$ (cf. 2.2.4.3) therefore the local algebra does not depend on the choice of representatives of the class of stable equivalence and is an invariant of a singularity.

3.1.3. REMARK. In a purely even situation (on manifolds) the multiplicity of a singular point equals the number of non-degenerate singular points into which it splits under a small perturbation. In the supercase the geometric meaning of the multiplicity is unclear as yet. The finite multiplicity of a germ on a manifold is equivalent to the isolatedness of a singular point. In general, on a supermanifold non-isolated singular points can be of finite multiplicity, e.g. the germ of $x^3 + x^2\xi_1\xi_2$ is singular on the hypersurface $x = 0$ though the dimension of its local ring is 3/2 and with a basis $1, x, \xi_1\xi_2$ (even) and ξ_1, ξ_2 (odd). A necessary and sufficient condition for finite multiplicity of an even singularity is isolatedness of its restriction onto the underlying manifolds.

3.1.4. PROPOSITION. *Let $f \in \mathcal{O}_{p/q,\bar{0}}$. Denote by $\tilde{f} \in \mathcal{O}_p$ the restriction of f onto the underlying manifold singled out by the equations $\xi_1 = \ldots = \xi_q = 0$. Then the origin is a singular point of f of finite multiplicity if and only if $\tilde{f}$ has an isolated singularity at the origin.*

PROOF. Under the reduction $\mathcal{O}_{p/q} \to \mathcal{O}_{p/q}/\langle\xi_1, \ldots, \xi_q\rangle = \mathcal{O}_p$ the gradient ideal $I_{\tilde{f}}$ and therefore an epimorphism $\mathcal{O}_f \to \mathcal{O}_{\tilde{f}}$ is defined and finite dimensionally of $\mathcal{O}_f$ implies that of $\mathcal{O}_{\tilde{f}}$ and this implies due to [A] that the singularity of $\tilde{f}$ is isolated.

Conversely, if the origin is an isolated singularity for $\tilde{f}$ then $\mu = \dim \mathcal{O}_{\tilde{f}} < \infty$. As we will see the finite dimensionality of the local algebra implies the nilpotency of its maximal ideal. Let the images of x_i in $\mathcal{O}_{\tilde{f}}$ satisfy $\tilde{x}_i^N = 0$, then $x_i^N \in \operatorname{Ker}(\mathcal{O}_f \to \mathcal{O}_{\tilde{f}})$. But this kernel is generated by the images of odd variables ξ_j for $\mathcal{O}_{p/q}$ and therefore $\operatorname{Ker}^{q+1} = 0$ implying $x_i^{N(q+1)} = 0$ in $\mathcal{O}_f$. Thus any elements from $\mathcal{O}_f$ can be presented as a linear combination of monomials in which the powers of variables x_i do not exceed $N(q + 1)$. Therefore $\dim \mathcal{O}_f < \infty$.

It remains to verify the following fact.

3.1.5. LEMMA. *Let A be a finite dimensional local algebra of dimension μ and $\mathfrak{m}$ its maximal ideal. Then $\mathfrak{m}^\mu = 0$.*

PROOF. Let $a_1, \ldots, a_\mu \in \mathfrak{m}$ be arbitrary. The elements $1, a_1, a_1a_2, \ldots, a_1 \ldots a_\mu$ are linearly dependent, say, $\lambda_0 + \lambda_1 a_1 + \ldots + \lambda_\mu a_1 \ldots a_\mu = 0$. Select the minimal i such that $\lambda_i \neq 0$ then $a_1 \ldots a_i(\lambda_i + \lambda_{i+1}a_{i+1} + \ldots + \lambda_r a_{i+1} \ldots a_\mu) = 0$. The element in parenthesis does not belong to $\mathfrak{m}$ and therefore is invertible implying $a_1 \ldots a_i = 0$ and $a_1 \ldots a_\mu = 0$.

3.2. *Sufficient jets.* The germs of finite multiplicity possess the following property of finite determinacy.

3.2.1. THEOREM. *A germ with the singular point of finite multiplicity is equivalent to a polynomial in x, ξ (notably, to a segment of its Taylor series).*

PROOF. Let the multiplicity of the singular point of f equal $\mu = \mu_0/\mu_1$. Then by Lemma 3.1.5 $I_f \supset \mathfrak{m}^{\mu_0+\mu_1}$, and denoting by $\tilde{f}$ the jet of f of order $\mu_0 + \mu_1 + 1$ we see that $\varphi = \tilde{f} - f$ belongs to $\mathfrak{m}^{\mu_0+\mu_1+2} \subset \mathfrak{m}^2 I_f$. Further, the partial derivatives of φ belong to $\mathfrak{m}^{\mu_0+\mu_1+1} \subset \mathfrak{m} I_f$. Therefore, we may apply Proposition 2.2.5 and deduce that f is equivalent to $\tilde{f}$.

3.2.2. We say that the k-jet of $f_0 \in \mathcal{O}/\mathfrak{m}^{k+1}$ is *sufficient* if any function f with $j^k(f) = f_0$ is equivalent to f_0.

In 2.2 we have proved that if the multiplicity of a germ is μ_0/μ_1 then its jet of order $\mu_0 + \mu_1 + 1$ is sufficient.

3.2.3. We say that a function f_1 is *adjacent* to a function f_2 if the orbit of f_1 belongs to the closure of the orbit of f_2.

In other words f_1 is adjacent to f_2 if in any neighbourhood f_1 a function equivalent to f_2 is contained.

The adjacency is a natural order relation on the equivalence classes of singularities. The local algebra enables us to trace this relation algebraically.

3.2.4. PROPOSITION. *The multiplicity is semicontinuous from below on the space of germs of finite multiplicity (i.e. under a small jiggling of a germ its multiplicity $\mu = \mu_0/\mu_1$ may only diminish).*

PROOF. Let f be a germ with a singular point of multiplicity $\mu = \mu_0/\mu_1$. Then its k-jet is sufficient for $k \geqq \mu_0 + \mu_1 + 1$ and under a small jiggling of f the image of the gradient ideal I_{f_t}, where f_t is the perturbed germ, in the finite dimensional space of k-jets $\mathcal{O}/\mathfrak{m}^{k+1}$ is the vector subsuperspace spanned by the products of partial derivatives of f_t. Now the proof follows from the obvious lemma:

3.2.5. LEMMA. *Let $v_1(t), \ldots, v_p(t), w_1(t), \ldots, w_q(t)$ be a family of p even and q odd vectors of a linear superspace U continuously depending on a parameter t. Then the function*

$$d(t) = (d_0(t) \mid d_1(t)) = \dim \langle v_1(t), \ldots, v_p(t), w_1(t), \ldots, w_q(t) \rangle$$

is semicontinuous from above ($d(t)$ cannot diminish under a small jiggling).

3.3. *Deformations.*

3.3.1. A *deformation* of a germ $f \in \mathcal{O}_{p/q, \bar{0}}$ with base $\mathbf{C}^{r/s}$ is a germ morphism of spaces with fixed points F: $(\mathbf{C}^{p/q} \times \mathbf{C}^{r/s}, 0) \to (\mathbf{C}, 0)$ whose restriction onto $\mathbf{C}^{p/q}$ coincides with f (where f is considered as a map $(\mathbf{C}^{p/q}, 0) \to (\mathbf{C}, 0)$). Two deformations F and $\tilde{F}$ with base $\mathbf{C}^{r/s}$ are called *equivalent* if F reduces to $\tilde{F}$ by an isomorphism of germs G: $\mathbf{C}^{p/q} \times \mathbf{C}^{r/s} \to \mathbf{C}^{p/q} \times \mathbf{C}^{r/s}$ of the form

$$G(x, \xi; t, \tau) = (g(x, \xi, t, \tau); t, \tau), \tag{3.3.1}$$

where g: $\mathbf{C}^{p+r/q+s} \to \mathbf{C}^{p/q}$ is a map such that $g(x, \xi; 0, 0) = (x, \xi)$.

A deformation F of f is called *versal* if any deformation $\tilde{F}$ of this germ with base $\mathbf{C}^{r'/s'}$ is equivalent to a deformation obtained from F by a reparametrization, i.e. if there exists a map of bases $h: (\mathbf{C}^{r'/s'}, 0) \to (\mathbf{C}^{r/s}, 0)$ such that $\tilde{F}'(x, \xi, t, \tau) = F(x, \xi; h(t, \tau))$.
(In the deformation theory of complex structures, see e.g. references in [W], such deformations are usually called *full* and what is called a versal deformation should in addition possess the unique differential of the inducing morphism. In singularity theory the deformations with this additional property are called *miniversal*).

As in the manifold case the finite multiplicity of f is equivalent to the existence of a versal deformation.

3.3.2. VERSALITY THEOREM. *Let the multiplicity of a germ $f: \mathbf{C}^{p/q} \to \mathbf{C}$ be finite and $e_1, \ldots, e_{\mu_0}$ be even and $\varepsilon_1, \ldots, \varepsilon_{\mu_1}$ be odd elements from $\mathcal{O}_{p/q}$ generating $\mathcal{O}_f$. Then the function*

$$F(x_1, \ldots, x_p, \xi_1, \ldots, \xi_q, a_1, \ldots, a_{\mu_0}, \alpha_1, \ldots, \alpha_{\mu_1}) = f + \sum a_i e_i + \sum \alpha_j \varepsilon_j \in \mathcal{O}_{p+\mu_0/q+\mu_1}$$

is a versal deformation of $f \in \mathcal{O}_{p/q}$. The dimension of the base of any other versal deformation of f is no less than μ_0/μ_1. Conversely, if f has a versal deformation then the multiplicity of f is finite.

PROOF. A deformation of an analytic function f is a particular case of the notion of a germ of a relative complex superspace $f: X \to B$. The latter is the triple consisting of a germ of a complex superspace S (the base of deformation), a germ of a relative superspace $F: \mathcal{X} \to B \times S$ flat over S and a morphism $i: X \to \mathcal{X}$ over B identifying X with $F^{-1}(B \times \{S_0\})$ (for definitions see [W]). The general deformation theory of complex structures on superspace (see [W]) implies that the set of equivalence classes of deformations over the germ of $D = (*, \mathcal{O}_D)$, where $\mathcal{O}_D = \mathbf{C}\{x, \xi\}/(x^2, x\xi)$ (such deformation is called *infinitesimal*) is endowed with a natural vector superspace structure (intuitively it may be considered as the tangent superspace at f to any transversal to the orbit of f in the space of all germs over $\mathbf{C}$). If the dimension a/b of this superspace in finite then the germ $f: X \to B$ has a versal deformation with an (a, b)-dimensional base S which is in general singular). The base of a versal deformation S is smooth if all the obstructions to the extension of infinitesimal deformations to deformations with base $\mathbf{C}^{1/0}$ (or $\mathbf{C}^{0/1}$) vanish. In our case when $X = \mathbf{C}^{p/q}$ and $B = \mathbf{C}$ any deformation of a germ of $\mathbf{C}^{p/q}$ is trivial ([W]) and we get the maps: $F: \mathbf{C}^{p/q} \times S \to \mathbf{C}$, i.e. a deformation of the function f. Let us calculate the space of infinitesimal deformations of f. Let $F(x, \xi; t, \tau)$ be a deformation depending on 1 even and 1 odd parameter. Its infinitesimality means that we are only interested in the linear terms of the decomposition in the power series in parameters:

$$F(x, \xi; t, \tau) = f(x, \xi) + ta(x, \xi) + \tau\alpha(x, \xi) + \ldots .$$

Let the change of variables (3.3.1) be given by the map

$$g\colon \mathbf{C}^{p+1/q+1} \to \mathbf{C}^{p/q}\colon g = 1_{p/q} + tb + \tau\beta + \ldots,$$

where $b = (b_1, \ldots, b_{p+q}), \beta = (\beta_1, \ldots, \beta_{p+q}); b_1, \ldots, b_p, \beta_{p+1}, \ldots$ $\ldots, \beta_{p+q} \in \mathcal{O}_{p/q,\bar{0}}$, $b_{p+1}, \ldots, b_{p+q}, \beta_1, \ldots, \beta_p \in \mathcal{O}_{p/q,\bar{1}}$, and dots denote the term of higher degrees in t and τ. Then

$$F(g(x,\xi;t,\tau);t,\tau) = f(g(x,\xi;t,\tau)) + ta(g(x,\xi;t,\tau) + \tau\alpha(g(x,\xi;t,\tau)) + \ldots =$$

$$= f(x,\xi) + t\left(\sum_{i=1}^{p} b_i\frac{\partial f}{\partial x_i} + \sum_{i=1}^{p} b_{p+i}\frac{\partial f}{\partial \xi_i}\right) + \tau\left(\sum_{i=1}^{p} \beta_i\frac{\partial f}{\partial x_i} + \sum_{i=1}^{q} \beta_{p+i}\frac{\partial f}{\partial \xi_i}\right) + \ldots$$
$$+ ta(x,\xi) + \tau\alpha(x,\xi) + \ldots$$

Therefore the infinitesimal deformations given by the pairs (a', α') and (a, α), where $a, a' \in \mathcal{O}_{p/q,\bar{0}}$; $\alpha, \alpha' \in \mathcal{O}_{p/q,\bar{1}}$ are equivalent if $a' - a \in I_{f,\bar{0}}$ and $\alpha' - \alpha \in I_{f,\bar{1}}$. Identifying the set of pairs (a, α) with $\mathcal{O}_{p/q}$ we finally see that the space of infinitesimal deformations is isomorphic to $\mathcal{O}_{p/q}/I_f = \mathcal{O}_f$. Therefore if $\mu = \dim \mathcal{O}_f < \infty$ the germ of the relative complex superspace $f\colon \mathbf{C}^{p/q} \to \mathbf{C}$ has a μ-dimensional base. For such a germ all the obstruction vanish (cf. [W]) therefore the base is the germ of the supermanifold $\mathbf{C}^{\mu}$.

3.3.3.4. REMARK. The above arguments hold if the analytic functions are replaced by smooth ones since by Theorem 2.1.2 a germ of finite multiplicity is equivalent to a polynomial.

4. Simple singularities

4.1. *Modality.* As is shown by V. Arnold the most natural results are obtained when singularities are classified with respect to the number of parameter (moduli) on which nearby orbits depend. Let us give an exact definition.

4.1.1. Let a Lie group G act on a manifold M. The *modality* of a point $f \in M$ is the minimal m such that a neighbourhood of f may be covered by a finite number of m-parameter families of G-orbits.

The points of modality 0, i.e. such that in their neighbourhoods there is only a finite number of orbits, are called *simple*.

The manifold of germs of even functions on $(\mathbf{C}^{p/q}, 0)$ with a singular point with zero critical value at the origin and the group $G_{p/q}$ which acts on this manifold are both infinite dimensional. Nevertheless, all the singularities of finite multiplicity are of finite modality. Indeed, by 3.2.1 any such germ has a sufficient k-jet therefore considering the action of k-jets of diffeomorphisms on the space $[\mathcal{O}/\mathfrak{m}^{k+1}]_{\bar{0}}$ for $k \geqq \mu_0 + \mu_1 + 1$ we reduce all the considerations to a finite dimensional situation.

(The germs close to these considered also have sufficient k-jets since the multiplicity does not grow under small jiggling thanks to 3.2.4. Therefore, the modality of a germ as calculated above does not depend on k).

Before we start describing simple germs let us make several remarks.

4.1.2. The above definition of modality is a mere surrogate of a correct definition (that will be given elsewhere) according to which and the general principles of supergeometry we should consider not only the manifold of even germs and jets but the supermanifolds of all germs on which the supergroup of automorphisms acts. Modality considered this way is not one number but a pair of numbers since the families of orbits may depend both on even and odd parameters. In the subsequent paper we will list the $(1, 0)$-modal singularities and hope to describe also $(0, 1)$-modal ones.

4.1.3. As is often the case when superizing different mathematical notions that of modality splits into two. The point is that the statement of Gabrielov's theorem to the effect that the modality of a (purely even) singularity equals the dimension of the stratum $\mu = \text{const}$ in the base of the miniversal (even if we disregard Remark 3.1.3) deformation is false in the supercase. Thus, the dimension of the stratum $\mu = \text{const}$ is also the legitimate pretender for the role of the superanalogue of modality. We will call this dimension the *inner modality* and denote it by μ_0. It is not difficult to show that $\mu_0 \leqq \mu$ so that in the list of singularities with inner modality 0 all the simple singularities will be found but not only they. In particular, this list contains the germs $\xi_1, \xi_2 \ldots \xi_q$, etc. Perhaps the classification of germs in terms of their inner modality leads to more final (algebraic) results.

4.2. *Lits of singularities.* In what follows (Tables 1–3) we list the normal forms of all the simple and bordering singularities (i.e. families of singularities that depend on at least one modulus, but all the other singularities in a neighbourhood fall into finitely many equivalence classes) and also the normal forms of germs of cork 1/4. Proofs of Theorems 4.2.1–4.2.5 follow from the Classifier of Singularities 4.3 and Lemmas 4.4.

REMARK. Our notations for the types of singularities unlike Arnold's ones have no deep meaning. In particular, the notations of Tables 2, 3 are compatible with those of Table 1, as far as corank is concerned. This explains the appearance of the types VIII in Table 2 between the types VI and VII of any places: the singularities of cork 1/8–1/9 do not occur in Table 1.

4.2.1. THEOREM. *Any singularity is either equivalent to one of the singularities from Table* 1 *or is adjacent to one of the singularities from Table* 2.

4.2.2. THEOREM. *The modality of any singularity from Table* 2 *is no less than* 1, *i.e. they are bordering ones.*

4.2.3. THEOREM. *The singularities from Table* 1 *are not adjacent to any of the singularities from Table* 2.

4.2.3.1. COROLLARY. *All the singularities from Table* 1 *are simple and any simple singularity is equivalent to one of them.*

4.2.4. THEOREM. *Any singularity of cork* 1/4 *is equivalent to one of the singularities from Table* 3.

4.2.5. THEOREM. *All the singularities from Tables* 1 *and* 2 *are pair-wise nonequivalent.*

TABLE 1.
Simple singularities

Type	Normal form	Cork	Comments
A_n	x^{n+1}	1/0–1/1	$n \geqq 2$
D_n	$x^{n-1} + xy^2$	2/0–2/1	$n \geqq 4$
E_6	$x^3 + y^4$	2/0–2/1	
E_7	$x^3 + xy^3$	2/0–2/1	
E_8	$x^3 + y^4$	2/0–2/1	
I_1	$\xi_1\xi_2\xi_3\xi_4$	0/4–0/7	
I_2	$\xi_1\xi_2\xi_3\xi_4 + \xi_1\xi_2\xi_5\xi_6$	0/6–0/7	
I_3	$\xi_1\xi_2\xi_3\xi_4 + \xi_1\xi_2\xi_5\xi_6 + \xi_3\xi_4\xi_5\xi_6$	0/6–0/7	
I_4	$\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6$	0/6–0/7	
I_5	$\xi_1\xi_2\xi_3\xi_4 + \xi_1\xi_5\xi_6\xi_7$	0/7	
I_6	$\xi_1\xi_2\xi_3\xi_4 + \xi_1\xi_2\xi_5\xi_6 + \xi_1\xi_3\xi_5\xi_7$	0/7	
I_7	$\xi_1\xi_2\xi_3\xi_4 + \xi_1\xi_2\xi_5\xi_6 + \xi_1\xi_3\xi_5\xi_7$	0/7	
I_8	$\xi_1\xi_2\xi_3\xi_4 + \xi_1\xi_2\xi_5\xi_6 + \xi_1\xi_3\xi_5\xi_7 + \xi_2\xi_4\xi_5\xi_7$	0/7	
I_9	$\xi_1\xi_2\xi_3\xi_4 + \xi_1\xi_2\xi_5\xi_6 + \xi_1\xi_3\xi_5\xi_7 + \xi_2\xi_4\xi_6\xi_7$	0/7	
I_{10}	$\xi_1\xi_2\xi_3\xi_4 + \xi_1\xi_2\xi_5\xi_6 + \xi_1\xi_3\xi_5\xi_7 + \xi_2\xi_3\xi_6\xi_7 + \xi_3\xi_5\xi_6\xi_7$	0/7	
I_{11}	0	0/1–0/7	

TABLE 1 (continued)

II	$x^n + x^m\xi_1\xi_2$	1/2–1/3	$1 \leqq m \leqq n-1$
$\mathrm{III}_1(n,m)$	$x^n + x\xi_1\xi_2 + x^m\xi_3\xi_4$	1/4	$1 \leqq m \leqq n-1$
$\mathrm{III}_2(n)$	$x^n + x\xi_1\xi_2$	1/4	$n \geqq 3$
III_3	$x^3 + \xi_1\xi_2\xi_3\xi_4$	1/4	
III_4	$x^3 + x\xi_1\xi_2\xi_3\xi_4$	1/4	
III_5	x^3	1/4	
$\mathrm{IV}_1(n,m)$	$x^n + x\xi_1\xi_2 + x^m\xi_3\xi_4 + \xi_1\xi_3\xi_4\xi_5$	1/5	$2 \leqq m \leqq n-1$
$\mathrm{IV}_2(n,m)$	$x^n + x\xi_1\xi_2 + x^m\xi_3\xi_4$	1/5	$1 \leqq m \leqq n-1$
$\mathrm{IV}_3(n)$	$x^n + x\xi_1\xi_2$	1/5	$n \geqq 3$
$\mathrm{IV}_4(n,m)$	$x^n + x^2\xi_1\xi_2 + x^m\xi_3\xi_4 + \xi_1\xi_3\xi_4\xi_5$	1/5	$2 \leqq m \leqq n-1$
IV_5	$x^3 + x\xi_1\xi_2\xi_3\xi_4$	1/5	
IV_6	x^3	1/5	
$\mathrm{V}(n)$	$x^n + x\xi_1\xi_2 + x\xi_3\xi_4 + x\xi_5\xi_6$	1/6	$n \geqq 3$
$\mathrm{VI}_1(n)$	$x^n + x\xi_1\xi_2 + x\xi_3\xi_4 + x\xi_5\xi_6$	1/7	$n \geqq 3$
$\mathrm{VI}_2(n)$	$x^n + x\xi_1\xi_2 + x\xi_3\xi_4 + x\xi_5\xi_6 +$ $+ \xi_7\xi_1\xi_3\xi_5$	1/7	$n \geqq 3$
$\mathrm{VI}_3(n)$	$x^n + x\xi_1\xi_2 + x\xi_3\xi_4 + x\xi_5\xi_6 +$ $+ \xi_7\xi_1(\xi_3\xi_5 + \xi_4\xi_6)$	1/7	$n \geqq 3$
$\mathrm{VI}_4(n)$	$x^n + x\xi_1\xi_2 + x\xi_3\xi_4 + x\xi_5\xi_6 +$ $+ \xi_7(\xi_1\xi_3\xi_5 + \xi_2\xi_4\xi_6)$	1/7	$n \geqq 3$
$\mathrm{VI}_5(n)$	$x^n + x\xi_1\xi_2 + x\xi_3\xi_4 + x\xi_5\xi_6 +$ $+ \xi_7(\xi_1\xi_3\xi_5 + \xi_1\xi_4\xi_6 + \xi_2\xi_4\xi_5)$	1/7	$n \geqq 3$
$\mathrm{VI}_6(n)$	$x^n + x\xi_1\xi_2 + x\xi_3\xi_4 + x^2\xi_5\xi_6 +$ $+ \xi_7\xi_5\xi_6\xi_1 + \xi_7\xi_5\xi_2\xi_3 +$ $+ \xi_7\xi_6\xi_2\xi_4$	1/7	$n \geqq 3$
VI_7	$x^3 + x\xi_1\xi_2 + x\xi_3\xi_4 + \xi_7\xi_6\xi_5\xi_1 +$ $+ \xi_7\xi_6\xi_2\xi_3 + \xi_7\xi_5\xi_3\xi_2$	1/7	
VI_8	$x^3 + x\xi_1\xi_2 + x\xi_3\xi_4 + \xi_7\xi_6\xi_5\xi_1 +$ $+ \xi_7\xi_6\xi_2\xi_3 + \xi_7\xi_5\xi_2\xi_4$	1/7	
VI_9	$x^3 + x\xi_1\xi_2 + x\xi_3\xi_4 +$ $+ \xi_7\xi_6(\xi_5\xi_1 + \xi_3\xi_4)$	1/7	
VI_{10}	$x^3 + x\xi_1\xi_2 + x\xi_3\xi_4 +$ $+ \xi_7\xi_6(\xi_5\xi_1 + \xi_3\xi_2)$	1/7	
VI_{11}	$x^3 + x\xi_1\xi_2 + x\xi_3\xi_4 + \xi_7\xi_6\xi_5\xi_1$	1/7	
$\mathrm{VII}_1(n)$	$x^2y + y^n + x\xi_1\xi_2 + y\xi_2\xi_3$	2/3	$n \geqq 3$
VII_2	$x^3 + y^4 + x\xi_1\xi_2 + y\xi_2\xi_3$	2/3	
VII_3	$x^3 + xy^3 + x\xi_1\xi_2 + y\xi_2\xi_3$	2/3	
VII_4	$x^3 + y^5 + x\xi_2\xi_2 + y\xi_2\xi_3$	2/3	

TABLE 2.
Bordering unimodal singularities

Type	Singularity	Cork	Modality
I'_1	generic $\Psi \in \Lambda^4 \langle \xi_1, \dots, \xi_8 \rangle$	0/8	$\geqq 6$
I'_2	generic $\Psi \in \Lambda^4 \langle \xi_1, \dots, \xi_9 \rangle$	0/9	$\geqq 45$
III′	$x^4 + x^2\xi_1\xi_2 + x^2\xi_3\xi_4 + a\xi_1\xi_2\xi_3\xi_4,\ a \neq \frac{1}{2}$	1/4	1
IV′	$x^4 + x^2\xi_1\xi_2 + x^2\xi_3\xi_4 + a\xi_1\xi_2\xi_3\xi_4,\ a \neq \frac{1}{2};\ 2$	1/4	1
V′	$x^3 + x\xi_1\xi_2 + x\xi_3\xi_4 + x^2\xi_5\xi_6 + a\xi_5\xi_6(\xi_1\xi_2 - \xi_3\xi_4)$	1/6	1
VI'_1	$x^3 + x\xi_1\xi_2 + x\xi_3\xi_4 + \xi_2\xi_3\xi_5\xi_7 + \xi_1\xi_4\xi_6\xi_7 + \xi_5\xi_6\left(\frac{1}{a}\xi_1\xi_2 + a\xi_3\xi_4\right)$	1/7	1
VI'_2	$x^3 + x\xi_1\xi_2 + a\xi_1\xi_2\xi_3\xi_4 + \xi_1\xi_2\xi_3\xi_5 + \xi_1\xi_2\xi_6\xi_7 + \xi_1\xi_4\xi_5\xi_7 + \xi_3\xi_5\xi_6\xi_7$	1/7	$\geqq 1$
VIII'_1	$x^3 + x(\xi_1\xi_2 + \xi_3\xi_4 + \xi_5\xi_6 + \xi_7\xi_8) + \Psi$; $\Psi \in \Lambda^4 \langle \xi_1, \dots, \xi_8 \rangle$	1/8	$\geqq 6$
VIII'_2	$x^3 + x(\xi_1\xi_2 + \xi_3\xi_4 + \xi_5\xi_6 + \xi_7\xi_8) + \Psi$; $\Psi \in \Lambda^4 \langle \xi_1, \dots, \xi_9 \rangle$	1/9	$\geqq 45$
VII'_1	$x^3 + y^6 + ax^2y^2$	2/0–2/1	1
VII'_2	$x^4 + y^4 + ax^2y^2$	2/0–2/1	1
VII'_3	$x^3 + xy^2 + x\xi_1\xi_2 + ay\xi_1\xi_2$	2/2–2/3	1
VII'_4	$x^3 + y^3 + z^3 + axyz$	3/0–3/1	1

TABLE 3.
Singularities of cork 1/4

Type	Normal form	Comments
$\mathrm{III}_a(n, m_1, m_2, l)$	$x^n + n^{m_1}\xi_1\xi_2 + x^{m_2}\xi_3\xi_4 + a\xi_1\xi_2\xi_3\xi_4 \cdot (1 + a_1x + \dots + a_kx^k),\ \neq 0$	$k = \begin{cases} \min(n - m_1 - 2, m_2 - 2 - l) \\ \text{for } m - m_1 - m_2 \neq l \text{ or } a \neq m_2/n \\ m_2 - l - 1 \text{ otherwise} \end{cases}$ $n - 1 > m_2 > m_1 > l + 1$
$\mathrm{III}_b(n, m_1, m_2)$	$x^n + x^{m_1}\xi_1\xi_2 + x^{m_2}\xi_3\xi_4$	$n - 1 > m_2 > m_1$
$\mathrm{III}_c(n, m, l)$	$x^n + x^m\xi_1\xi_2 + x^l\xi_1\xi_2\xi_3\xi_4$	$n - > m \geqq l$
$\mathrm{III}_d(n, m)$	$x^n + x^m\xi_1\xi_2$	$n - 1 > m$
$\mathrm{III}_e(n, m)$	$x^n + x^m\xi_1\xi_2\xi_3\xi_4$	$n - 1 \geqq m$

4.3. *Classifier of singularities.* Like the classifiers of plants, minerals, etc., the classifier of singularities is a device that helps one to assign a singularity its place in the accepted classification system.

Its other aim is to help organizing the proof of the right classification.

NOTATIONS:

$\mathfrak{m}$ the maximal ideal of the ring of germs $\mathcal{O}_{p/q}$;

$\mathfrak{n}$ the ideal of $\mathcal{O}_{p/q}$ generated by the odd elements;

(M) the ideal of $\mathcal{O}_{p/q}$ generated by a set M; in particular

(x) the ideal generated by the even generators of $\mathcal{O}_{p/q}$ (it is *not* G-invariant);

$\langle M \rangle$ the linear span of a set M (over C);

$\operatorname{ord} f = \min\{k \mid f \in m^k\}$ the degree of f in the grading adjoined to the filtration $\mathcal{O} \supset \mathfrak{m} \supset \mathfrak{m}^2 \supset \ldots$;

$\operatorname{ord}_x f$ the degree of f in the grading adjoined to the filtration $\mathcal{O} \supset (x) \supset (x)^2 \supset \ldots$ (this degree is *not* G-invariant);

$f = f^{(0)} + f^{(2)} + f^{(4)} + \ldots, f^{(2k)} \in \Lambda^{2k}[\xi_1, \ldots, \xi_q]$, the decomposition of an even function in the power series in odd variables;

$\operatorname{rk} r$ the minimal number of summands in the representations of $r \in \Lambda^k[\xi_1, \ldots, \xi_q]$ as the sum of decomposible polyvectors

$\rightarrow$ is adjacent to

$\Rightarrow$ implies, yields

$\mapsto$ see

$\twoheadrightarrow$ only one of the following possibilities may occur

In parenthesis we indicate the numbers of the lemmas in which the statements of Classifier are proved; the number at the end of a subsection is the number of a subsection that refers to this one (unless the one that refers is the immediately preceding one).

Let $\operatorname{cork} f = p/q$

1. If $\mu(f) < \infty$ then $\twoheadrightarrow$
   1) $q = 0$ or $1 \Rightarrow f = f^{(0)}$ and f does not depend on odd variables and belongs to one of the types A_n, D_n, E_n, VII'_1, VII'_2, VII'_3, VII'_4 ([A]);
   2) $p \geqq 3 \Rightarrow f \rightarrow \mathrm{VII}'_4$ ([A]);
   3) $p = 2, q = 2$ or $q \geqq 4 \Rightarrow \mathrm{VII}'_3$ (5.19–5.20);
   4) $p = 2, q = 3 \mapsto 24$
   5) $p = 1, q \geqq 8 \Rightarrow f \rightarrow \mathrm{VIII}'_1$ or VIII'_2 depending on the parity of q
   6) $p = 1; 2 \leqq q \leqq 7 \mapsto 2$;
   7) $p = 0, q \geqq 8 \Rightarrow f \rightarrow \mathrm{I}'_1$ or I'_2 depending on the parity of q
   8) $p = 0, 2 \leqq q \leqq 7 \mapsto 26$.

2. If $p = 1, 2 \leqq q \leqq 7$ then $\twoheadrightarrow$
   1) $q = 2$ or $q = 3 \Rightarrow f \sim (n, m)$ (5.1.1);
   2) $q = 4 \mapsto 3$;
   3) $q \geqq 5 \mapsto 7$.

3. If $p = 1, q = 4$ then ➡ (5.1)
   1) $f \sim x^n + x^{m_1}\xi_1\xi_2 + x^{m_2}\xi_3\xi_4 + f^{(4)}, m_1, m_2 \leqq n - 2 \Mapsto 4$;
   2) $f \sim x^n + x^{m_1}\xi_1\xi_2 + f^{(4)}, m_1 \leqq n - 2 \Mapsto 5$;
   3) $f \sim x^n + f^{(4)} \Mapsto 6$.

4. If $f \sim x^n + x^{m_1}\xi_1\xi_2 + x^{m_2}\xi_3\xi_4 + f^{(4)}, m_1 \leqq m_2 \leqq n - 2$, then ➡
   1) $\mathrm{ord}_x f^{(4)} \geqq m_1 - 1 \Rightarrow f \sim \mathrm{III}_b(n, m_1, m_2)$ (5.2a);
   2) $\mathrm{ord}_x f^{(4)} \leqq m_1 - 2 \Rightarrow f \sim \mathrm{III}_a(n, m_1, m_2, l)$ (5.2a).

5(3). If $f \sim x^n + x^{m_1}\xi_1\xi_2 + f^{(4)}$ then ➡
   1) $\mathrm{ord}_x f^{(4)} \geqq m_1 - 1 \Rightarrow f \sim \mathrm{III}_d(n, m_1)$ (5.2b);
   2) $\mathrm{ord}_x f^{(4)} = l \leqq m_1 - 2 \Rightarrow f \sim \mathrm{III}_c(n, m_1, l)$ (5.2b).

6(3). If $f \sim x^n + f^{(4)}$ then ➡
   1) $\mathrm{ord}_x f^{(4)} \geqq n - 1 \Rightarrow f \sim \mathrm{III}_e(n, n - 1)$ (5.2c);
   2) $\mathrm{ord}_x f^{(4)} = l \leqq n - 2 \Rightarrow f \sim \mathrm{III}_e(n, l)$ (5.2c).

7(2). If $p = 1, 5 \leqq q \leqq 7$ then ➡
   1) $q = 5 \Mapsto 8$;
   2) $q = 6 \Mapsto 17$;
   3) $q = 7 \Rightarrow 18$.

8. If $p = 1$, $q = 5$ then $f \sim x^n + x^{m_1}\xi_1\xi_2 + x^{m_2}\xi_3\xi_4 + f^{(4)}$, $m_1 \leqq m_2$ (5.1) and ➡
   1) $m_1 \geqq 3 \Mapsto 9$;
   2) $m_1 = 2 \Mapsto 10$;
   3) $m_1 = 1 \Mapsto 13$.

9. If $f \sim x^n + x^{m_1}\xi_1\xi_2 + x^{m_2}\xi_3\xi_4 + f^{(4)}$, $3 \leqq m_1 \leqq m_2$ then ➡
   1) $n = 3 \Mapsto 16$;
   2) $n \geqq 4 \Rightarrow f \to \mathrm{IV}'$ (5.4.1)

10(8). If $f \sim x^n + x^2\xi_1\xi_2 + x^m\xi_3\xi_4 + f^{(4)}, m \geqq 2$ then ➡
   1) $n = 3 \Mapsto 16$;
   2) $n \geqq 4, m = 2 \Mapsto 11$;
   3) $n \geqq 4, m \geqq 3 \Mapsto 12$.

11. If $f \sim x^n + x^2(\xi_1\xi_2 + \xi_3\xi_4) + f^{(4)}, n \geqq 4$ then ➡
   1) $f^{(4)} \in (\xi_1\xi_2\xi_3\xi_4) + (x) \Rightarrow f \to \mathrm{IV}'$ (5.4.3);
   2) $f^{(4)} \in (\xi_1\xi_2\xi_3\xi_4) + (x) \Rightarrow f \sim \mathrm{IV}_4(n, 2)$ (5.5).

12(10). If $f \sim x^n + x^2\xi_1\xi_2 + x^m\xi_3\xi_4 + f^{(4)}, n \geqq 4, m \geqq 3$ then ➡
   1) $f^{(4)} \in (\xi_1\xi_2) + (x) \Rightarrow f \to \mathrm{IV}'$ (5.4.2);
   2) $f^{(4)} \notin (\xi_1\xi_2) + (x)$ and $m \leqq n - 1 \Rightarrow f \sim \mathrm{IV}_4(n, m)$ (5.5);
   3) $f^{(4)} \notin (\xi_1\xi_2) + (x)$ and $m \geqq n \Rightarrow f \sim \mathrm{IV}_4(n, n - 1)$ (5.5).

13(8). If $f \sim x^n + x\xi_1\xi_2 + x^m\xi_3\xi_4 + f^{(4)}$ then $\Rrightarrow$
1) $m = 1 \Rightarrow f \sim \mathrm{IV}_2$ (n. 1) (5.6a);
2) $2 \leqq m \leqq n - 2 \Rrightarrow 14$;
3) $m \geqq n - 1 \Rrightarrow 15$.

14. If $f \sim x^n + x\xi_1\xi_2 + x^m\xi_3\xi_4 + f^{(4)}, 2 \leqq m \leqq n - 2$ then $\Rrightarrow$
1) $f^{(4)} \in (\xi_1\xi_2) + (x) \Rightarrow f \sim \mathrm{IV}_2\,(n, m)$ (5.6a);
2) $f^{(4)} \notin (\xi_1\xi_2) + (x) \Rightarrow f \sim \mathrm{IV}_1\,(n, m)$ (5.6b);

15(13). If $f \sim x^n + x\xi_1\xi_2 + x^m\xi_3\xi_4 + f^{(4)} +$ then
$f \sim x^n + x\xi_1\xi_2 + f^{(4)}$ (5.1) and $\Rrightarrow$
1) $f^{(4)} \in (x) \Rightarrow f \sim \mathrm{IV}_3\,(n)$ (5.7a);
2) $f^{(4)} \notin (x)$ and $f^{(n)} \in (x) + (\xi_1\xi_2) \Rightarrow f \sim \mathrm{IV}_2(n, n-1)$ (5.7b);
3) $f^{(4)} \notin (x) + (\xi_1\xi_2) \Rightarrow f \sim \mathrm{IV}_1\,(n, n-1)$ (5.7c).

16(9 and 10). If $f \sim x^3 + x^{m_1}\xi_1\xi_2 + x^{m_2}\xi_3\xi_4 + f^{(4)}, m_1, m_2 \geqq 2$ then
$f \sim x^3 + f^{(4)}$ (5.1) and $\Rrightarrow$
1) $f^{(4)} \in (x^2) \Rightarrow f \sim \mathrm{IV}_6$ (5.8a);
2) $f^{(4)} \in (x)$ and $f^{(4)} \notin (x^2) \Rightarrow f \sim \mathrm{IV}_5$ (5.8b);
3) $f^{(4)} \notin (x) \Rightarrow f \sim \mathrm{IV}_4$ (3.2) (5.8c).

17(7). If $p = 1, q = 6$ then
$f \sim x^n + x^{m_1}\xi_1\xi_2 + x^{m_2}\xi_3\xi_4 + x^{m_3}\xi_5\xi_6 + f^{(4)} + f^{(6)}$,
$m_1 \leqq m_2 \leqq m_3$ (5.1)
and $\Rrightarrow$
1) $m_1 = m_2 = m_3 = 1 \Rightarrow f \sim \mathrm{V}(n)$ (5.9);
2) $m_3 \geqq 2 \Rightarrow f \to \mathrm{V}'$ (5.10).

18(7). If $p = 1, q = 7$ then $f \sim x^n + x^{m_1}\xi_1\xi_2 + x^{m_2}\xi_3\xi_4 + x^{m_3}\xi_5\xi_6 + f^{(4)} + f^{(6)}$,
$m_1 \leqq m_2 \leqq m_3$. (5.1) and $\Rrightarrow$
1) $m_1 = m_2 = m_3 = 1 \Rightarrow 19$;
2) $m_1 = m_2 = 1, m_3 \geqq 2 \Rightarrow 20$;
3) $m_2 \geqq 2 \Rightarrow f \to \mathrm{VI}'_2$ (5.15).

19. If $f \sim x^n + x(\xi_1\xi_2 + \xi_3\xi_4 + \xi_5\xi_6) + f^{(4)} + f^{(6)}$ then
$f \sim x^n + x(\xi_1\xi_2 + \xi_3\xi_4 + \xi_5\xi_6) + \xi_7 r$, where $r \in \Lambda^3\langle\xi_1, \ldots, \xi_6\rangle$ (5.13).
and $\Rrightarrow$ (5.11):
1) $r = 0 \Rightarrow f \sim \mathrm{VI}_1\,(n)$ (5.12 and 5.14);
2) $\mathrm{rk}\, r = 1 \Rightarrow f \sim \mathrm{VI}_2\,(n)$ (5.12 and 5.14);
3) $\mathrm{rk}\, r = 2$ and r is divisible by a vector $v \in \langle\xi_1, \ldots, \xi_6\rangle \Rightarrow$
$\Rightarrow f \sim \mathrm{VI}_3\,(n)$ (5.12 and 5.14);
4) $\mathrm{rk}\, r = 2$ and r is not divisible by a vector $v \Rightarrow f \sim \mathrm{VI}_4\,(n)$ (5.12 and 5.14);
5) $\mathrm{rk} = 3 \Rightarrow f \sim \mathrm{VI}_5\,(n)$ (5.12 and 5.14).

20(18). If $f \sim x^n + x\xi_1\xi_2 + x\xi_3\xi_4 + x^m\xi_5\xi_6 + f^{(4)} + f^{(6)}$, $m \geqq 2$ then ⇛
1) $n = 3 \Rightarrow f \sim x^3 + x(\xi_1\xi_2 + \xi_3\xi_4) + f^{(4)} + f^{(6)}$ (5.1) ⇛ 22;
2) $n \geqq 4, m = 2$ ⇛ 21;
3) $n \geqq 4, m \geqq 3 \Rightarrow f \to \mathrm{IV}'$ (5.16.1).

21. If $f = x^n + x(\xi_1\xi_2 + \xi_3\xi_4) + x^2\xi_5\xi_6 + f^{(4)} + f^{(6)}$ and $f^{(4)} \equiv \xi_7 r \bmod ((x) + (\Lambda^4\langle\xi_1,\dots,\xi_6\rangle) + (\xi_1\xi_2 + \xi_3\xi_4))$, where $r \in \Lambda^3\langle\xi_1,\dots,\xi_6\rangle$ then ⇛
1) $\operatorname{rk} r = 3 \Rightarrow f \sim \mathrm{VI}_6(n)$ (5.16a);
2) $\operatorname{rk} r \leqq 2 \Rightarrow f \to \mathrm{IV}'$ (5.16b).

22(20). If $f \sim x^3 + x(\xi_1\xi_2 + \xi_3\xi_4) + f^{(4)} + f^{(6)}$ then ⇛
1) $f^{(4)} \in (\Lambda^2(\xi_1,\dots,\xi_n)) + (x) \Rightarrow f \to \mathrm{VI}'_1$ (5.17a)
2) $f^{(4)} \notin (\Lambda^2(\xi_1,\dots,\xi_n)) + (x) \Rightarrow f \sim x^3 + x(\xi_1\xi_2 + \xi_3\xi_4) + a\xi_3\xi_4\xi_6\xi_7 + b\xi_2\xi_4\xi_5\xi_7 + c\xi_2\xi_3\xi_5\xi_6$ (5.17b) ⇛ 23.

23. If $f \sim x^3 + x(\xi_1\xi_2 + \xi_3\xi_4) + a\xi_3\xi_4\xi_6\xi_7 + b\xi_2\xi_4\xi_5\xi_7 + c\xi_2\xi_3\xi_5\xi_6$ then ⇛ (5.18):
1) $a, b, c \neq 0 \Rightarrow f \sim \mathrm{VI}_4(3)$;
2) $a, c \neq 0$ or $a, b \neq 0, c = 0 \Rightarrow f \sim \mathrm{VI}_7$;
3) $a = 0, b, c \neq 0 \Rightarrow f \sim \mathrm{VI}_8$;
4) $a \neq 0, b = c = 0 \Rightarrow f \sim \mathrm{VI}_9$;
5) $a = b = 0, c \neq 0$ or $a = c = 0, b \neq 0 \Rightarrow f \sim \mathrm{VI}_{10}$;
6) $a = b = c = 0 \Rightarrow f \sim \mathrm{VI}_{11}$.

24(1). If $p = 2$, $q = 3$ then $f = f_0 + r \bmod m^4$, where $r \in \langle x, y\rangle \otimes \Lambda^2\langle\xi_1,\xi_2,\xi_3\rangle$ then ⇛
1) r is decomposible $\Rightarrow f \to \mathrm{VII}'_3$ (5.21a);
2) r is not decomposible and f_0 is a simple even singularity ⇒ 25;
3) r is not decomposible and f_0 is not simple $\Rightarrow f \to \mathrm{VII}'_1$ or VII'_2 (5.21b).

25. If $f = f_0 + r \bmod \mathfrak{m}^4$, where $r \in \langle x, y\rangle \otimes \Lambda^2\langle\xi_1,\xi_2,\xi_3\rangle$, cork $k\, f = 2$ then ⇛ (5.21c).
1) f^0 is of type $D_n \Rightarrow f \sim \mathrm{VII}_1(n-1)$;
2) f^0 is of type $E_6 \Rightarrow f \sim \mathrm{VII}_2$;
3) f^0 is of type $E_7 \Rightarrow f \sim \mathrm{VII}_3$;
4) f^0 is of type $E_8 \Rightarrow f \sim \mathrm{VII}_4$.

26(1). If $p = 0$ $2 \leqq q \leqq 7$ then $f = f^{(4)} + f^{(6)}$ and ⇛ (5.22)
1) $f = 0 = I_{11}$;
2) $f^{(4)} = 0$; $f^{(6)} = 0$; $q = 6$ or 7 and $f \sim I_4$
3) $f^{(4)} \neq 0 \Rightarrow f \sim f^{(4)}$ which belongs to one of the types $I_1 = I_3, I_5 - I_{10}$. The classification in this case is the description of the orbits of $\mathrm{GL}(V)$ in $\Lambda^4(V)$ in $\Lambda^4(V)$ where $\dim V \leqq 7$. The most difficult part of this classifi-

cation is the case of 3-vectors in 7-dimensional space solved by Schouten, see [VE].

5. Proof of lemmas of the classifier

5.1. LEMMA. *Let* cork $f = (1/r)$, $s = [r/2]$, $h = \operatorname{ord}_x f^{(0)}$, $m_1 = \operatorname{ord}_x f^{(2)}$, $m_2 = \operatorname{ord}_x(f^{(2)})^2 - m_1, \ldots, m_s = \operatorname{ord}_x(f^{(2)})^s - \ldots - m_{s-1}$. *Then* $m_1 \leqq m_2 \leqq \ldots \leqq m_s$. L et i be such that $m_i < n - 1$, $m_{i+1} \geqq n - 1$. *Then*

$$f \sim x^n + x^{m_1}\xi_1\xi_2 + \ldots + x^{m_i}\xi_{2i-1}\xi_{2i} + \varphi$$

for some $\varphi \in n^4$.

PROOF. Since $\operatorname{ord}_x a^l \geqq \operatorname{ord}_x a^{l-1} + \operatorname{ord}_x a$, then $m_1 \leqq m_2 \leqq \ldots \leqq m_s$. As follows from [A], $f^{(0)} \sim x^n$. Therefore f reduces to the form:

$$f = x^n + f^{(2)} + f^{(4)} + f^{(4)} + \ldots.$$

Let $f^{(2)} = \omega \cdot x^{m_1} + \tau$, where $\omega \in \Lambda^2(\xi_1, \ldots, \xi_7)$ and $\tau \in m^{m_1+1}$. Reduce ω to the form

$$\xi_1\xi_2 + \xi_3\xi_4 + \ldots + \xi_{2p-1}\xi_{2p}.$$

Then $m_1 = m_2 = \ldots = m_p$. By Morse's lemma

$$f^{(2)} \sim x^{m_1}(\xi_1\xi_2 + \xi_3\xi_4 + \ldots + \xi_{2p-1}\xi_{2p} + \mathfrak{k}_1(x, \xi_{2p+1}, \ldots, \xi_r))$$

and $\mathfrak{k}_1 \in m^{m_{p+1}-m_p}$. Similarly we may show that

$$f^{(2)} \sim x^{m_1}(\xi_1\xi_2 + \ldots + \xi_{2p-1}\xi_{2p}) + x^{m_2}(\xi_{2p+1}\xi_{2p+2} + \ldots$$
$$\ldots + \xi_{2q+1}\}_{2q+2} + \mathfrak{k}_2(x, \xi_{2q+3}, \ldots, \xi_7),$$

where $\mathfrak{k}_2 \in m^{m_{q+1}-m_q}$, and $m_q = \ldots = m_{p+1}$. Repeating this procedure several times we finally reduce $f^{(2)}$ to the form

$$f^{(2)} = x^{m_1}\xi_1\xi_2 + \ldots + x^{m_1}\xi_{2-s-1}\xi_{2s}.$$

We have only used the changes of coordinates preserving x. Therefore

$$f \sim x^n + x^{m_1}\xi_1\xi_2 + \ldots + x^{m_s}\xi_{2s-1}\xi_{2s} + \varphi, \text{ where } \varphi \in n^4.$$

Now applying the change of coordinates

$$x \mapsto x - \frac{1}{n}(x^{m_i+1-n+1}\xi_{2i+1}\xi_{2i+2} + \ldots + x^{m_s-n+1}\xi_{2s-1}\xi_{2s}),$$

we reduce f to the form indicated in the lemma.

5.1.1. COROLLARY. *Let* cork$f$ = 1/2 (*or* 1/3). *Then* $f \sim \mathrm{II}(n, m)$, *where* $n = \operatorname{ord}_x f^{(0)}$, $m = \min(\operatorname{ord}_x f^{(2)}, n - 1)$

5.2. LEMMA. *Let* $\operatorname{cork} f = 1/4$.

a) If $f \sim x^n + x^{m_1}\xi_1\xi_2 + x^{m_2}\xi_3\xi_4 + \varphi$,
where $m_1, m_2 < n - 1$, *and*

$$\varphi = ax^l\xi_1\xi_2\xi_3\xi_4(1 + x\Psi(x)),$$

then $f \sim III_a$ *for* $l < m_1 - 1$ *and* $f \sim III_b$ *for* $l \geqq m_1 - 1$.

b) *If* $f \sim x^n + x^{m_1}\xi_1\xi_2 + \varphi$ *and* $\deg\varphi = 1$, *then* $f \sim III_c$ *for* $l \leqq m_1$ *and* $f \sim III_d$ *for* $l > m_1$.

c) *If* $f \sim x^n + \varphi$ *and* $\deg\varphi = l$, *then* $f \sim III_l\,(n, l)$ *for* $l < n - 1$, *and* $f \sim III_l\,(n, n-1)$ *for* $l \geqq n - 1$.

PROOF. Let

$$f = x^n + x^{m_1}\xi_1\xi_2 + x^{m_2}\xi_3\xi_4 + ax^l\xi_1\xi_2\xi_3\xi_4(1 + x\Psi(x))\ (a \neq 0).$$

Notice that $f \sim f + \mathfrak{k}(x)\xi_1\xi_2\xi_3\xi_4$, if $\operatorname{ord}\mathfrak{k}(x) \geqq m_1$, since f turns into $f + \mathfrak{k}(x)\xi_1\xi_2\xi_3\xi_4$ under the change $\xi_1 \to \xi_1\xi_3\xi_4\mathfrak{k}(x)/x^{m_1} + \xi_1$. Consecutively, if $l \geqq m_1$, then $f \sim \mathrm{III}_b$.

Let $l = m_1 - 1$. Then the composition of the changes $x \mapsto x - a\xi_3\xi_4/m_1$ and $\xi_3 \mapsto \xi_3(1 - nx^{n-1}a)^{-1}$ reduces f to the same form but with $a = 0$, implying $f \sim \mathrm{III}_b$.

Let $l < m_1 - 1$. Consider the change of coordinates

$$x \mapsto x + \theta(x)\xi_3\xi_4 \text{ and } \xi_3 \mapsto \xi_3(1 - n\theta(x)nx^k)^{-1},$$

where $f_1 = f + \varphi(x)\xi_1\xi_2\xi_3\xi_4$. Then f_1 turns into f for

$$\theta(x) = \varphi(x)/(m_1x^{m_1-1} - ax^l(1 + x\Psi(x)).$$

Notice that for $l + k \neq m_1 - 1$ or $a \neq m_1/n$ the power of the numerator does not exceed $\min(m_1 - 1, l + k)$. Therefore

$$f \sim f + \varphi(x)\xi_1\xi_2\xi_3\xi_4 \text{ if } \deg\varphi \geqq \min(m_1 - 1, l + k).$$

Let $l + k = m_1 - 1$ and $a = m_1/n$. Then $f \sim f + \varphi(x)\xi_1\xi_2\xi_3\xi_4$, if $\deg\varphi \geqq m_1$. This makes it clear that f reduces to the form III_a.

b) Let $f = x^n + x^{m_1}\xi_1\xi_2 + \Psi(x)\xi_1\xi_2\xi_3\xi_4$, where $\operatorname{ord}\Psi(x) = l$. The change of variables $\xi_3 \mapsto \xi_3\Psi(x)/x^l$ reduces f to the form $f = x^n + x^{m_1}\xi_1\xi_2 + x^l\xi_1\xi_2\xi_3\xi_4$. If $l > m_1$ then $f \sim x^n + x^m\xi_1\xi_2$ and therefore $f \sim \mathrm{III}_c(n, m_1)$. If $l \leqq m_1$, then $f \sim \mathrm{III}_d$.

c) Let $f = x^n + \Psi(x)\xi_1\xi_2\xi_3\xi_4$.
This case is similar to the preceding one.

5.2.1. Corollary. *Let* $\operatorname{cork} f = (1/4)$ *and* $f \in \mathfrak{m}^4$. *Then* f *is adjacent to* III'.

Proof. Let g be a singularity of the form III′ and $h\lambda = f + \lambda g$. For all λ/y except perhaps three values the germ h_λ falls into the case a) of Lemma 5.2 with $n = 4, m_1 = m_2 = 2$. Then by Lemma 5.2 we get $h_\lambda \sim$ III′ for almost all λ sufficiently close to 0.

5.2.2. Corollary. *Let* $f = x^n + x\xi_1\xi_2 + x^m\xi_3\xi_4 + \varphi$, *where* $m < n - 1$, $\varphi \in n^4$. *Then* $f \sim III_1(n, m)$.

5.2.3. Corollary. *Let* $f = x^n + x\xi_1\xi_2 + \varphi$, *where* $\varphi \in n^4$ *for* $\operatorname{ord}_x \varphi = 0$, *then* $f \sim III_1(m, n-1)$ *and if* $\operatorname{ord}_\varphi > 0$ *then* $f \sim III_2(n)$.

5.2.4. Corollary. *Let* $f = x^3 + \varphi$, *where* $\varphi \in n^4$. *Then* $f \sim III_5$, *if* $\deg \varphi \geqq 2$, $f \sim III_4$, *if* $\operatorname{ord} \varphi = 1$, $f \sim III_3$ *if* $\operatorname{ord} \varphi = 0$.

5.3. Lemma. *All the singularities listed in Table* 3 *are pairwise inequivalent.*

Proof. Since the ideals $\mathfrak{m}$ and $\mathfrak{n}$ are invariant with respect to automorphism of $\mathcal{O}$, then the singularities from different lines of the table and different n, m_1, m_2, l are pairwise inequivalent. Let us show that two singularities f and g of type III_a (n, m_1, m_2, l) with different sets of parameters $a, a_1, \ldots, a_k$, and $b, b_1, \ldots, b_k$ are inequivalent.

On the contrary, let $f \sim g$. Any change of variables sending f into g preserves $f^{(0)} + f^{(2)} = g^{(0)} + g^{(2)}$. Any such change is of the form $x \mapsto x + \tau$, where $\tau \in \mathfrak{n}^2$, $\xi_i \mapsto \xi_i + \sum a_{ij}(x)\xi_j$, where $\deg a_{ij}(x) \geqq n - m_2 - 1$.

Under this change $f^{(4)}$ turns into

$$(f^{(4)} + (m_2\xi_1\xi_2\tau x^{m_2-1} + m_1\xi_1\xi_2\tau x^{m_1-1})\cdot\xi_1\xi_2\xi_3\xi_4(1 + tr(a_{ij})) = f^{(4)} + \varphi,$$

where $\operatorname{ord}_x \varphi \geqq \min(n - m_1 - 1, m_2 - 1)$.

Therfore $a = b$, $a_i = b_i$ for $i < \min(n - m_1 - 2, m_2 - 2 - b)$.

It remains to consider the case $n - m_1 - m_2 = l$, $a = m_2/n$. To preserve $f^{(0)} + f^{(2)}$ we need that $a_{ij}(x)(\omega) = -\tau_n x^{n-1}$, where $\omega = x^{m_1}\xi_1\xi_2 + x^{m_2}\xi_3\xi_4$, and $a_{ij}(x)$ act on ω as the derivative $\sum a_{ij}(x)\,\xi_i(\partial/\partial\xi_j)$.

This implies that $\operatorname{tr} a_{ij} = -nx^{n-m_1-1} + 0(x^{n-m_1})$, and therefore $\operatorname{ord}_x \varphi \geqq m_2$ for $a = m_2/n, n - m_1 = m_2 - l$ as required.

5.4. Lemma. *Let* $\operatorname{cork} f = (1/5)$ *and* $f = x^4 + x^2\xi_1\xi_2 + x^2\xi_3\xi_4 + f^{(4)}$, *where* $f^{(4)} = a\xi_1\xi_2\xi_3\xi_4 \bmod (x)$. *Then* $f \sim IV'$ *for* $a \neq 2, 1/2$.

Proof. Consider the change

$$x \mapsto x + d\xi_1\xi_5 + \beta\xi_2\xi_5 + \gamma\xi_3\xi_5 + \delta\xi_4\xi_5,$$

$$\xi_1 \mapsto \xi_1 - 4\beta x\xi_5, \xi_2 \mapsto \xi_2 - 4\alpha x\xi_5, \xi_3 \mapsto \xi_3 - 4\delta x\xi_5, \xi_4 \mapsto \xi_4 - 4\gamma x\xi_5.$$

Then f turns into

$$g = f + 2x(2-a)(\alpha\xi_1\xi_5\xi_3\xi_4 + \beta\xi_5\xi_2\xi_3\xi_4 + \gamma\xi_1\xi_2\xi_3\xi_5 + \\ + \delta\xi_1\xi_2\xi_5\xi_4) + x^2\varphi, \text{ where } \varphi \in n^4.$$

If $a \neq 2$ we may select α, β, γ and δ so that

$$g = x^4 + x^2\xi_1\xi_2 + x^2\xi_3\xi_4 + (a + \mu x)\xi_1\xi_2\xi_3\xi_4 + x^2\varphi.$$

Now with the help of Morse's lemma we get rid of the term φ and the term $\mu x\xi_1\xi_2\xi_3\xi_4$ for $a \neq 1/2$ may be killed thanks to Lemma 5.2.

5.4.1. Corollary. *Let* cork $f = 1/5$ *and* $f = x^n + ax^{m_1}\xi_1\xi_2 + bx^{m_2}\xi_3\xi_4 + f^{(4)}$, *where* $n > 3, m_1, m_2 > 2$. *Then* f *is adjacent to* IV'.

5.4.2. Corollary. *Let* cork $f = 1/5$ *and* $f = x^n + x^2\xi_1\xi_2 + bx^m\xi_3\xi_4 + f^{(4)}$, *where* $n > 3$, $m > 2$, $f^{(4)} \in (\xi_1\xi_2)n^2 + (x)$. *Then* f *is adjacent to* IV'.

5.4.3. Corollary. *Let* cork $f = 1/5$ *and*

$$f = x^n + x^2\xi_1\xi_2 + x^2\xi_3\xi_4 + a\xi_1\xi_2\xi_3\xi_4 + \varphi,$$

where $n > 3$, $\varphi \in n^4$. *Then* f *is adjacent to* IV'.

5.5. Lemma. *Let* cork $f = 1/5$, $f = x^n + x^2\xi_1\xi_2 + x^m\xi_3\xi_4 + f^{(4)}$, *where* $n > 3$, $m \leqq n-1$, $f^{(4)} \notin (x) + \xi_1\xi_2 n^2$.

Then $f \sim IV_4(n, m)$. *If* $m = 2$ *and* $f^{(4)} \notin (\xi_1\xi_2\xi_3\xi_4) + (x)$, *then* $f \sim IV_4(n, 2)$, $f \sim IV_4(n, m)$

Proof. Making a linear change of variables ξ_1, ξ_2 and multiplying ξ_5 by a constant we may reduce $f^{(4)}$ to the form $f^{(4)} = \xi_1\xi_2\xi_3\xi_4 + \varphi$, where $\varphi \in n^4(x)$. Let us show that $f \sim f - \varphi$. Let for $\gamma \in n^4$

$$\varphi = ax\xi_1\xi_2\xi_3\xi_4 + bx\xi_1\xi_2\xi_3\xi_5 + cx\xi_1\xi_2\xi_4\xi_5 + dx\xi_1\xi_3\xi_4\xi_5 + \\ + ex\xi_2\xi_3\xi_4\xi_5 + \gamma.$$

With the change $\xi_5 \mapsto \xi_5 - a\xi_2$ equate a to zero and with the change $x \mapsto x - (b\xi_3\xi_5 + c\xi_4\xi_5)/2$, $\xi_3 \mapsto \xi_3 + \dfrac{1}{an}\xi_5 x^{n-1-m}$, $\xi_4 \mapsto \xi_4 + \dfrac{1}{an}\xi_5 x^{n-1-m}$ equate b to c. After that with the change $\xi_2 \to \xi_2 - \dfrac{d}{e}\xi_1$ let us equate d to 0. Finally, with the change $\xi_1 \to \xi_1 - x\xi_2$ we equate e to 0, i.e. reduce $f = \varphi$ to the form $f - \mathfrak{k}$, with $\mathfrak{k} \in (x^2)n^4$. Further with Morse's lemma we get $f \sim f - \mathfrak{k}$.

The case $m = 2$ is similar.

5.6. Lemma. *Let* cork $f = 1/5$ *and* $f = x^n + x\xi_1\xi_2 + x^m\xi_3\xi_4 + f^{(4)}$, *where* $m < n - 1$.

a) *If* $f^{(4)} \in (\xi_1\xi_2) + (x)$ *or* $m = 1$ *then* $f \sim IV_2\,(n, m)$
b) *If* f *fails to satisfy* a) *then* $f \sim IV_1\,(n, m)$.

PROOF. a) If $m \neq 1$ then the change $x \mapsto x - v$, where v is found from the condition $\xi_1\xi_2 v \equiv f^{(4)} \bmod (x)$, reduces f to the form $f = x^n + x\xi_1\xi_2 + x^m\xi_3\xi_4 + \varphi$, with $\varphi \in \mathfrak{n}^2(x)^2 + \mathfrak{n}^4(x)$ and if $m = 1$ then $f^{(4)} = v_1 \cdot \xi_1\xi_2 + v_2\xi_4 + \varphi_1$, with $\varphi_1 \in \mathfrak{n}^4(x)$. The change $x \mapsto x - v_1 v_2$ also reduces f to the same form.

Let $m > 1$ and $f = x^n + x\xi_1\xi_2 + x^m\xi_3\xi_4 + \varphi_1$. Then $\varphi = \xi_1\Psi_1 + \xi_2\Psi_2$ and the change $\xi_1 \mapsto \xi_1 + \Psi_2$, $\xi_2 \mapsto \xi_2 + \Psi_1$ establishes the equivalence of f and $f - \varphi$. This yields $f \sim \mathrm{IV}_2(n, m)$. The case $m = 1$ follows from Morse's lemma.

b) If f fails to satisfy the condition a) then $f^{(4)} = a\xi_1\xi_3\xi_4\xi_5 + b\xi_2\xi_3\xi_4\xi_5 + \xi_1\xi_2 V + w$, where $v \in \Lambda^2(\xi)$, $w \in \mathfrak{n}^4(x)$. The linear change $\xi_1 \to a\xi_1 - b\xi_2$, $\xi_2 \to a^{-1}\xi_1$ reduces the germ of f to the form where $b = 0$, $a = 1$. Applying the same changes of coordinates as in the case a) we can reduce f to the form $\mathrm{IV}_4(n, m)$.

5.7. LEMMA. *Let* cork $f = 1/5$ *and* $f = x^n + x\xi_1\xi_2 + f^{(4)}$
a) *If* $f^{(4)} \in (x)$ *then* $f \sim IV_3(n)$
b) *If* $f^{(4)} \notin (x)$ *but* $f^{(4)} \in (\xi_1\xi_2) + (x)$, *then* $f \sim IV_2(n, n - 1)$.
c) *If* $f^{(4)} \notin (x) + (\xi_1\xi_2)$, *then* $f \sim IV_1(n, n - 1)$.

PROOF. a) follows from Morse's lemma and b) and c) are proved as Lemma 5.6.

5.8. LEMMA. *Let* cork $f = 1/5$ *and* $f = x^3 + f^{(4)}$.
a) *If* $\mathrm{ord}_x f^{(4)} \geqq 2$, *then* $f \sim IV_6$
b) *If* $\mathrm{ord}_x f^{(4)} = 1$, *then* $f \sim IV_5$
c) *If* $\mathrm{ord}_x f^{(4)} = 0$, *then* $f \sim IV_4(3, 2)$

PROOF. a) The change $x \mapsto x - f^{(4)}/3x^2$ turns f into x^3.

b) Let $\mathrm{ord}_x f^{(4)} = 1$. A linear change of variables $\xi_1, \ldots, \xi_5$ reduces f to the form $f = x^3 + x\xi_1\xi_2\xi_3\xi_4 + \varphi$, where $\varphi \in (x^2)\mathfrak{n}^4$. Further apply the change of variables $x \mapsto x - \varphi/3x^2$

c) Let $\mathrm{ord}_x f^{(4)} = 0$. A linear change in $\langle \xi_1, \ldots, \xi_5 \rangle$ reduces f to the form $f = x^3 + \xi_1\xi_3\xi_4\xi_5 + \varphi$ with $\varphi \in (x)\mathfrak{n}^4$; then kill φ by a change of the form $\xi_i \mapsto \sum a_{ij}(x)\xi_j$.

It is not difficult to verify that under the change

$$x \mapsto x + \xi_1\xi_2/3 + \xi_3\xi_4/3,\ \xi_5 \mapsto \xi_5 - \xi_2/3$$

$x^3 + \xi_1\xi_3\xi_4\xi_5$ turns into $\mathrm{IV}_4\,(3, 2)$.

5.9. LEMMA. *Let* cork $f = 1/6$ *and*

$$f = x^n + x\xi_1\xi_2 + x\xi_3\xi_4 + x\xi_5\xi_6 + f^{(4)} + f^{(6)}.$$

Then $f \sim V(n)$.

PROOF. Let $f^{(4)} + f^{(6)} = v + w + tx$, where

$$v \in \Lambda^4(\xi_1, \ldots, \xi_6),\ w \in \Lambda^6(\xi_1, \ldots, \xi_6).$$

It is easy to see that

$$v = (\xi_1\xi_2 + \xi_3\xi_4 + \xi_5\xi_6)v_0,\quad w = (\xi_1\xi_2 + \xi_3\xi_4 + \xi_5\xi_6)w_0.$$

The change $x \mapsto x + v_0 + w_0$ reduces f to the form

$$f = x^n + x(\xi_1\xi_2 + \xi_3\xi_4 + \xi_5\xi_6) + \Psi$$

for some $\Psi \in (x^2)\mathfrak{n}^2 + (x)\mathfrak{n}^4$, which thanks to Morse's lemma yields $f \sim V(n)$.

5.10. LEMMA. *Let* cork $f = 1/6$ *and*

$$f = x^n + ax^{m_1} + bx^{m_3} + f^{(4)} + f^{(6)},$$

where $m_i > 1$ *for at least one* i. *Then* f *is adjacent to* V'.

PROOF. Let $m_1 < m_2 < m_3$,

$$g = x^3 + x\xi_1\xi_2 + x\xi_3\xi_4 + x^2\xi_5\xi_6 + a\xi_5\xi_6(\xi_1\xi_2 - \xi_3\xi_4),$$

and $h_\lambda = f + \lambda g$. Then

$$h_\lambda \sim x^3 + x\xi_1\xi_2 + x\xi_3\xi_4 + x^2\xi_5\xi_6 + v_0 + \varphi,$$

where $\varphi \in (x)\mathfrak{n}^4 + \mathfrak{n}^6$. Express v_0 in the form

$$v_0 = \xi_5\xi_6 w_1 + \xi_5 w_2 + \xi_6 w_3 + \xi_1\xi_2\xi_3\xi_4.$$

The changes of the form $x \mapsto x + \alpha$ reduce h_λ to the form with $v_0 = \xi_5\xi_6 w_1$ and $w_1(\xi_1\xi_2 + \xi_3\xi_4) = 0$. Notice that $w_1^2 \neq 0$ for almost all λ and therefore the linear changes in $\xi_1\xi_2\xi_3\xi_4$ send w_1 into $\mu(\xi_1\xi_2 - \xi_3\xi_4)$. Thus,

$$h_\lambda = x^3 + (x\xi_1\xi_2 + \xi_3\xi_4) + x^2\xi_5\xi_6 + \alpha\xi_5\xi_6(\xi_1\xi_2 - \xi_3\xi_4) + \varphi,$$

where $\varphi \in (x)\mathfrak{n}^4 + \mathfrak{n}^6$. Killing φ as usual (Morse's lemma) we get $h_\lambda \sim V'$.

5.11. LEMMA. *Let* $V = \langle \xi_1, \ldots, \xi_6 \rangle$.

Then in $\Lambda^3(V)$ *there is exactly five orbits of* $\mathrm{GL}(V)$ *and any element* $w \in \Lambda^3 V$ *reduces to one of the following:*

a) 0,

b) $\xi_1\xi_5\xi_3$,

c) $\xi_1(\xi_3\xi_5 + \xi_4\xi_6)$,

d) $\xi_1\xi_3\xi_5 + \xi_2\xi_4\xi_6$,
e) $\xi_1\xi_3\xi_5 + \xi_1\xi_4\xi_6 + \xi_2\xi_4\xi_5$.

5.12. Lemma. *Let $\omega \in \Lambda^2(V)$, with* $\operatorname{rk}\omega = 6$ *and $w \in \Lambda^3(V)$ such that $\omega \wedge w = 0$. Then by linear transformations of V we may reduce w to one of the forms* a) – e) *and ω to the form: $\xi_1\xi_2 + \xi_3\xi_4 + \xi_5\xi_6$.*

Proof. Consider all the cases. For a 3-dimensional subspace $L \subset V$, $L = \langle \xi_i, \xi_j, \xi_k \rangle$ set $\Delta_L = \xi_i\xi_j\xi_k$ and $L^\perp = \{f \in V^*: f|_L = 0\}$ and for $f \in V^*$ denote by f^c an element of $\Lambda^5 V$ such that if $f(v) = 0$ then $f^c \wedge v = 0$ (f^c is defined up to a factor). It is not difficult to see that $\omega\Delta_L = 0$ if $L^\perp$ is Lagrangean with respect to ω and $\omega\Delta_L \in \operatorname{Ker}\omega \,|\, L^\perp$ and nonzero otherwise. Let $\mathfrak{k}_1, \ldots, \mathfrak{k}_6$ be the dual basis of $\xi_1, \ldots, \xi_6$.

a) Trivially.

b) $L^\perp = \langle \xi_1, \xi_3, \xi_5 \rangle^\perp$ is Lagrangean with respect to ω and therefore ω reduces to the indicated form.

c) Let $L_1 = \langle \xi_1, \xi_3, \xi_5 \rangle$, $L_2 = \langle \xi_1, \xi_4, \xi_6 \rangle$.

Since $(\Delta_{L_1} + \Delta_{L_2})\omega = 0$, then $\operatorname{Ker}\omega|_{L_1^\perp} = \operatorname{Ker}\omega|_{L_2^\perp}$ or, in other words, $L_1^\perp$ and $L_2^\perp$ are Langrangean. In any case $\mathfrak{k}_2 \in \operatorname{Ker}\omega|_{L_1^\perp + L_2^\perp}$. This makes it clear that $\mathfrak{k}_2 \perp \langle \mathfrak{k}_3, \mathfrak{k}_4, \mathfrak{k}_5, \mathfrak{k}_6 \rangle$ and therefore ω is expressed in the form $a\xi_1\xi_2 + \omega'$, where $\omega' \in \Lambda^2(\xi_3, \xi_4, \xi_5, \xi_6)$, $\omega'(\xi_3\xi_5 + \xi_4\xi_6) = 0$. But the changes in $\langle \xi_3, \xi_4, \xi_5, \xi_6 \rangle$ preserving $\xi_3\xi_5 + \xi_4\xi_6$ send ω' to the form $\lambda(\xi_3\xi_4 + \xi_5\xi_6)$. After that the change

$$\xi_3 \to \xi_3/\lambda,\ \xi_6 \to \xi_6/\lambda,\ \xi_1 \to \lambda\xi_1,\ \xi_2 \to \xi_2/a\lambda$$

reduces ω to the form: $\omega = \xi_1\xi_2 + \xi_3\xi_4 + \xi_5\xi_6$.

d) Let $L_1 = \langle \xi_1, \xi_3, \xi_5 \rangle$, $L_2 = \langle \xi_2, \xi_4, \xi_6 \rangle$. Since $L_1 \bigcap L_2 = 0$ and $(\Delta_{L_1} + \Delta_{L_2}) \cdot \omega = 0$, then $L_1^\perp$ and $L_2^\perp$ are Lagrangean with respect to ω. And therefore ω reduces to the form $\xi_1\xi_2 + \xi_3\xi_4 + \xi_5\xi_6$ by the changes of variables in $\langle \xi_2, \xi_4, \xi_6 \rangle$.

e) Let $v = \xi_1\xi_3\xi_5 + \xi_1\xi_4\xi_6 + \xi_2\xi_4\xi_5$, $L_1 = \langle \xi_1, \xi_3, \xi_5 \rangle$, $L_2 = \langle \xi_1, \xi_4, \xi_6 \rangle$, $L_3 = \langle \xi_2, \xi_4, \xi_5 \rangle$, Then $\dim L_i \bigcap L_j = 1$ if $i \neq j$ and $L_1 \bigcap L_2 \bigcap L_3 = 0$. Since $(\Delta_{L_1} + \Delta_{L_2} + \Delta_{L_3})\omega = 0$ then at least one $L_i^\perp$ is Langrean with respect to ω. Without loss of generality we may assume $i = 1$. Then since $(\Delta_{L_2} + \Delta_{L_3})\omega = 0$ the by c) $L_2^\perp$ and $L_3^\perp$ are also Lagrangean. But then $\langle \xi_1, \xi_4, \xi_5 \rangle^\perp$ is Lagrangean and ω is of the form

$$\omega = a\xi_1\xi_2 + b\xi_3\xi_4 + c\xi_5\xi_6.$$

The change of variables

$$\xi_1 \mapsto \sqrt{bc}\,\xi_1,\ \ \xi_2 \mapsto \xi_2/a\sqrt{bc}$$

$$\xi_4 \mapsto \sqrt{ac}\,\xi_4,\ \ \xi_3 \mapsto \xi_3/b\sqrt{ac}$$

$$\xi_5 \mapsto \sqrt{ab}\,\xi_5,\ \ \xi_6 \mapsto \xi_4/c\sqrt{ab}$$

reduces ω to the form: $\omega = \xi_1\xi_2 + \xi_3\xi_4 + \xi_5\xi_6$.

5.13. LEMMA. *Let* cork $f = 1/7$ *and*

$$f = x^n + x\xi_1\xi_2 + x\xi_3\xi_4 + x\xi_5\xi_6 + f^{(4)} + f^{(6)}.$$

Then

$$f \sim x^n + x\xi_1\xi_2 + x\xi_3\xi_4 + x\xi_5\xi_6 + \xi_7 w,$$

where

$$w \in \Lambda^3(\xi_1, \dots, \xi_6), \quad w(\xi_1\xi_2 + \xi_3\xi_4 + \xi_5\xi_6) = 0.$$

PROOF. Let $f^{(4)} = \xi_7 w_1 + w_2 \bmod (x)\mathfrak{n}^4$. The change of variables of the form $x \mapsto x + \xi_7 v_1 + v_2 + v_3 + v_4$, where $v_i \in \Lambda^i(\xi_1, \dots, \xi_6)$, reduces f to the form

$$f = x^n + x(\xi_1\xi_2 + \xi_3\xi_4 + \xi_5\xi_6) + \xi_7 w + \varphi, \text{ where } \varphi \in (x^2)\mathfrak{n}^2 + (x)\mathfrak{n}^4,$$
$$w \in \Lambda^3(\xi_1, \dots, \xi_6), w(\xi_1\xi_2 + \xi_3\xi_4 + \xi_5\xi_6) = 0.$$

Morse's lemma implies that $f \sim f - \varphi$.

5.14. LEMMA. *If* cork $f = 1/7$ *and*

$$f = x^n + x(\xi_1\xi_2 + \xi_3\xi_4 + \xi_5\xi_6) + f^{(4)} + f^{(6)}$$

then f is equivalent to one of the singularities $VI_1(n) - VI_5(n)$.

PROOF follows from Lemmas 5.11–5.13.

5.15. LEMMA. *Let* cork $f = 1/7$ *and*

$$f = x^n + x\xi_1\xi_2 + x^{m_1}\xi_3\xi_4 + x^{m_2}\xi_5\xi_6 + f^{(4)} + f^{(6)}, \; m_1, m_2 > 1.$$

Then f is adjacent to VI'_2.

PROOF. Set $f + \lambda \mathrm{VI}'_2 = g_\lambda$. Then $g_\lambda \sim x^3 + x\xi_1\xi_2 + f^{(4)} + f^{(6)}$ for a sufficiently small λ. Let $q = f^{(4)} \bmod \mathfrak{n}^4(x) + \mathfrak{n}^6$. Present q in the form

$$q = v_0 + \xi_1 v_1 + \xi_2 v_2 + \xi_1\xi_2 v_3,$$

where $v_0 \in \Lambda^4(\xi_7, \xi_6, \xi_5, \xi_4, \xi_3)$, v_1, $v_2 \in \Lambda^3(\xi_7, \xi_6, \xi_5, \xi_4, \xi_3)$ and $v_3 \in \Lambda^2(\xi_7, \xi_6, \xi_5, \xi_4, \xi_3)$. For a sufficiently small λ the vector q is generic in $\Lambda^4(\xi_1, \dots, \xi_7)$. Without loss of generality we may set $v_0 = \xi_7\xi_6\xi_5\xi_4$.

The changes of the form $\xi_i \mapsto \xi_i + a\xi_1 + b\xi_2, i \geqq 4$, enable us to reduce q to the form

$$q = \xi_7\xi_6\xi_5\xi_4 + \xi_1\xi_3 w_1 + \xi_2\xi_3 w_2 + \xi_1\xi_2 w_3,$$

without affecting $g_\lambda^{(2)}$. The changes

$$\xi_1 \mapsto a\xi_1 + b\xi_2, \xi_2 \mapsto c\xi_1 - a\xi_1, \xi_3 \mapsto \xi_3 + d_1\xi_1 + d_2\xi_2$$

reduce q to the form

$$q = \xi_7, \xi_6, \xi_5, \xi_4 + \xi_1, \xi_3, \xi_7, \xi_6 + \xi_2, \xi_3, \xi_5, \xi_4 + \xi_1\xi_2\xi_3 z_0 +$$
$$+ \xi_1\xi_2(\gamma_1\xi_5\xi_6 + \gamma_2\xi_5\xi_7 + \gamma_3\xi_4\xi_6 + \gamma_4\xi_4\xi_7) +$$
$$+ \xi_1\xi_2\xi_3(\delta_1\xi_7 + \delta_2\xi_6 + \delta_3\xi_5 + \delta_4\xi_4).$$

The composition of the changes from the group $\mathrm{SL}(\xi_5, \xi_4) \times \mathrm{SL}(\xi_7, \xi_6)$ with $\xi_3 \mapsto \lambda\xi_3$ reduces q to the form

$$\xi_7\xi_6\xi_5\xi_4 + \xi_3\xi_1\xi_7\xi_6 + \xi_3\xi_2\xi_5\xi_4 + \xi_1\xi_2\xi_7\xi_5 + \xi_1\xi_2\xi_4\xi_6 + a\xi_1\xi_2\xi_3\xi_4.$$

Therefore, $g_\lambda \sim \mathrm{VI}'_2 + \varphi$, where $\varphi \in n^4(x) + n^6$. Let us show that $g_\lambda \sim g\mathrm{VI}'_2$. First, by Morse's lemma

$$g_\lambda \sim x^3 + \xi_7\xi_6\xi_5\xi_4 + \xi_1\xi_3\xi_7\xi_6 + \xi_2\xi_3\xi_5\xi_4 +$$
$$+ a\xi_1\xi_2\xi_3\xi_4 + \xi_1\xi_2\xi_7\xi_5 + \xi_1\xi_2\xi_4\xi_6 + x \cdot \Psi(x, \xi_3 \cdot \xi_4 \cdot \xi_5 \cdot \xi_6, \xi_7).$$

Since $\xi_7\xi_6\xi_5\xi_4$ belongs to the open orbit of GL(5) in $\Lambda^4(\xi_7, \ldots, \xi_3)$ then there exists $g \in \mathrm{gl}(5) \cdot x$ such that

$$g(\xi_7\xi_6\xi_5\xi_4) = x \cdot \Psi(0, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7).$$

Therefore $g_\lambda \sim \mathrm{VI}'_2 \cdot \Psi(\xi_3, \ldots, \xi_5, x) \cdot x^2$, but then $g_\lambda \sim \mathrm{VI}'_2$.

5.16. LEMMA. *Let* cork $f = 1/7$ *and*

$$f = x^n + x\xi_1\xi_2 + x\xi_3\xi_4 + x^2\xi_5\xi_6 + f^{(4)} + f^{(6)},$$

where $n > 3$, $t = f^{(4)} \bmod (\xi_1\xi_2 + \xi_3\xi_4) + (x) + (\Lambda^4(\xi_1, \ldots, \xi_6))$.

a) *If* $\mathrm{rk}\, t = 3$, *then* $f \sim \mathrm{VI}_6(n)$,

b) *If* $\mathrm{rk}\, t < 3$, *then* f *is adjacent to* IV'.

PROOF. Let us express t in the form

$$t = \xi_7\xi_5\xi_6 t_0 + \xi_7\xi_5 w_1 + \xi_7\xi_6 w_2,$$

where $t_0\langle\xi_1, \ldots, \xi_4\rangle$, $w_1, w_2 \in \Lambda^2(\xi_1, \ldots, \xi_4)$
and $w_2(\xi_1\xi_2 + \xi_3\xi_4) = w_2(\xi_1\xi_2 + \xi_3\xi_4) = 0$.

Let $t_0 \neq 0$. Then by the changes of variables in $\langle\xi_1, \ldots \xi_4\rangle$ we may reduce t_0 to ξ_1 without affecting $f^{(2)}$.

The change $\xi_5 \mapsto \xi_5 + \sum_{1 \leq i \leq 4} a_i\xi_i$, $\xi_6 \mapsto \xi_6 + \sum_{1 \leq i \leq 4} b_i\xi_i$, reduces t to the form

$$t = \xi_7\xi_5\xi_6\xi_1 + \xi_7\xi_5(a\xi_2\xi_3 + b\xi_2\xi_4) + \xi_7\xi_6(c\xi_2\xi_3 + d\xi_2\xi_4).$$

Then the composition of the changes from $\mathrm{SL}(\xi_5, \xi_6)$ and the transformation

$$\xi_7 \mapsto \lambda\xi_7, \ \xi_1 \mapsto \lambda^{-1}\xi_1, \ \xi_2 \mapsto \lambda\xi_2$$

reduces t to the form $t = \xi_7\xi_5\xi_6\xi_1 + \xi_7\xi_5\xi_2\xi_3 + \xi_7\xi_6\xi_2\xi_4$.

Let $f^{(4)} = t + t_1 \bmod (x)$. Since $t_1 \in (\xi_1\xi_2 + \xi_3\xi_4) + \Lambda^4(\xi_1, \xi_2, \dots, \xi_6)$, then t_1 splits into the sum $t_1 = (\xi_1\xi_2 + \xi_3\xi_4)v_0 + \xi_5\xi_6 v_1$, where $v_1 \in \Lambda^2(\xi_1, \dots, \xi_4)$, and $v_1 \cdot \omega = 0$.

The first summand may be put to zero with the change $x \mapsto x = v_0$, and the second one with the change $\xi_7 \mapsto \xi_7 + \sum_{i\leqq 4} a_i\xi_i$, $\xi_5 \mapsto \xi_5 + \sum_{i\leqq 4} b_i\xi_i$, $\xi_6 \mapsto \xi_6 + \sum_{i\leqq 4} c_i\xi_i$.

Now making use of Morse's lemma let us kill all terms that belong to $\mathfrak{n}^2(x^2) + \mathfrak{n}^4(x) + \mathfrak{n}^6$, and get $f \sim \mathrm{VI}_6(n)$.

Let $\operatorname{rk} t < 3$. Then t reduces to the form $t = \xi_7\xi_5\xi_6\xi_1 + k\xi_7\xi_5\xi_2\xi_3$. Consider the singularity $h_\lambda = f + \lambda\xi_1\xi_3$. For $\lambda \neq 0$ it satisfies the conditions of Corollary 5.2.4 and therefore it is adjacent to IV′.

Let $t_0 = 0$. Then the composition of the changes from $\mathrm{SL}_2(\xi_5, \xi_6)$ with $\xi_7 \mapsto \lambda\xi_7$, reduces to the form $t = \xi_7\xi_5\xi_1\xi_3 + \xi_7\xi_6\xi_2\xi_4$, therefore $\operatorname{rk} t = 0$. Let $h_\lambda = f + \lambda\xi_1\xi_4$. Then, by the reasons similar to the preceding case h_λ is adjacent to IV′.

5.16.1. COROLLARY. *Let*

$$f = x^n + x\xi_1\xi_2 + x\xi_3\xi_4 + x^m\xi_5\xi_6 + f^{(4)} + f^{(6)}.$$

If $m > 2$, *then* f *is adjacent to* IV'.

PROOF. Let t be the same as in Lemma. It suffices to prove Corollary for the case $\operatorname{rk} t = 3$. As had been proved in the proof of Lemma in this case

$$f \sim x^n + x\xi_1\xi_2 + x\xi_3\xi_4 + x^m\xi_5\xi_6 + \xi_7\xi_5\xi_6\xi_1 + \xi_7\xi_5\xi_2\xi_3 + \xi_7\xi_6\xi_2\xi_4.$$

Let $g_\lambda = f + \lambda\xi_1\xi_3$, where $\lambda \neq 0$. Then

$$g_\lambda \sim x^n + x^2\xi_2\xi_4 + x^m\xi_5\xi_6 + \xi_7\xi_6 \cdot \xi_2\xi_4.$$

Since $m > 2$, then g_λ falls into the conditions of Corollary 5.2.2 and therefore is adjacent to IV′.

5.17. LEMMA. *Let* $\operatorname{cork} f = 1/7$ *and*

$$f = x^3 + x\xi_1\xi_2 + x\xi_3\xi_4 + f^{(4)} + f^{(6)}$$

with $f^{(4)} = t_0 \bmod (\Lambda^2(\xi_1, \dots, \xi_4)) + (x)$.

a) *If* $t_0 = 0$, *then* f *is adjacent to* VI'.

b) *If* $t_0 \neq 0$, *then*

$$f \sim x^3 + x\xi_1\xi_2 + x\xi_3\xi_4 + \xi_7\xi_6\xi_5\xi_1 +$$
$$+ a\xi_7\xi_6\xi_2\xi_3 + b\xi_7\xi_5\xi_2\xi_4 + c\xi_6\xi_5\xi_3\xi_4.$$

PROOF. Let

$$f = x^3 + x\xi_1\xi_2 + x\xi_3\xi_4 + f^{(4)} + f^{(6)}.$$

Similarly to the case of Lemma 4.4.16, $f \sim f + \varphi$, if $\varphi \in (\Lambda^3(\xi_1, \dots, \xi_4)) + (x)$. Therefore without affecting $f^{(0)}, f^{(2)}$, we may reduce f to the form with

$$f^{(4)} + f^{(6)} \in L = \langle \xi_1, \dots, \xi_4 \rangle \otimes \Lambda^3(\xi_7, \xi_5, \xi_6) \oplus \Lambda^2(\xi_1, \dots, \xi_4) \otimes \Lambda^2(\xi_7, \xi_5, \xi_6).$$

On the other hand, the two reduced singularities f_1 and f_2 are equivalent only if so are $f_2^{(4)}$ and $f_1^{(4)}$ with respect to the subgroup G of linear transformations in $\langle \xi_1, \dots, \xi_7 \rangle$ preversing $\xi_1\xi_2 + \xi_3\xi_4$. Let

$$f^{(4)} = \xi_7\xi_6\xi_5 v_0 + \xi_7\xi_6 w_1 + \xi_7\xi_5 w + \xi_6\xi_5 w_3$$

where $w_i \in \Lambda^2(\xi_1, \dots, \xi_4)$, $v_0 \in \langle \xi_1, \dots, \xi_4 \rangle$.

a) Let $v_0 = 0$. Then a generic element $f^{(4)}$ reduces with the help of transformations from G to the form

$$\xi^{(4)} = \xi_7\xi_5\xi_2\xi_3 + \xi_7\xi_6\xi_1\xi_4 + \xi_7\xi_6(\xi_1\xi_2/a + a\xi_3\xi_4).$$

implying that f is adjacent to VI'_7.

b) Let $v_0 \neq 0$. Then v_0 reduces to ξ_1 with the help of transformations from $\mathrm{SP}(\xi_1, \dots, \xi_4)$ and w_1, w_2, w_3 could be forced to belong to $\Lambda^2(\xi_2, \xi_3, \xi_4)$ with the help of transitions $\xi_i \mapsto \xi_i + \sum_{1 \leqq j \leqq 4} a_{ij}\xi_j$ for $i = 5, 6, 7$.

Therefore

$$f^{(4)} \sim \xi_7\xi_6\xi_5\xi_1 + \xi_7\xi_6(a_1\xi_2\xi_3 + b_1\xi_2\xi_4 + c_1\xi_3\xi_4) +$$
$$+ \xi_7\xi_5(a_2\xi_2\xi_3 + b_2\xi_2\xi_4 + c_2\xi_3\xi_4) + \xi_6\xi_5(a_3\xi_2\xi_3 + b_3\xi_2\xi_4 + c_3\xi_3\xi_4).$$

The linear changes from $\mathrm{GL}(\xi_5, \xi_6, \xi_7)$ enable us to kill b_1, c_1, a_2, c_2; a_3 and b_3.

5.18. LEMMA. *If f corresponds to case 23 of Classifier and $a, b, c \neq 0$, then $f \sim VI_4(3)$. If $a \neq 0$ and either b or c are nonzero then $f \sim VI_7$. If $a = 0$, b, $c \neq 0$, then $f \sim VI_8$. If $a \neq 0$, $b = c = 0$, then $f \sim VI_9$. If $a = 0$, and either b or c are nonzero, then $f \sim VI_{10}$. If $a = b = c = 0$, then $f \sim VI_{11}$.*

PROOF is straightforward.

5.19. LEMMA. *Let* cork $f = (2/2\,k)$. *Then f is adjacent to VII'_3.*

PROOF. Consider

$$g_\lambda = \lambda(x^3 + xy^2 + x\xi_1\xi_2 + ay\xi_1\xi_2 + \xi_3\xi_4 + \dots + \xi_{2k-1}\xi_{2k}) + f.$$

Then g_λ is a singularity of corank(2, 2). The change of coordinates (x, y) reduces g_λ for a sufficiently small λ to the form

$$g_\lambda = x^3 + xy^2 + (bx + dy)\xi_1\xi_2 + \varphi(x, y)\xi_1\xi_2,$$

where $\varphi(x, y) \in m^2$ (see [A]) Let $\varphi(x, y) = (bx + dy)\Psi(x, y) + xy\mathfrak{f}(x, y)$. The change

$$\xi_\lambda \mapsto \xi_2/b, \ \xi_1 \mapsto \xi_1(1 + \Psi(x, y)), \ y \mapsto y + \mathfrak{f}(x, y)\xi_1\xi_2,$$

reduces g_λ to the form VII'_3.

5.20. LEMMA. *Let* cork $f = (2/2k + 1)$. *If* $k > 1$, *then* f *is adjacent to* VII'_2.

PROOF. It suffices to prove the case $k = 2$. Let

$$f^{(2)} = \varphi x + \Psi y \bmod m^4, \text{ where } \varphi, \Psi \in \Lambda^2(\xi_1, \ldots, \xi_5).$$

Reduce φ to the form $\varphi = \xi_1\xi_3 + a\xi_2\xi_4$ by linear changes in $\langle \xi_1, \ldots, \xi_5 \rangle$. Let

$$g_\lambda = \lambda(x^3 + xy^2 + ay\xi_1\xi_2 + \xi_3\xi_4) + f.$$

Then

$$g_\lambda \sim x^3 + xy^2 + x\xi_1\xi_2 + \varphi, \text{ where } \varphi \in \mathfrak{n}^2\mathfrak{m}^2.$$

The term φ is killed in the same way as in Lemma 5.19. Adding $\mu y\xi_1\xi_2$ to g_λ we see that g_λ is adjacent to VII'_3.

5.21. LEMMA. *Let* cork $f = (2/3)$, *and* $t = f^{(2)} \bmod \mathfrak{m}^4$.

a) *If* rk $t = 1$, *then* f *is adjacent to* VII'_3.

b) *If* rk $t = 2$, *then* $f^{(0)}$ *is nonsimple, and* f *is adjacent to either* VII'_1 *or* VII'_2.

c) *If* rk $t = 2$ *and* $f^{(0)}$ *is simple then* f *is equivalent to one of* VII_1 – VII_4 *depending on the equivalence class of* $f^{(0)}$.

PROOF. If rk $t = 1$, then t reduces by a linear change of coordinates ξ_1, ξ_2, ξ_3 to the form $t = (ax + by)\xi_1\xi_2$. By the reasons similar to those from Lemma 5.20, f is adjacent to VII'_2.

If rk $t = 2$, then t reduces to the form $t = x\xi_1\xi_2 + y\xi_2\xi_3$ by a linear change of coordinates ξ_1, ξ_2, ξ_3. Let $f = f^{(0)} + x\xi_1\xi_2 + y\xi_2\xi_3 + \varphi$, where $\varphi \in \mathfrak{n}^2\mathfrak{m}^2$. Morse's lemma implies that we may kill φ without affecting $f^{(0)}$. Therefore adjacency and equivalence for singularities of the form f are equivalent to adjacency and equivalence for the singularities of the form $f^{(0)}$. Now the statements b) and c) follow from the classificational results for the germs in even variables (see [A]).

5.22. LEMMA. *Let* cork $f = 0/q$, $q \leqq 7$. *Then*

a) *if* $f^{(4)} = 0$ *then* $f = f^{(6)} \sim I_4$ *if* $f^{(6)} \neq 0$ (*or* I_{11} *if* $f^{(6)} = 0$);

b) *if* $f^{(4)} \neq 0$ *then* $f \sim f^{(4)}$.

PROOF. a) If $q = 6$ the statement is obvious. If $q = 7$ then in $\Lambda^6 \langle \xi_1, \dots, \varepsilon_7 \rangle$ there is only one nonzero GL(7)-orbit: that of $\xi_1 \xi_2 \xi_3 \xi_4 \xi_5 \xi_6$.

b) If $f^{(6)} = 0$ we have nothing to prove. Let $f^{(6)} \neq 0$. Then reduce $f^{(6)}$ to the form $I_4 = \xi_1 \xi_2 \xi_3 \xi_4 \xi_5 \xi_6$ and represent $f^{(4)}$ in the form $f^{(1)} = \xi_7 \varphi + r$, where $\varphi \in \Lambda^3 \langle \varepsilon_1, \dots, \varepsilon_6 \rangle$, $r \in \Lambda^4 \langle \varepsilon_1, \dots, \varepsilon_6 \rangle$. If $\varphi \neq 0$ then there exists $\psi \in \Lambda^3 \langle \xi_1, \dots, \xi_6 \rangle$ such that $\varphi\psi = f^{(6)}$ and the change of variables $\xi_7 \mapsto \xi_7 + \psi$ kills $f^{(6)}$.

If $\varphi = 0$ then for at least one of $\xi_1, \dots, \xi_6$ (say for ξ_1) we have the decomposition $f^{(4)} = \xi_1 \varphi + r$ for $r \in \Lambda^4 \langle \xi_2, \xi_3, \dots, \xi_6 \rangle$ with a nonzero φ. Taking now ψ such that $\varphi\psi = f^{(6)}$ we reduce f to $f^{(4)}$ by the change $\xi_1 \mapsto \xi_1 + \psi$.

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