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CONTINUITY AND LINEAR EXTENSIONS OF QUANTUM MEASURES ON JORDAN OPERATOR ALGEBRAS

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Introduction.

Let M be a W^*-algebra on a JBW-algebra. A (finitely additive) *quantum measure* on M is a non-negative real valued function μ on the projections on M which satisfies the property

$$\mu(p_1 + \dots p_n) = \mu(p_1) + \dots + \mu(p_n), \quad \mu(1) = 1$$

whenever $p_1, \dots, p_n$ are orthogonal projections of M.

The work of Christensen [3], for Type I_n and properly infinite W^*-algebras, combined with that of Yeadon [15] for arbitrary finite W^*-algebras proves the conjecture of Mackey that every quantum measure on a W^*-algebra without Type I_2 part is the restriction of a linear state. We showed how to extend these results to JBW-algebras without Type I_2 part in [2]. However, somewhat unreasonably, for the general Type I finite case, countable additivity of the quantum measure was assumed.

The proof of 'linearity' of a quantum measure in the properly infinite case in [3], and accordingly in [2] which draws heavily upon [3], makes essential use of the continuity of the measure. This latter property was regarded as established by the methods pioneered in [4]. However, S. Maeda subsequently pointed out a difficulty with the argument given by Gunson in [4].

Nevertheless, in a personal communication to S. Maeda, Christensen showed how to obtain a proof of continuity (thereby settling the linearity question beyond doubt) by modifying the crucial argument of [4] with the aid of a (σ-finite) diagonalisation theorem of Kadison [7 Theorem 3.18]. The interested reader will find these matters documented in Maeda's scholarly article [8].

We show here that it is not necessary to call upon the full power of Kadison's recent Theorem [7] nor is it necessary to require the underlying algebra to be

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σ-finite. For our purposes, it suffices to use a twenty year old result of Fillmore [5] to obtain the required diagonalisation result, see 1.4. Fillmore's theorem was obtained without any cardinality restrictions.

We will prove that every finitely additive quantum measure on a JBW-algebra without Type I_2 part is the restriction of a linear state. In view of the remarks above and the results of [2], only the unbounded Type I finite algebras and the properly infinite algebras still need to be considered. In the former case there is very little for us to do because Yeadon's methods transfer readily to the non-associative situation. A proof of continuity will settle the latter case. This is obtained with the aid of a Jordan algebra generalization of Fillmore's theorem.

Instead of an 'ab initio' proof of this generalization we are able to deduce it from Fillmore's theorem by first developing some results on equivalence of projections and enveloping W^*-algebras which may have some independent interest.

§1. Equivalence of Projections

Let M be a JW-algebra and let Φ be the canonical involutory $*$-antiautomorphism of $W^*(M)$, the universal enveloping W^*-algebra of M. We may suppose that $M \subset W^*(M)$, so that Φ restricts to the identity on M. The real W^*-algebra

$$R^*(M) = \{x \in W^*(M);\quad \Phi(x) = x^*\} \quad \text{satisfies}$$

$$R^*(M) \cap i\,R^*(M) = \{0\}$$

and gives

$$W^*(M) = R^*(M) + iR^*(M), \quad \Phi(x + iy) = x^* + iy^* \quad (x, y \in R^*(M)).$$

Moreover, if M has no type I_2 part, then

$$M = R^*(M)_{sa} = \{x \in W^*(M)_{sa};\quad \Phi(x) = x\}.$$

This notation will be retained throughout. The relevant reference is [6, chapter 7].

We want to consider the behaviour of projections of M relative to their behaviour in $W^*(M)$. The following is useful for this. Afterwards it will be used without comment.

1.1. Lemma *Let e be a projection in a* JW*-algebra M without Type I_2 part. Then*
(a) *e is finite in M if and only if e is finite in $W^*(M)$.*
(b) *e is properly infinite in M and only if e is properly infinite in $W^*(M)$.*

Proof. (a) This is [1, corollary 3.2].

(b) If e is proprly infinite in M then it is properly infinite in $W^*(M)$ by [2, Proposition 4.5] and [11, V. 1.36]. The converse is immediate from (a) and the fact that the centre of M is contained in the centre of $W^*(M)$, [6, 4.3.8].

1.2. PROPOSITION. *Let M be a JW-algebra without Type I_2 part and let e, f be projections of M such that $e \sim f$ in $W^*(M)$. Then $e \sim f$ in $R^*(M)$.*

PROOF. There is a central projection z in M (and hence central in $W^*(M)$) such that ez is finite and $e(1-z)$ is properly infinite or zero. So we can suppose either that e is finite or that e is properly infinite.

Suppose that e is finite. Then so is $p = e \vee f$. Let T be the faithful normal centre valued trace of $pW^*(M)p$. By [6, 5.2.17] the centre of pMp is contained in the centre of $pW^*(M)p$ and so T restricts to the faithful normal centre valued trace of pMp for the same reasons as those given in [2, §5]. Obviously $T(e) = T(f)$ and so $ses = f$ for some symmetry s of M, by [2, Lemma 5.1]. In particular, $e \sim f$ in $R^*(M)$.

Suppose now that e is properly infinite and consider the real $*$-algebra homomorphism

$$W^*(M) \to M_2(R^*(M)) \text{ given by } x + iy \to \begin{pmatrix} x & y \\ -y & x \end{pmatrix}, x, y \in R^*(M).$$

Clearly $\begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \sim \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}$ in $M_2(R^*(M))$. By [2, Proposition 4.5], there are projections e_1, e_2 in M such that $e = e_1 + e_2 \sim e_1 \sim e_2$ in $R^*(M)$. Since also $\begin{pmatrix} e_2 & 0 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 \\ 0 & e_2 \end{pmatrix}$ in $M_2(R^*(M))$ being exchanged by the symmetry $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we see that

$$\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} e_2 & 0 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & e_2 \end{pmatrix} \sim \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix},$$

where all the equivalences are implemented in $M_2(R^*(M))$. Similarly,

$$\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} \text{ in } M_2(R^*(M)).$$

Hence there is an element $v = \begin{pmatrix} u_1 & u_3 \\ u_3 & u_4 \end{pmatrix}$ in $M_2(R^*(M)$ such that $\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} = v^*v$, $\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} = vv^*$. Calculation given $u_2 = u_3 = u_4 = 0$ so that $e = u_1^*u_1$, $f = u_1u_1^*$. Hence $e \sim f$ in $R^*(M)$, thereby completing the proof.

In passing, we observe the following:

1.3 COROLLARY. *Let M be a JW-algebra without Type I_2 part and let e, f be projections of M which are unitarily equivalent in $W^*(M)$. Then there are symmetries $s_1, \ldots, s_n$ in M such that $s_1 \ldots s_n e s_n \ldots s_1 = f$ (i.e. e and f are Jordan equivalent in M).*

PROOF. Let zM be the finite part of M, z being a central projection of M. By 1.2 together with [13, Corollary 22] there is a symmetry s in M such that $sezs = fz$. Notice that if $t_1, \ldots, t_n$ are symmetries of $(1 - z)M$ such that $t_1 \ldots t_n e(1 - z) t_n \ldots t_1 = f(1 - z)$ then the elements of M, $s_1 = sz + t_1$, $s_i = z + t_i$, $i = 2, \ldots, n$, are symmetries of M satisfying the requirements. In order to complete the proof we may therefore suppose that M is properly infinite.

By assumption we must have $e \sim f$, $1 - e \sim 1 - f$ in $W^*(M)$, and hence in $R^*(M)$ by 1.2. In turn, this implies that $ueu^* = f$ for some unitary u in $R^*(M)$. But then [14, Theorem –.6] implies that it equals a finite product of symmetries in $R^*(M)_{sa} = M$. This completes the proof.

The following result of Fillmore can be deduced directly from Kadison's theorem [7, Theorem 3.18] in the σ-finite case and from K. Saitô's version [8] of Kadison's theorem in the general case. Kadison's theorem gives much more information than Fillmore's theorem. However, Fillmore's article predates Kadison's by several years and his results are powerful enough for our purposes.

1.4. THEOREM (Fillmore [5]). *Let W be a properly infinite W^*-algebra and let u be a normal element in W. Then there is a projection e in W commuting with u such tht $e \sim 1 - e$.*

We will deduce the corresponding theorem for Jordan algebras as a direct corollary. To facilitate the ease of reading (and writing) the proof some prior comments are appropriate.

1.5. (a) Let W be a properly infinite W^*-algebra with $1 = p + q$ where p is a finite projection and q a properly infinite projection in W. If, in W, $q = e + f \sim e \sim f$, then $e + p \sim 1 - (e + p)$. Indeed, q has central suport 1 and so $p \lesssim q$ by [II, V 1.3.9, 2-9].

So, $p + e \lesssim f + e = q \sim e \leqq p + e$, and $1 - (p + e) = q - e = f \sim q$.

(b) Let M be a JW-algebra without Type I_2 part, so that $M = R^*(M)_{sa}$, and let x be an element of M. Put

$$B = W^*(M) \cap \{x\}', N = R^*(M) \cap \{x\}', A = M \cap \{x\}'$$

It is easy to see that $B = N + iN$, so that $\Phi(B) = B$, and that $A = N_{sa}$. Also, if p is a projection in B then

$$pBp = pW^*(M)p \cap \{pxp\}' = pW^*(M)p \cap \{x\}'$$

with the corresponding equalities holding for A and N for p in A. Note that B is abelian if and only if N is abelian.

1.6. PROPOSITON *Let M be a properly infinite JW-algebra and let x be an element of M. Then there is in M a projection e and a symmetry s with $ex = xe$, $ses = 1 - e$.*

PROOF. It is enough to find a projection e in M commuting with x such that $e \sim 1 - e$ in $W^*(M)$. For then $e = u^*u$, $1 - e = uu^*$, for some u in $R^*(M)$, so $s = u + u^*$ is a summetry in$M$ and $ses = 1 - e$.

Since there is nothing to prove if M is isomorphic to the self-adjoint part of a W^*-algebra we may suppose that M and $W^*(M)_{sa}$ have the same centre, by virtue of [6, 7.3.4, 7.3.5]. Let A, N and B be as in 1.5 (b).

Suppose first that A, is abelian. Then, by [10, Proposition 2], there are projections p, q in $\underline{A}$ which are central in N with $p + q = 1$ and $pN \simeq C(X, \mathrm{R})$, $qN \simeq C(Y, \mathrm{C}) + C(Z, \mathrm{H})$ for certain compact Hausdorff spaces X, Y, Z. It follows that $pA = pN = pB_{sa}$ and that there is an element v in qN with $v^* = -v$, $v^2 = -q$.

Let zp, tq be the largest finite projection in pMp, qMq, respectively, where z, t are central projections of M. It is immediate from 1.4 and the above that if $\bar{p} = (1 - z)p \neq 0$ (which obviously lies in pA with $\bar{p} = g_1 + g_2 \sim g_1 \sim g_2$.

On the other hand suppose that $\bar{q} = (1 - t)q \neq 0$. Note that $f = \frac{1}{2}(q + iv)$ is a projection in B and that $f + \Phi(f) = q$. Since M and $W^*(M)_{sa}$ have the same centre, [11, Lemma 3.3] implies that $\bar{q}f \sim \Phi(\bar{q}f) = \bar{q}\,\Phi(f)$. In particular, $\bar{q}f$ must be properly infinite (in $W^*(M)$). So passing to $\bar{q}fB\bar{q}f$, 1.4 implies that there are projections h_1, h_2 in B with $\bar{q}f = h_1 + h_2 \sim h_1 \sim h_2$. Using [11, Lemma 3.3.] again we get $\Phi(h_1) \sim h_1 \sim h_2 \sim \Phi(h_2)$. It follows that, for $i = 1, 2$, the projections $r_i = h_i + \Phi(h_i)$ are in A with $\bar{q} = r_1 + r_2$ and $r_1 \sim r_2$.

Since $\bar{p} \neq 0$ or $\bar{q} \neq 0$ we see that the previous two paragraphs combined with 1.5. (b) show the existence of a projection e in A with $e \sim 1 - e$.

Now suppose that A is non-abelian. Let $\{f_\alpha, g_\alpha\}$ be a maximal family of orthogonal projections in A with $f_\alpha \sim g_\alpha$ for each α. Then $f = \vee f_\alpha, g = \vee g_\alpha$ are orthogonal equivalent projections in A. If $h = 1 - (f + g)$ is finite (in M) then $f + g$ is properly infinite and the desired result follows from 1.5. (a). Otherwise there is a central projection z of M such that zh is properly infinite. But this leads to a contradiction. Indeed if $zhAzh$ is non-abelian then the maximality of $\{f_\alpha, g_\alpha\}$ must be contradicted, as indeed it must be if $zhAzh$ is abelian, by the first part of the proof. This completes the proof.

§2 Quantum Measures

The proof of continuity (and hence linearity) for a quantum measure on a properly infinite JW-algebra can now be obtained along the lines of Christensen's argument for properly infinite W^*-algebras. But since the latter has not been published we will outline a version of the proof.

Let μ be a (finitely additive) *quantum* measure on a properly infinite JW-algebra M and lt f, g be projections in M with $\|f - g\| < \frac{1}{2}$. The aim is to show that $|\mu(f) - \mu(g)| \leqq k\,\|f - g\|$ for some constant k independent of f and g.

It is enough to suppose that f (and hence g) is properly infinite. This is because there is a central projection z of M for which zf is finite and $(1 - z)f$ is properly infinite (or zero) and so it is surely sufficient to consider the cases f finite and f properly infinite separately. But f finite implies $1 - f$ properly infinite and, as always, $\mu(f) - \mu(g) = \mu(1 - g) - \mu(1 - f)$.

The polar decomposition $(1 - f)gf = v|(1 - f)gf|$ (in $R^*(M)$) gives $fv = 0$, $vf = v$, $f = w^*w$, $g = ww^*$ where $w = x^{\frac{1}{2}} + v(1 - x)^{\frac{1}{2}}$ and $x = fgf$. Application of 1.6 gives rise to projections $f_1 \cdot f_2$ in M satisfying

$$f = f_1 + f_2 \sim f_1 \sim f_2, f_i x = x f_i \quad (i = 1,2).$$

Now with $g_i = wf_iw^*$, calculation gives (for $i = 1,2$)

$$\|f_i - g_i\| < 1 \text{ and } f_1 g_2 = f_2 g_1 = 0, \text{ so that } f_i \lesssim (1 - f_i) \wedge (1 - g_i).$$

The proof can now be seen to be completed upon application of the Jordan analogue of [3, Proposition 2.3] which is readily obtained with the aid of [2, Lemma 3.6] which is itself the Jordan analogue of [3, Lemma 2.3].

Turning to the question of the linearity of a measure on a general JBW-algebra without Type I_2 part it remains only to consider the unbounded Type I finite case, which was unreasonably omitted from [2]. By the structure of such a JBW-algebra M (e.g., [6, §6.4]) together with [2, Theorem 3.8] and Yeadon's theorem [15], M can be regarded as a JW-algebra of the form $M_{\mathsf{R}} + M_{\mathsf{H}}$ where, for $\mathsf{F} = \mathsf{R}$ and H respectively, M_{F} is a direct sum of algebras of the form $C(X, M_n(\mathsf{F}))_{sa}$ where X is a compact hyperstonean space. Obviously it is enough to consider the real and quaternionic parts separately. Now the fact is that Yeadon's methods [15] transfer with virtually no change. We might remark that the non-abelian part of a JW-subalgebra generated by two projections (in any JBW-algebra) is isomorphic to $C(X, M_2(\mathsf{R}))_{sa}$ for some compact hyperstonean space X and the splitting argument in [15, Proposition I] is independent of the underlying division ring. The rest is transparent from an inspection of Yeadon's article. We therefore have:

2.1. THEOREM Let M be a JBW-*algebra without Type I_2 part and let μ be a finitely quantum measure on M. Then μ extends to a linear functional on M.*

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