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WEAKLY UNCONDITIONALLY CONVERGENT SERIES IN M-IDEALS

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Abstract.

Every Banach space X which is an M-ideal in its bidual has the proprty (V) of Pełczynski. If E is separable complex Banach space with the approximation property and $K(E)$ is an M-ideal in $L(E)$, then E is isomorphic to a complemented subspace of a space with a shrinking unconditional finite dimensional decomposition.

Introduction.

The concept of an M-ideal has been introduced by Alfsen and Effros in 1972 ([1], [2]) and has attracted a lot of attention since then (see e.g. [4], [17], [28]). The present work is a contribution to the study of their structure in Banach spaces and Banach algebras. We first show that if a Banach space X is an M-ideal in its bidual X^{**} then X has the property (V) of Pełczynski [30], that is if $T: X \to E$ is a non weakly compact operator from X into a Banach space E, then X contains a subspace Y isomorphic to c_0 such that the restriction of T to Y is an isomorphism between Y and $T(Y)$. We use different techniques for showing that if E is a separable complex Banach space and $K(E)$ is an M-ideal in $L(E)$, then an operator $T \in L(E^{**})$ is a conjugate operator if and only if it is the weak*-sum of a weakly unconditionally convergent series of compact operators of $K(E)$; this applies of course to the identity operator, and this permits to show that if E is separable complex Banach space with the approximation property and $K(E)$ is an M-ideal in $L(E)$ then E is isomorphic to a complemented subspace of a space with a shrinking unconditional finite dimensional decomposition. We also show that if a separable complex space E is reflexive and $K(E)$ is an M-ideal in $L(E)$, then $K(E)$ has the property (u) of Pełczynski.

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NOTATIONS. The Banach spaces we consider are real or complex. The space of compact (resp. bounded) linear operators on a Banach space E is denoted by $K(E)$ (resp. $L(E)$). The closed unit ball of a Banach space X will be denoted by X_1, and the unit sphere is $S_1(X)$. A weakly unconditionally convergent series is a sequence $(x_n)_{n \geqq 1}$ in X such that

$$\sum_{n=1}^{\infty} |y^*(x_n)| < \infty \quad \text{for every } y^* \in X^*.$$

It is an easy consequence of the uniform boundedness principle that this condition is equivalent to

$$\sup \left\|\sum \varepsilon_i x_i\right\| < \infty,$$

where the supremum is taken over the finite sequences of ε_i with $|\varepsilon_i| = 1$.

Since the sequence $S_n = \sum_{i=1}^{n} x_i$ is clearly weakly Cauchy, we can let

$$\sum^* x_n = \lim_{n \to \infty} S_n \text{ in } (X^{**}, \text{weak}^*).$$

A space X is said to have property (u) [30] if every $z \in X^{**}$ which is in the sequential closure of X in (X^{**}, weak^*) may be written

$$z = \sum^* x_n,$$

where $(x_n)_{n \geqq 1}$ is a weakly unconditionally convergent series in X. A subspace X of a Banach space E is an M-ideal in E if there exists a subspace Z of E^* such that $E^* = X^{\perp} \oplus_1 Z$, where $X^{\perp}$ denotes the annihilator of X in E^*, and $\oplus_1$ means $\|u + v\| = \|u\| + \|v\|$ for every $u \in X^{\perp}$ and every $v \in Z$. The conjugate of an operator T is denoted by T^*, and we note $T^{**} = (T^*)^*$. All subspaces we consider are supposed to be norm closed.

Results.

Our first result was announced in [11].

THEOREM 1. *Let X be a Banach spce which is an M-ideal in its bidual X^{**}. Then X has the proprty* (V) *of Pelczynski.*

PROOF. We consider the set

$$D = \{z \in X^{**} \mid \|z\| = 1 = \operatorname{dist}(z, X)\}.$$

Since X is proximinal in X^{**} [18], the linear span of $(X \cup D)$ is X^{**}. Hence if $T: X \to Y$ is a non weakly compact operator, there exists $z \in D$ such that $T^{**}(z) \notin Y$. We let $\alpha = \operatorname{dist}(T^{**}(z), Y) > 0$, and pick $\varepsilon \in (0, \alpha)$.

We stop now to prove the following lemma

LEMMA 2. *For every finite subset* $(x_i)_{1 \leqq i \leqq n}$ *of* X *with* $\|x_i\| < 1$ *for* $1 \leqq i \leqq n$, *there exists* $x \in X$ *such that* $\|x\| < 1$, $\|x - x_i\| < 1$ *for* $1 \leqq i \leqq n$ *and* $\|T(x)\| > \varepsilon$.

PROOF OF LEMMA 2. Pick $\eta > 0$ such that $(1 + \eta)\|x_i\| \lesssim 1$ for $1 \leqq i \leqq n$ and $\varepsilon < \alpha(1 + \eta)^{-2}$. For $z \in D$ such that $\operatorname{dist}(T^{**}(z), Y) = \alpha > 0$, let

$$P(z) = \{x \in X \mid \|z - x\| = 1\}.$$

The set $P(z)$ is a pseudo ball [4] and thus there exists $x_0 \in P(z)$ such that $(x_0 + (1 + \eta)x_i) \in P(z)$ for $1 \leqq i \leqq n$ [4]. Since $x_0 \in X$, we have

$$\|T^{**}(z - x_0)\| \geqq \operatorname{dist}(T^{**}(z), Y) = \alpha,$$

hence there exists $y \in Y^*$ with $\|y\| = 1$ such that

$$\langle T^{**}(z - x_0), y\rangle = \langle z - x_0, T^*(y)\rangle > \alpha(1 + \eta)^{-1}.$$

Let V be the linear span of $\{z, x_0, x_1, \ldots, x_n\}$; by the local reflexivity principle [22], there is an operator $A\colon V \to X$ such that

(i) $\|A\| < 1 + \eta$.
(*ii*) $A(u) = u$ for every $u \in V \cap X$.
(iii) $\langle A(z - x_0), T^*(y)\rangle > \alpha(1 + \eta)^{-1}$.

We let now

$$x = (1 + \eta)^{-1} A(z - x_0)$$

This element x works; indeed

$$\|x\| \leqq (1 + \eta)^{-1} \|A\| \|z - x_0\| < 1$$

Moreover, for $1 \leqq i \leqq n$, we have

$$\|z - x_0 - (1 + \eta)x_i\| \leqq 1$$

and thus

$$\|A(z - x_0) - (1 + \eta)x_i\| < 1 + \eta,$$

which implies that $\|x - x_i\| < 1$; finally, the condition (iii) implies

$$\langle x, T^*(y)\rangle = \langle T(x), y\rangle > \alpha(1 + \eta)^{-2} > \varepsilon$$

and therefore

$$\|T(x)\| > \varepsilon.$$

Let us now resume the proof of Theorem 1. Since $\|T^{**}(z)\| > \varepsilon$, there exists $u_0 \in X$ with $\|u_0\| < 1$ and $\|T(u_0)\| > \varepsilon$. We apply the lemma 2 to the family $\{u_0, -u_0\}$ to find $u_1 \in X$ with

(i) $$\|u_1\| < 1, \|u_0 + u_1\| < 1, \|u_1 - u_0\| < 1$$

and

(ii) $$\|T(u_1)\| > \varepsilon.$$

We apply now the lemma 2 to

$$\{\varepsilon_0 u_0 + \varepsilon_1 u_1 \mid \varepsilon_i = \pm 1\},$$

and we continue in this way to construct by induction a sequence $(u_i)_{i \geqq 1}$ such that

(i) $$\left\| \sum_{i=1}^{n} \varepsilon_i u_i \right\| < 1 \ \forall n \text{ and } \forall \varepsilon_i = \pm 1$$

and

(ii) $$\|T(u_i)\| > \varepsilon \ \forall i;$$

the result follows easily.

Let us observe that the above proof is making a crucial use of the techniques of [4] and [17].

REMARKS 3.

1) If X is an M-ideal in its bidual, then X^* is weakly sequentially complete [12] and since, by Theorem 1 X has (V), this implies that every operator from X to X^* is weakly compact. In particular if X a Banach algebra, then X is Arens-regular [13].

2) If X has (V), then X^* has (V*) [30] and thus by Theorem 2, if Y is such that $Y^{**} = Y \oplus_1 Y_s$ with Y_s weak* closed, then Y has the property (V*). It is an open question to know whether the assumption put on Y_s to be weak* closed is actually necessary.

3) The space $X = (\sum \oplus l_n^1)_{c_0}$ is an M-ideal in its bidual; however X^{**} contains a complemented copy of l^1 [19] and thus X^{**} does not have the property (V).

4) It is an open question to know whether a separable Banach space that is an M-ideal in its bidual has the property (u); This question will be answered below in the affirmative in an important special case (Corollary 8), see: Added in proof.

We will now prove a structural theorem for the complex spaces E such that $K(E)$ is an M-ideal in $L(E)$. Let us state our main result.

THEOREM 4. *Let E be separable complex Banach space such that $K(E)$ is an M-ideal in $L(E)$. Then $K(E)^{**}$ is canonically isometric to $L(E^{**})$ and for $T \in L(E^{**})$ the following are equivalent:*

1) *There exists $T_0 \in L(E^*)$ such that $T_0^* = T$.*

2) *There is a sequence $(K_n)_{n \geqq 1}$ in $K(E)$ such that:*

(i) $\|\sum \varepsilon_i K_i\| \leqq M$ *for every finite sequence of* $|\varepsilon_i| = 1$.

(ii) $\langle T(z), y\rangle = \sum_{n=1}^{\infty} \langle z, K_n^* y\rangle \ \forall y \in E^*$ *and* $\forall z \in E^{**}$.

PROOF. If $K(E)$ is an M-ideal in $L(E)$, then E and E^* have the compact approximation property (C.A.P) [17] and E^* has the Radon Nikodym Property (RNP). The Feder-Saphar technique [9] permits to show that $K(E)^{**}$ is canonically isometric to $L(E^{**})$ [14]; where canonical means that the diagram

$$\begin{array}{ccc} L(E) & \xrightarrow{t^{**}} & L(E^{**}) \\ \uparrow{\scriptstyle j} & & \uparrow{\scriptstyle I} \\ K(E) & \xrightarrow{i} & K(E)^{**} \end{array}$$

is commutative, where i and j are the canonical injections, I is the isometry and $t^{**}(T) = T^{**}$.

Let us now proceed to the proof of the equivalence. To show that 1) implies 2) we need to prove the following crucial lemma which relies heavily on ([28], lemma 2.4.).

LEMMA 5. *Let A be a complex Banach algebra with unit e. Let X be a separable subspace of A which is an M-ideal in A; if we write $A^* = X^\perp \oplus_1 Y$, then there is a weakly unconditionally convergent series $(x_n)_{n \geqq 1}$ in X such that*

$$e(y) = \sum_{n=1}^{\infty} x_n(y) \quad \forall y \in Y.$$

PROOF. Let

$$S = \{y \in A^* \mid \|y\| = 1 = y\{e\}\}$$

be the state space of A. Since X is an M-ideal in A, the sets

$$F = X^\perp \cap S \text{ and } F' = Y \cap S$$

form a pair of split faces of S such that $S = \operatorname{conv}(F \cup F')$ and moreover $X^\perp$, (resp., Y) is algebraically spanned by F (resp., F'), [28]. Let

$$\Pi: A^* \to Y$$

be the projection having as kernel $X^\perp$, and let $z = \Pi^*(e) \in A^{**}$. It is clear that $z_{|F} = 0$ and $z_{|F'} = 1$. Since $S = \operatorname{conv}(F \cup F')$ we have $0 \leqq z \leqq 1$ on S and for every $\lambda \in [0,1]$ the set

$$S_\lambda = S \cap z^{-1}((-\infty, \lambda])$$

may be written

$$S_\lambda = \{\mu t + (1 - \mu)t' \mid t \in F, t' \in S, 1 - \lambda \leqq \mu \leqq 1\}.$$

Since F is w^*-compact, the set S_λ is w^*-closed. The projection Π is continuous from (A^*, w^*) to $(Y, \sigma(Y, X))$ and therefore the set $S_0 = \Pi(S)$ is $\sigma(Y, X)$-compact. Moreover since $z = e \circ \Pi = \Pi^*(e)$ we have $0 \leqq z \leqq 1$ on S_0 and

$$S_0 \cap z^{-1}((-\infty, \lambda]) = \Pi(S_\lambda),$$

and this shows that z is lower semi-continuous on $(S_0, \sigma(Y, X))$; therefore there exists an increasing sequence $(f_n)_{n \geqq 1}$ of continuous functions on $(S_0, \sigma(Y, X))$ which converges pointwise to z; in particular we have

(i) $\sum_{n=1}^{\infty} |f_n(y)| < \infty \quad \forall y \in S_0$,

(ii) $z(y) = \sum_{n=1}^{\infty} f_n(y) \quad \forall y \in S_0$.

But we also have $z \in X^{\perp\perp}$ and a fortiori z belongs to the pointwise closure on S_0 of X_1. Hence by a classical lemma (see [24], p. 32) there is a sequence $(x_n)_{n \geqq 1}$ in X_1 such that

(iii) $\sum_{n=1}^{\infty} |\lambda_n(y)| < \infty \quad \forall y \in S_0$,

(iv) $z(y) = \sum_{n=1}^{\infty} x_n(y) \quad \forall y \in S_0$.

The numerical radius defines an equivalent norm on A [3], thus $A^* = \operatorname{span}(S)$ and $Y = \operatorname{span}(S_0)$; hence the conditions (iii) and (iv) hold also for $y \in Y$; this finishes the proof of the lemma since $z(y) = e(y) \ \forall y \in Y$.

Let us now proceed to the proof of Theorem 4. We apply Lemma 5 to $A = L(E)$ and $X = K(E)$. For every $y \in E^*$ and every $z \in E^{**}$ with $\|y\| = \|z\| = 1$, let us consider the linear form $z \otimes y$ in $L(E)^*$ where

$$\langle z \otimes y, T\rangle = \langle z, T^*(y)\rangle;$$

clearly

$$\|z \otimes y\| = 1 \text{ in } L(E)^*,$$

but also in

$$K(E)^* = L(E)^*/K(E)^\perp.$$

Hence if

$$L(E)^* = K(E)^\perp \oplus_1 Y$$

we have

$$z \otimes y \in Y \text{ if } \|y\| \, \|z\| = 1$$

and thus

$$z \otimes y \in Y \text{ for every } y \in E^* \text{ and } z \in E^{**}.$$

Hence by Lemma 5, there is a sequence $(S_n)_{n \geqq 1}$ in $K(E)$ such that

$$\|\sum \varepsilon_i S_i\| \leqq M$$

for every finite sequence $|\varepsilon_i| = 1$, and such that

$$(*) \qquad \langle z, y\rangle = \sum_{n=1}^{\infty} \langle z, S_n^*(y)\rangle \;\; \forall y \in E^* \text{ and } \forall z \in E^{**};$$

consider now $T \in L(E^{**})$ such that there is $T_0 \in L(E^*)$ with $T = T_0^*$ and apply (*) to $z = T(z')$ to get

$$\langle T(z'), y\rangle = \sum_{n=1}^{\infty} \langle T(z'), S_n^*(y)\rangle \;\; \forall y \in E^* \text{ and } \forall z' \in E^{**}$$

and thus

$$\langle T(z'), y\rangle = \sum_{n=1}^{\infty} \langle z', T_0 S_n^*(y)\rangle,$$

but since S_n^* is compact, it is weak* to norm continuous on bounded sets and so is $T_0 S_n^*$, hence $T_0 S_n^*$ is weak* to weak* continuous and compact and thus there exists $K_n \in K(E)$ such that $K_n^* = T_0 S_n^*$. Finally we have

$$\begin{aligned}\|\sum \varepsilon_i K_i\| &= \|\sum \varepsilon_i K_i^*\| \\ &= \|\sum \varepsilon_i T_0 S_i^*\| \\ &\leqq \|T_0\| \, \|\sum \varepsilon_i S_i^*\| \\ &\leqq \|T_0\| M,\end{aligned}$$

and this concludes the proof of 1) implies 2) in Theorem 4.

Conversely, we will prove a much stronger result than 2) implies 1), namely if there exists a sequence $(V_n)_{n \geqq 1}$ in $L(E^*)$ such that

$$(**) \qquad \langle T(z), y\rangle = \lim_{n \to \infty} \langle z, V_n(y)\rangle \;\; \forall y \in E^* \text{ and } \forall z \in E^{**},$$

then there is $T_0 \in L(E^*)$ such that $T_0^* = T$. Indeed (**) implies that

$$T^*(y) = \lim_{n \to \infty} V_n(y) \text{ in } (E^{***}, \text{weak*}),$$

but $K(E)$ being M-ideal in $L(E)$ implies that E is an M-ideal in E^{**} [21] and thus E^* is weakly sequentially complete [12]; hence $T^*(y) \in E^*$ and if we define T_0 to be the restriction of T^* to E^* we have $T_0^* = T$.

Our first application of theorem 4 is a structural result for the complex spaces E for which $K(E)$ is an M-ideal in $L(E)$. Let us observe that such a space has always the metric compact approximation property [17]; it is unknown whether it has necessarily the approximation property (A.P). Our next result asserts that if the A.P. holds, then a much stronger property is satisfied.

COROLLARY 6. *Let E be a separable complex Banach space such that $K(E)$ is an M-ideal in $L(E)$. Then the following statements are equivalent:*

1) *E has the* A.P.
2) *E has the metric* A.P.
3) *E^* has the* A.P.
4) *E is isomorphic to a complemented subspace of a space a with a shrinking unconditional finite dimensional decomposition.*

PROOF. The implications 4) implies 3) and 2) implies 1) are obvious. To see that 3) implies 2) notice that E^* has the RNP and therefore E^* has the metric A.P. if it has A.P. (see [23]); and it is always true that E has the metric A.P. if E^* does (see [23]).

For 1) implies 4), apply Theorem 4, to find a sequence $(S_n)_{n\geqq 1}$ in $K(E)$ such that

(i) $\|\sum \varepsilon_i S_i\| \leqq M$ for every finite sequence of $|\varepsilon_i| = 1$

(ii) $\langle z, y\rangle = \sum_{n=1}^{\infty} \langle z, S_n^* y\rangle \ \forall y \in E^*$ and $z \in E^{**}$.

Since E has the A.P. there exists a sequence $(R_n)_{n\geqq 1}$ of finite rank operators such that

$$\|S_n - R_n\| < 2^{-n-1}.$$

Following the lines of ([27], proposition 3), we observe that for every $x \in E$, the series

$$S(x) = \sum^* R_n(x)$$

is weakly unconditionally convergent and thus defines an operator from E into E^{**}; but S actually takes its value in E; indeed for every N, we have

$$\left\| \sum_{i>N}^* R_i(x) - \sum_{i>N}^* S_i(x) \right\| \leqq \|x\| \sum_{i>N} \|R_i - S_i\|$$
$$\leqq \|x\| 2^{-N-1}$$

and thus

$$\left\| S(x) - \sum_{n=1}^{N} R_i(x) - x + \sum_{n=1}^{N} S_i(x) \right\| \leqq \|x\| 2^{-N-1},$$

which shows that

$$\operatorname{dist}(S(x), E) \leqq \|x\| 2^{-N-1} \text{ for every } N$$

and thus $S(x) \in E$. Moreover we clearly have that $\|\mathrm{Id}_E - S\| \leqq 2^{-1}$ and thus $S = U^{-1}$ is an invertible operator; if we consider now the finite rank operators $U_n = UR_n$ we have:

$$x = \sum_{n=1}^{\infty} U_n(x) \ \forall x \in E$$

and the convergence is unconditonal. In other words E has the unconditional approximation property (in the terminology of [8]). Thus by [31], the space E is isomorphic to a complemented subspace of the unconditional sum [31]

$$X = \sum_u U_n(E).$$

For completing the proof let us observe that E is an M-ideal in E^{**} [21] and thus E is an Asplund space; that is E is an Asplund complemented subspace of a space X which has an unconditional finite dimensional decomposition. Under these assumptions, it is possible [20] to adapt the proof of Theorem 3.3 of [10] to show that E is isomorphic to a complemented subspace of a space with a shrinking unconditional finite dimensional decomposition: in the notation of ([10], with $E = A(Z)$), one needs to observe that the set

$$W = \overline{\mathrm{conv}}\left\{\left(\sum_{n=1}^{k} \varepsilon_n P_n\right) E_1 \mid \varepsilon_n \in \{-1, 1\}^{\mathbb{N}}, k \geqq 1\right\}$$

where the P_n's are the "coordinate projections" associated with the finite dimensional decomposition is weak*-sequentially compact and apply the interpolation techique of [5].

REMARKS 7.

1) The above condition 4) implies in particular that E^* is complemented in a space with an unconditional boundedly complete finite dimensional decomposition.

2) If moreover E is reflexive, we can show like in ([10], Theorem 3.3) that E is isomorphic to a complemented subspace of a reflexive space with an unconditional finite dimensional decomposition. It suffices indeed to reproduce th above proof and to observe [20] that the corresponding set W is weakly compact.

In the case where E is reflexive we obtain without assuming the A.P. the following corollary:

COROLLARY 8. *Let E be a separable reflexive complex Banach space such that $K(E)$ is and M-ideal in $L(E)$. Then $K(E)$ has the property* (u).

PROOF. By [17], $K(E)^{**}$ is canonically isometric to $L(E)$. By Theorem 4, there exists a sequence $(K_n)_{n \geqq 1}$ in $K(E)$ such that

(i) $$\left\| \sum_{i=1}^{n} \varepsilon_i K_i \right\| \leqq M \quad \forall |\varepsilon_i| = 1, \ \forall n \geqq 1$$

(ii) $$\langle y, x \rangle = \sum_{n=1}^{\infty} \langle y, K_n(x) \rangle \quad \forall x \in X, \text{ and } \forall y \in X^*.$$

Condition (ii) means that $\mathrm{Id}_E = \sum^* K_n$ in $K(E)^{**} = L(E)$: if now $T \in L(E)$ is any operator, then we have

$$T = \sum^* TK_n = \sum^* S_n$$

since the multiplication in $L(E)$ is weak*-separately continuous if E is reflexive; and it is clear that $S_n \in K(E)$ and the S_n's satisfy condition (i).

Examples, remarks and questions.

1) It is easy to deduce from the resuls of [8] and [14] that if X has a shrinking unconditional finite dimensional decomposition such that the weak* and the weak topology coincide on the unit sphere $S_1(X^*)$ *of* X^* and E is a subspace of X then saying that E has the approximation property is equivalent to saying that E^* has the metric approximation property and this in turn is equivalent to asserting that E has the unconditional approximation property.

2) Let A be a subset of an abelian discrete group $\Gamma = \hat{G}$; let $\mathscr{C} = \mathscr{C}(G)$ and Λ' be the complement of $(-\Lambda)$ in Γ, then the following statements were shown to be equivalent in [15]:

(i) $\mathscr{C}/\mathscr{C}_{\Lambda'}$ is an M-ideal in its bidual.

(ii) The unit ball B_Λ of $L^1_\Lambda(G)$ is closed for the topology τ of convergence in measure, and the Fourier coefficients $\mathscr{F}_\alpha(f) = \hat{f}(\alpha)$ are continuous on (B_Λ, τ).

It is not known if these M-ideals have the property (u) in general. This is true if $\Gamma = \mathsf{Z}$ and $\Lambda = \mathsf{N}$ since $\mathscr{C}(\mathsf{T})/A_0(\mathsf{D})$ is isometric to a subspace of $K(l_2)$ [15], see: Added in proof.

Observe that the convolution induces a structure of Banach algebra on $\mathscr{C}/\mathscr{C}_{\Lambda'}$, since $\mathscr{C}_{\Lambda'}$ is an ideal of $(\mathscr{C}, *)$, but the bidual space $L^\infty/L^\infty_{\Lambda'}$ has no unit in general.

3) If X is a separable complex Banach algebra such that:

a) X is an M-ideal in its bidual X^{**}

b) X is an ideal of the algebra X^{**}

c) X^{**} is a Banach algebra with unit

then it is easy to deduce from Lemma 5 that X has the property (u). We do not know whether the statement a) implies the statement b); this is true if X is commutative [29], see: Added in proof.

4) Any space which has an unconditional finite dimensional decomposition is a subspace of a space with an unconditional basis ([23] Theorem 1.g.5), hence by Corollary 6 and [10], any separable complex Banach space with the approxi-

mation property such that $K(E)$ is an M-ideal in $L(E)$ is a subspace of a space X with a shrinking unconditional basis. If moreover E is reflexive, the space X can be taken reflexive as well.

5) On which separable spaces E does there exist an equivalent norm such that $K(E)$ is an M-ideal in $L(E)$ when $L(E)$ is equipped with the operator norm? Observe that by Corollary 6 and [26] (resp. [23]) the spaces

$$E = l_p \check{\otimes} l_p;\ 1 < p < 2$$

(resp.)

$$F = (\sum \oplus L^{1+1/n})_2$$

which are reflexive spaces with basis, do not admit such a renorming. Note also that if a complex space E is reflexive, separable and $K(E)$ is an M-ideal in $L(E)$, then Corollary 8 permits to show easily that $K(E)^* = E^* \hat{\otimes} E$ has the property (X) [16] or equivalently $K(E)^* \prec l_1$ in Edgar's ordering [6].

6) If $E = l_2$, let $N(E)$ be the space of nuclear operators on E. it is well known that $N(E) = K(E)^* = E \hat{\otimes} E$; let H be the subspace of "uper triangular operators", that is the closed linear span of $\{e_i \otimes e_j \mid j \geqq i\}$ where $(e_n)_{n \geqq 1}$ the usual basis of E. It is easily seen that H is weak* closed in $K(E)^*$, hence $N(E)/H \equiv (H^{\top})^*$ and sine $H^{\top}$ is a subspace of $K(E)$ which has the property (u), $H^{\top}$ has (u) as well; therefore $N(E)/H$ has (X) (see [6] and [16]), so it has (V*) and hence it is weakly sequentially complete; actually the space $N(E)/H$ shares most of the infinite dimensional geometrical properties of its "commutative relative" $L^1(\mathrm{T})/H^1(\mathrm{D})$.

7) If E is an M-ideal in E^{**} and thus if $K(E)$ is an M-ideal in $L(E)$ then E is weakly compactly generated [7]. Hence the assumptions of separability we made can be deleted mutatis mutandis with standard but tedious technicalities.

8) If A is a real Banach algebra, the state space S does not separate A in general; a classical example is $A = l_2^2 \check{\otimes} l_2^2$. Hence for being able to apply ou crucial Lemma 5, we have to limit ourselves to the complex situation. This restriction is probably unnecessary; however, it seems technically uneasy to complexify the Banach algebras we are using while respecting the M-ideal structure.

Added in proof. After this paper was accepted, D. Li and the first-named author showed that any M-ideal in its bidual has property u (Ann. Inst. Fourier 39(1989), 361–371). It follows in particular that our results on $K(E)$ are still valid it E is a real Banach space.

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