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GALOIS ACTIONS ON RINGS AND FINITE GALOIS COVERINGS

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Introduction.

Bongartz, Gabriel and Riedtmann introduced in [BG, G, R] galois coverings of k-categories which have become an important tool in studying the representation theory of finite dimensional algebras over an algebraically closed field k. Although the general theory of galois coverings of k-categories obviously lies outside ring theory, this is not the case for finite galois coverings. One of the main aims of this paper is to develop a notion of galois extensions for general rings which when suitably specialized gives a purely ring theoretic formulation of finite galois covers (see section 6). Another requirement we had in mind for this general theory was that it gives the usual theory also when specialized to commutative rings. We recall that a commutative ring extension S of R is said to be a galois extension with galois group G if $a)$ G is a finite group of R-automorphisms of S such that $R = S^G$, the fixpoint ring, b) S is a finitely generated projective R-module and c) the natural ring morphism from $S(G)$ to $\mathrm{End}_R(S)$ is an isomorphism where $S[G]$ is the skew group ring (see [AG2]). Using Morita theory it is not difficult to see that b) and c) together are equivalent to S being a projective $S[G]$-generator. It is this latter formulation of commutative galois ring extensions that we adapt for arbitrary rings.

Specifically, let Γ be a subring of a ring Λ and G a finite group of automorphisms of Λ such that Γ is equal to the fixpoint ring Λ^G, and Λ is a finitely generated Γ-module. Λ is defined to be a pregalois extension of Γ with group G if Λ is a projective $\Lambda[G]$-generator, and we then say that the pair (Λ, G) is pregalois. It is an easy consequence of Morita theory that (Λ, G) being pregalois is equivalent to the fixpoint functor from $\mathrm{Mod}\,\Lambda[G]$ to $\mathrm{Mod}\,\Lambda^G$ being an equivalence of categories.

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While pregalois ring extensions are interesting in their own right, they are not restrictive enough to give a suitable theory of finite galois coverings. This aim is accomplished by the following definition.

We say that a pregalois extension Λ of Γ with group G is galois if for every simple left or right Λ-module S, $\Gamma/\mathrm{ann}_\Gamma S$ is a semisimple artin ring. Here $\mathrm{ann}_\Gamma S$ is the annihilator of S viewed as a Γ-module.

It should be observed that when Λ is commutative, this last condition is automatically satisfied, so that this definition of galois extension coincides with the usual one for commutative rings.

We now discuss the content of this paper section by section.

Let Λ be a ring and G a finite group acting on Λ. In section 1 we give descriptions of Λ being a projective $\Lambda[G]$-module and of Λ being a $\Lambda[G]$-generator in terms of the existence of certain elements in $\Lambda[G]$. We apply this to show that (Λ, G) is pregalois if and only if $(\Lambda^{\mathrm{op}}, G)$ is.

In section 2 we show that if (Λ, G) is pregalois and H is a subgroup of G, then (Λ, H) is pregalois via the induced action of H on Λ. Also if H is a normal subgroup we show that $(\Lambda^H, G/H)$ is pregalois. If I is a G-ideal in Λ, we prove in section 3 that if (Λ, G) is pregalois, then $(\Lambda/I, G)$ is, and if $(\Lambda/\mathrm{rad}\,\Lambda, G)$ is pregalois, then (Λ, G) is.

The notion of (Λ, G) being galois is introduced in section 4, and we prove that results similar to those in sections 2 and 3 hold for galois extensions.

In section 5 (Λ, G) being galois is related to the condition that G acts freely on the isomorphism classes of simple Λ-modules. Section 6 is devoted to showing that finite galois coverings for finite dimensional algebras correspond to our notion of galois extensions.

In section 7 we return to considering pregalois (Λ, G). First we recall various well known interpretations in terms of relative global dimension and derivations of the fact that the multiplication morphism $\Lambda \otimes_{\Lambda^G} \Lambda \to \Lambda$ is a split epimorphism as Λ-bimodules. We also give charaterizations of when the multiplication morphism $\Lambda[G] \otimes_\Lambda \Lambda[G] \to \Lambda[G]$ is a split epimorphism of $\Lambda[G]$-bimodules.

1. Preliminaries.

Let Λ be a ring and let G be a finite group operating on Λ as a group of automorphims. Let $\Lambda[G]$ denote the skew group ring of Λ by G, i.e. $\Lambda[G]$ is a free left Λ-module with the elements of G as basis and the multiplication in $\Lambda[G]$ is defined by the rule $\lambda_g g \cdot \lambda_h h = \lambda_g g(\lambda_h) gh$ for $\lambda_g, \lambda_h \in \Lambda$, and $g, h \in G$. A left $\Lambda[G]$-module M may be thought of as a Λ-module M with an action of G such that the equality $g(\lambda m) = g(\lambda) g(m)$ holds for each $g \in G$, $\lambda \in \Lambda$ and $m \in M$. In the same way a $\Lambda[G]$-morphism between two $\Lambda[\mathrm{G}]$-modules M and N is a Λ-morphism $f\colon M \to N$ such that $f(g(m)) = g(f(m))$ for each $m \in M$ and $g \in G$.

We say that the action of G on Λ is pregalois if Λ is a projective $\Lambda[G]$-generator, where Λ is considered as a left $\Lambda[G]$-module by $\lambda g \cdot x = \lambda g(x)$ for $\lambda, x \in \Lambda$ and $g \in G$. In this situation we also say that the pair (Λ, G) is pregalois.

Let Γ be a subring of Λ. We also say that (Λ, Γ) is pregalois if Γ is the fixpoint set $\Lambda^G = \{\lambda \in \Lambda \mid g(\lambda) = \lambda, \textit{for all } g \in G\}$ of a finite group G acting on Λ with (Λ, G) pregalois.

When G operates on Λ as a group of automorphisms, the same action induces an action of G as a group of automorphisms on Λ^{op}, the opposite ring of Λ. The map $f\colon \Lambda[G] \to \Lambda^{\mathrm{op}}[G]$ defined by $f(\lambda_g g) = g^{-1}(\lambda g)g^{-1}$ is easily seen to be an antiisomorphism. Hence, with this map we may identify $(\Lambda[G])^{\mathrm{op}}$ with $\Lambda^{\mathrm{op}}[G]$. The left $\Lambda^{\mathrm{op}}[G]$-structure on Λ^{op} will by this identification correspond to the right $\Lambda[G]$-structure on Λ given by $\lambda \cdot \lambda_g \mathrm{g} = \mathrm{g}^{-1}(\lambda\lambda_g)$ for each λ in Λ and each $\lambda_g g$ *in* $\Lambda[G]$.

The main aim of this first section is to establish the fact that (Λ, G) is pregalois if and only if $(\Lambda^{\mathrm{op}}, G)$ is pregalois. By the above discussion, this may be reformulated as Λ is a projective left $\Lambda[G]$-generator if and only if Λ is a projective right $\Lambda[G]$-generator. This will be done separately for the two properties involved in the definition of pregalois. In order to do this we need different characterizations of the fixpoint set $M^G = \{m \in M \mid g \cdot m = m \text{ for all } g \in G\}$ of a $\Lambda[G]$-module M. The subgroup M^G of M is easily seen to be a left module over the fixpoint ring Λ^G of Λ. We begin with the following well known result.

PROPOSITION 1.1. a) *The evaluation map* $\phi_M\colon \mathrm{Hom}_{\Lambda[G]}(\Lambda, M) \to M$ *defined by* $\phi_M(f) = f(1)$ *induces a functorial isomorphism* $\phi_M\colon \mathrm{Hom}_{\Lambda[G]}(\Lambda, M) \to M^G$ *for all left* $\Lambda(G)$*-modules* M.

b) $\phi_\Lambda\colon \mathrm{End}_{\Lambda[G]}(\Lambda)^{\mathrm{op}} \to \Lambda^G$ *is a ring isomorphism.*

As a consequence of the isomorphism $\mathrm{End}_{\Lambda[G]}(\Lambda)^{\mathrm{op}} \simeq \Lambda^G$ we have from general Morita theory the following connection between $\mathrm{Mod}\,\Lambda[G]$, the category of left $\Lambda[G]$-modules, and $\mathrm{Mod}\,\Lambda^G$, the category of left Λ^G-modules, when (Λ, G) is pregalois.

PROPOSITION 1.2. (Λ, G) *is pregalois if and only if* $\Lambda \otimes_{\Lambda^G} -\colon \mathrm{Mod}\,\Lambda^G \to \mathrm{Mod}\,\Lambda[G]$ *and* $\mathrm{Hom}_{\Lambda[G]}(\Lambda\ -)\colon \mathrm{Mod}\,\Lambda[G] \to \mathrm{Mod}\,\Lambda^G$ *are inverse equivalences of categories.*

PROOF. Assume first that (Λ, G) is pregalois. Hence Λ is a finitely generated projective $\Lambda[G]$-generator. Then from general Morita theory, $\mathrm{Hom}_{\Lambda[G]}(\Lambda, -)$ induces an equivalence of categories between $\mathrm{Mod}\,\Lambda[G]$ and $\mathrm{Mod}\,(\mathrm{End}_{\Lambda[G]}(\Lambda)^{\mathrm{op}})$ with $\Lambda \otimes_{\mathrm{End}_{\Lambda[G]}(\Lambda)^{\mathrm{op}}} -$ as inverse (see [AF, Section 22]). But from the above proposition $\mathrm{End}_{\Lambda[G]}(\Lambda)^{\mathrm{op}}$ is naturally isomorphic to Λ^G, which completes the proof of the only if part of the proposition. From Moritas characterization of equivalence the converse follows (see [AF, Section 22]).

We now use the obvious embedding of Λ as a subring of $\Lambda[G]$ to give several descriptions of $(\Lambda[G])^G$ which will be useful in describing when Λ is a $\Lambda[G]$-generator as a left $\Lambda[G]$-module. We also include a description of the set of elements x in $\Lambda[G]$ such that $xg = x$ for all $g \in G$ which will be needed later on.

PROPOSITION 1.3. a) *The following three equalities hold:*

$$\left(\sum_{g\in G} g\right)\cdot \Lambda = \left(\sum_{g\in G} g\right)\Lambda[G] = (\Lambda[G])^G = \left\{\sum_{g\in G} g(\lambda)g \mid \lambda\in\Lambda\right\}.$$

b) $(\Lambda[G])^G$ *is isomorphic to* Λ *as a right* Λ*-module with* $\sum_{g\in G} g$ *as a generator.*

c) *The following two equalities hold:*

$$\Lambda\cdot\left(\sum_{g\in G} g\right) = \Lambda[G]\cdot\left(\sum_{g\in G} g\right) = \{x\in\Lambda[G] \mid x\cdot g = x, \forall g\in G\}.$$

This $\Lambda[G]$-Λ^G*-bimodule is again isomorphic to* $\mathrm{Hom}_{\Lambda[G]}(\Lambda, \Lambda[G])$ *by the map* $\phi(f) = f(1)$, *where* Λ *and* $\Lambda[G]$ *are considered as right* $\Lambda[G]$*-modules.*

PROOF. The proof of a) and c) is straightforward and b) follows directly from a).

Let Γ be any ring and M and N be left Γ-modules. The trace, $\tau_M(N)$, of M in N is defined as the subgroup of N generated by $\{\mathrm{Im}\, f \mid f\in \mathrm{Hom}_\Gamma(M,N)\}$. Obviously, $\tau_M(N)$ is a left Γ-submodule of N. Using this definition, Proposition 1.3 gives the following result.

COROLLARY 1.4. a) *Consider* Λ *and* $\Lambda[G]$ *as left* $\Lambda[G]$*-modules. Then* $\tau_\Lambda(\Lambda[G])$ *is equal to the twosided* $\Lambda[G]$*-ideal generated by* $\sum_{g\in G} g$ *which is equal to the abelian group generated by the elements* $\{\lambda_1\cdot\sum_{g\in G}\eta\cdot\lambda_2 \mid \lambda_1,\lambda_2\in\Lambda\}$.
b) *Consider* Λ^{op} *and* $\Lambda^{\mathrm{op}}[G]$ *as left* $\Lambda^{\mathrm{op}}[G]$*-modules. Then* $\tau_\Lambda(\Lambda[G]) = \tau_{\Lambda^{\mathrm{op}}}(\Lambda^{\mathrm{op}}[G])$ *as twosided* $\Lambda[G]$*-ideal.*

Let Γ be any ring and M a left Γ-module. Then M is called a Γ-generator if $\tau_M(\Gamma) = \Gamma$. Using the description of $\tau_\Lambda(\Lambda[G])$ given above we get the following description of when Λ is a $\Lambda[G]$-generator which will be used throughout this paper.

COROLLARY 1.5. Λ *is a* $\Lambda[G]$*-generator if and only if there exist* λ_i *and* γ_i *in* Λ *such that* $1 = \sum_{i=1}^{n} \lambda_i\left(\sum_{g\in G} g\right)\gamma_i$. *Moreover,* Λ *is a* $\Lambda[G]$*-generator if and only if* Λ^{op} *is a* $\Lambda^{\mathrm{op}}[G]$*-generator.*

PROOF. This is a direct conequence of the description of $\tau_\Lambda(\Lambda[G])$ given in the previous corollary.

We have some general consequences of Λ being a $\Lambda[G]$-generator. Part b) establishes a relationship with [V].

PROPOSITION 1.6. *Let Λ be a $\Lambda[G]$-generator. Then we have the following.*

a) *Λ is both a left and a right finitely generated projective Λ^G-module.*

b) *The natural map $\alpha: \Lambda \otimes_{\Lambda^G} \Lambda \to \Lambda[G]$ given by $\alpha(x \otimes y) = x \cdot \sum_{g \in G} g \cdot y$ is an isomorphism of $\Lambda[G]$-bimodules, where the left $\Lambda[G]$-structure on $\Lambda \otimes_{\Lambda^G} \Lambda$ is induced by the left $\Lambda[G]$-structure on the first Λ and the right $\Lambda[G]$-structure by the right $\Lambda[G]$-structure on the second Λ.*

c) *The natural map $\beta: \Lambda[G] \to \mathrm{End}_{\Lambda^G}\Lambda$ given by $\beta(xg)(y) = xg(y)$ is a ring isomorphism, where $g \in G$, x and y in Λ and Λ is considered as a right Λ^G-module.*

d) *If Λ and Λ^G are considered as right Λ^G-modules, then the natural map $\mu: \Lambda \to \mathrm{Hom}_{\Lambda^G}(\Lambda, \Lambda^G)$ given by $\mu(\lambda) = \sum_{g \in G} g \cdot \lambda$ is an isomorphism of right Λ-modules.*

e) *The natural multiplication map $\delta: \Lambda \otimes_{\Lambda^G} \Lambda \to \Lambda$ is a split epimorphism as Λ-bimodule map.*

PROOF. Part a) follows by standard Morita theory (see [AF, p. 195]).

The natural isomorphism of Λ^G-$\Lambda[G]$-bimodules $\gamma: \Lambda \to \mathrm{Hom}_{\Lambda[G]}(\Lambda, \Lambda[G])$ given by $\gamma(\lambda)(1) = \sum_{g \in G} g \cdot \lambda$ induces an isomorphism

$$\xi: \Lambda \otimes_{\Lambda^G} \Lambda \to \mathrm{Hom}_{\Lambda[G]}(\Lambda, \Lambda[G]) \otimes_{\mathrm{End}_{\Lambda[G]}(\Lambda)} \Lambda$$

of $\Lambda[G]$-bimodules given by $\xi(x \otimes y) = \gamma(y) \otimes x$. From [AG1, Th. A.2] we then get a commutative diagram

$$\begin{array}{ccc} \Lambda \otimes_{\Lambda^G} \Lambda & \xrightarrow{1 \otimes \mu} & \Lambda \otimes_{\Lambda^G} \mathrm{Hom}_{\Lambda^G}(\Lambda, \Lambda^G) \\ \downarrow \alpha & & \downarrow \\ \Lambda[G] & \xrightarrow{\beta} & \mathrm{End}_{\Lambda^G}(\Lambda) \end{array}$$

Since Λ is a $\Lambda[G]$-generator, α is surjective. It then follows from [AG1, Th. A.2] that all maps in the diagram are isomorphisms. It is easy to see that α is a $\Lambda[G]$-bimodule homomorphism, so that b) and c) follow.

The map $t = \sum_{g \in G} g$ is a twosided Λ^G-homomorphism from Λ to Λ^G. Since Λ is a $\Lambda[G]$-generator, there are elements x_i and y_i in Λ $(1 \leqq i \leqq n)$, such that $1 = \sum_{i=1}^{n} x_i t y_i$. By the definition of the right Λ-structure on $\mathrm{Hom}_{\Lambda^G}(\Lambda, \Lambda^G)$, we have $\lambda = \sum_{i=1}^{n} x_i + (y_i \lambda)$. Let now $f: \Lambda \to \Lambda^G$ be a right Λ^G-homomorphism. For

$\lambda \in \Lambda$ we get

$$f(\lambda) = \sum_{i=1}^{n} f(x_i t(y_i \lambda) = \sum_{i=1}^{n} f(x_i) t(y_i \lambda) = \sum_{i=1}^{n} t(f(x_i) y_i \lambda).$$

This shows that $f = t \cdot \sum_{i=1}^{n} f(x_i) y_i$, so that $\mathrm{Hom}_{\Lambda^G}(\Lambda, \Lambda^G)$ is generated by t as a right Λ-module. It follows from [AG1, Th. A.2] that μ is a monomorphism. This finishes the proof of d).

The map $p_1 : \Lambda[G] \to \Lambda$ given by $p_1\left(\sum_{g \in G} \lambda_g g\right) = \lambda_1$, where 1 is the identity of G, is clearly a split epimorphism as Λ-bimodule map. Since $f = p_1 \circ \alpha$, it follows from a) that $\delta : \Lambda \otimes_{\Lambda^G} \Lambda \to \Lambda$ is a split epimorphism as Λ-bimodule map.

We now show that Λ is a projective $\Lambda[G]$-module if and onlu if Λ^{op} is a projective $\Lambda^{\mathrm{op}}[G]$-module, obtaining the same symmetry as for generators. This proof also uses an elementwise description of when Λ is a projective $\Lambda[G]$-module which will be used repeatedly throughout this paper without further references.

Define $\varepsilon_l : \Lambda[G] \to \Lambda$ by $\varepsilon_l\left(\sum_{g \in G} \lambda_g g\right) = \sum_{g \in G} \lambda_g$. This map is called the left augmentation map, and is a left $\Lambda[G]$-morphism. Similarly, the map $\varepsilon_r : \Lambda[G] \to \Lambda$ defined by $\varepsilon_r\left(\sum_{g \in G} \lambda_g G\right) = \sum_{g \in G} g(\lambda_{g^{-1}})$ is called the right augmentation map, and is a right $\Lambda[G]$-morphism.

With these definitions we have the following result.

Proposition 1.7. *The following statements are equivalent:*

i) *Λ is a projective left $\Lambda[G]$-module;*

ii) *ε_l is a split epimorphism;*

iii) *there exists a λ in Λ such that $\sum_{g \in G} g(\lambda) = 1$;*

iv) *ε_r is a split epimorphism;*

v) *Λ is a projective right $\Lambda[G]$-module.*

Proof. That i) and ii) are equivalent is obvious. So we want to prove that ii) and iii) are equivalent. The rest then follows since iii) is left right symmetric. Assume first that ε_l is a split epimorphism. Then there exists a $\Lambda[G]$-morphism $\delta : \Lambda \to \Lambda[G]$ such that $\varepsilon_l \circ \delta = \mathrm{id}_\Lambda$. However, δ is uniquely determined by $\delta(1)$ which is in $(\Lambda[G])^G$. Now $(\Lambda[G])^G = \left\{\sum_{g \in G} g(\lambda) g \mid \lambda \in \Lambda\right\}$ by Proposition 1.3, so $\delta(1) = \sum_{g \in G} g(\lambda) g$ for a λ in Λ. Therefore $1 = (\varepsilon_l \circ \delta)(1) = \sum_{g \in G} g(\lambda)$ which shows that iii) follows from ii).

To prove the converse, let λ be in Λ such that $\sum_{g\in G} g(\lambda) = 1$ and define $\delta: \Lambda \to \Lambda[G]$ by $\delta(x) = x \cdot \sum_{g\in G} g(\lambda)g$. Since $\sum_{g\in G} g(\lambda)g \in (\Lambda[G])^G$ we have that δ is a left $\Lambda[G]$-morphism and obviously $(\varepsilon_l \circ \delta)(x) = x$ for all x in Λ. Hence ii) follows from iii) and the proof of the proposition is complete.

As a consequence of this proposition and Corollary 1.5 we get that the notion of pregalois is left right symmetric, which was the main goal of this section.

COROLLARY 1.8. *(Λ, G) is pregalois if and only if (Λ^{op}, G) is pregalois.*

The following result connected with Proposition 1.7 will be needed later on.

PROPOSITION 1.9. *If Λ is a projective $\Lambda[G]$-module, then $\Lambda^G \subset \Lambda$ is a Λ^G-summand both as a left and as a right Λ^G-submodule of Λ. Moreover, if there exists a λ in the centralizer of Λ^G in Λ such that $\sum_{g\in G} g(\lambda) = 1$, then Λ^G is a twosided summand of Λ.*

PROOF. By Proposition 1.7 we have that Λ being a projective $\Lambda[G]$-module is equivalent to that there exists a λ in Λ such that $\sum_{g\in G} g(\lambda) = 1$. Define $v: \Lambda \to \Lambda^G$ by $v(x) = \sum_{g\in G} g(\lambda x)$ and $u: \Lambda \to \Lambda^G$ by $u(x) = \sum_{g\in G} g(x\lambda)$. Then it is easy to verify that v is a right Λ^G-morphism and that u is a left Λ^G-morphism, both being left inverses of the inclusion of Λ^G in Λ. Further, if λ can be choosen in the centralizer of Λ^G in Λ with $\sum_{g\in G} g(\lambda) = 1$, then both u and v as defined above are in fact twosided Λ^G-morphisms.

We now apply Propositions 1.7 and 1.9 to obtain the following description of when (Λ, G) is pregalois.

PROPOSITION 1.10. *If Λ is a left (right) $\Lambda[G]$-generator, then the following are equivalent.*
a) *$\Lambda^G \to \Lambda$ is a split monomorphism as right (left) Λ^G-modules.*
b) *Λ is a right (left) Λ^G-generator.*
c) *Λ is a right (left) $\Lambda[G]$-projective module.*

PROOF. By Proposition 1.9 we have that c) implies a). That a) implies b) is obvious. So we only need to prove that b) implies c).

Since Λ is a Λ^G-generator, we have a surjection $h: \Lambda^n \to \Lambda^G$, and hence Λ^G-morphisms $h_i: \Lambda \to \Lambda^G$ and elements λ_i, for $1 \leqq i \leqq n$, such that $\sum_{i=1}^{n} h_i(\lambda_i) = 1$. By Proposition 1.6 we have that $\mathrm{Hom}_{\Lambda^G}(\Lambda, \Lambda^G)$ is generated as a right Λ^G-module by

$t = \sum_{g \in G} g$. Hence for each $i, h_i = tx_i$ for some x_i in Λ so that $1 = \sum_{i=1}^{n} tx_i(\lambda_i) = \sum_{i=1}^{n} t(x_i\lambda_i)$. This shows by Proposition 1.7 that Λ is a projective $\Lambda[G]$-module.

COMMENTS. The above proposition can be used to get sufficient conditions for Λ being a left $\Lambda[G]$-generator implying that (Λ, G) is pregalois. Consider the class of rings Σ with the property that if $i: \Sigma \to \Gamma$ is an inclusion of rings such that Γ is a finitely generated left projective Σ-module, then $i: \Sigma \to \Gamma$ is a split monomorphism as left Σ-modules. If Λ is such that Λ^G is in this class of rings, then we see that (Λ, G) is pregalois if Λ is a $\Lambda[G]$-generator, by using Proposition 1.10. It is not hard to see that Λ^G is in this class of rings if either a) Λ^G is a finite product of local rings, b) Λ^G is commutative, or c) Λ^G is self-injective (for example semisimple artin).

However, the following example shows that Λ may be a $\Lambda[G]$-generator without (Λ, G) being pregalois: Let $\Lambda = \begin{bmatrix} k & k & k \\ k & k & k \\ k & k & k \end{bmatrix}$, where $k = \mathbf{Z}/2\mathbf{Z}$, and let $g: \Lambda \to \Lambda$ be the automorphism given by conjugation with the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

Then $G = \langle g \rangle$ is a group of order 2. One can show that Λ is a $\Lambda[G]$-generator, but not a projective $\Lambda[G]$-module, by using Corollary 1.5 and Proposition 1.7.

We now give an example where (Λ, G) is pregalois, showing that Λ^G is not always a twosided Λ^G-summand of Λ.

EXAMPLE 1.11. Let k be a field and let Λ be the subring of the lower 4 by 4 matrix ring over k described by

$$\Lambda = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & c & a & 0 \\ d & 0 & 0 & b \end{pmatrix} \,\middle|\, a, b, c, d \in k \right\}.$$

Now, conjugation by the matrix

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

is obviously of order two, and acts as an automorphism ϕ of Λ. Let $G = \{\mathrm{id}, \phi\}$. Using the elementwise description of when Λ is a projective $\Lambda[G]$-module we have from the equality

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \phi \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

that Λ is a projective $\Lambda[G]$-module. Similarly, it is easy to verify that Λ is a $\Lambda[G]$-generator using the elementwise description of when Λ is a $\Lambda[G]$-generator. Further, direct calculations gives that

$$\Lambda^G = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & c & a & 0 \\ c & 0 & 0 & a \end{pmatrix} \middle| a, c \in k \right\}.$$

If char $k \neq 2$, then $\frac{1}{2} + \phi(\frac{1}{2}) = 1$, so by the proposition above we always have that Λ^G is a twosided Λ^G-summand of Λ. However, if char $k = 2$ it is easy to show that there is no twosided Λ^G-complement of Λ^G in Λ.

The final result of this section is also a consequence of Proposition 1.7.

COROLLARY 1.12. *If Λ is a projective $\Lambda[G]$-module, then $M^G = \left(\sum_{g \in G} g\right) \cdot M$ for all left $\Lambda[G]$-modules M.*

PROOF. Obviously $\left(\sum_{g \in G} g\right) \cdot M \subseteq M^G$. In order to prove the other inclusion, let $m \in M^G$. Then $m = \sum_{g \in G} g(\lambda) \cdot m$ for a $\lambda \in \Lambda$ by the elementwise description of projectivity in Proposition 1.7. Hence we get $m = \sum_{g \in G} g(\lambda m)$ because $g(m) = m$ by assumption. Therefore $m \in \left(\sum_{g \in G} g\right) \cdot M$ which completes the proof.

2. Pregalois correspondence.

In the first section we showed that the notion of pregalois action of a finite group G on a ring Λ was left and right symmetric. In this section we show that if (Λ, G) is pregalois then the induced ation of any subgroup H of G makes (Λ, H) pregalois. We also prove that the induced action of G/H on Λ^H for any normal subgroup H of G is pregalois if (Λ, G) is pregalois.

LEMMA 2.1. *Let H be a subgroup of G. If Λ is a projective $\Lambda[G]$-module, then Λ is a projective $\Lambda[H]$-module.*

PROOF. We have that $\Lambda[G]$ is a free $\Lambda[H]$-module, and that any $\Lambda[G]$-summand of $\Lambda[G]$ is also a $\Lambda[H]$-summand. From this it follows that Λ is a projective $\Lambda[H]$-module if Λ is a projective $\Lambda[G]$-module.

LEMMA 2.2. *Let H be a subgroup of G. If Λ is a $\Lambda[G]$-generator, then Λ is a $\Lambda[H]$-generator.*

PROOF. Since $\Lambda[G]$ is clearly a $\Lambda[H]$-generator, any $\Lambda[G]$-generator is also a $\Lambda[H]$-generator.

These two lemmas then immediately give the following.

PROPOSITION 2.3 *Let H be a subgroup of G. If (Λ, G) is pregalois, then (Λ, H) is pregalois.*

As a consequence of this we have the following connection between the rings Λ^G, Λ^H and Λ for H a subgroup of G when (Λ, G) is pregalois.

COROLLARY 2.4. *Let H be a subgroup of G. If (Λ, G) is pregalois, then Λ is both a projective left and a right Λ^H-module, and Λ^H is projective both as a left and as a right Λ^G-module.*

PROOF. The first of these statements is a restatement of Proposition 1.6 in a weaker form.

The second statement uses Proposition 1.9. From the above proposition (Λ, H) is pregalois, so in particular Λ is a projective $\Lambda[H]$-module. Hence, by Proposition 1.9, Λ^H is a summand of Λ both as a left and as a right Λ^H-submodule. But we already know that that Λ is a projective left as well as a right Λ^G-module, and since any Λ^H-summand of Λ is also a Λ^G-summand we have that Λ^H is a projective Λ^G-module on both sides.

Before obtaining the final results about quotient groups, we need some preliminary results. In the following lemma we view Λ and Λ^H as right Λ^G-modules when H is a subgroup of G.

LEMMA 2.5. *Let H be a subgroup of G. If (Λ, G) is pregalois, then $\mathrm{Hom}_{\Lambda^G}(\Lambda^H, \Lambda^G)$ is generated as a right Λ^H-module by the map σ defined by $\sigma(x) = \sum_i g_i(x)$ where the elements g_i are representatives of the left cosets of H in G.*

PROOF. Obviously σ is a Λ^G-morphism from Λ^H to Λ^G since $\psi\colon \Lambda \to \Lambda^H$ defined by $\psi(x) = \sum_{h\in H} h(x)$ is surjective. Let f be a Λ^G-morphism from Λ^H to Λ^G. By Proposition 1.9 Λ^H is a summand of Λ as a right Λ^H-module, so the restriction map $\mathrm{Hom}_{\Lambda^G}(\Lambda, \Lambda^G) \to \mathrm{Hom}_{\Lambda^G}(\Lambda^H, \Lambda^G)$ is a split epimorphism. Hence, there is a Λ^G-morphism $f'\colon \Lambda \to \Lambda^G$ such that $f'|_{\Lambda^H} = f$. From a result of Auslander and Goldman (see [AG2]) we have that $\mathrm{Hom}_{\Lambda^G}(\Lambda, \Lambda^G)$ is generated as a right Λ-module by the map ϕ defined by $\phi(x) = \sum_{g \in G} g(x)$. Therefore there is a λ in Λ with $f' = \phi \cdot \lambda$. Now $f(x) = f'(x) = \phi(\lambda x) = \sum_{g \in G} g(\lambda x) = \sum_i g_i\left(\sum_{h \in H} h(\lambda x)\right) = \sum_i g_i\left(\left(\sum_{h \in H} h(\lambda)\right) x\right)$. Therefore $f = \sigma \cdot \left(\sum_{h \in H} h(\lambda)\right)$ which completes the proof of the lemma.

LEMMA 2.6. *Let H be a normal subgroup of G. Then Λ^H is a faithful $\Lambda^H[G/H]$-module if Λ is a faithful $\Lambda[G]$-module.*

PROOF. Let $\sum_{\tau \in G/H} \lambda_\tau \tau$ be a nonzero element of $\Lambda^H[G/H]$. Consider the element $\sum_{\tau \in G/H} \lambda_\tau\left(\sum_{g \in \tau} g\right)$ in $\Lambda[G]$. Since by assumption Λ is a faithful $\Lambda[G]$-module, there is an x in Λ such that $\sum_{\tau \in G/H} \lambda_\tau\left(\sum_{g \in \tau} g\right)(x) \neq 0$. But $\sum_{h \in H} h(x) \in \Lambda^H$ and we have that $\sum_{\tau \in G/H} \lambda_\tau\left(\sum_{g \in \tau} g\right)(x) = \left(\sum_{\tau \in G/H} \lambda_\tau g_\tau \sum_{h \in H} h\right)(x)$ by choosing one representative g_τ from each coset τ. Hence, $\left(\sum_{\tau \in G/H} \lambda_\tau g_\tau\right)\left(\sum_{h \in H} h(x)\right) = \sum_{\tau \in G/H} \lambda_\tau \tau\left(\sum_{h \in H} h(x)\right) \neq 0$, with $\sum_{h \in H} h(x) \in \Lambda^H$. From this we conclude that Λ^H is a faithful $\Lambda^H[G/H]$-module.

In order to apply these results we need the following.

LEMMA 2.7. *Λ is a $\Lambda[G]$-generator if the following statements hold.*
a) *Λ is a finitely generated projective Λ^G-module.*
b) $\mathrm{Hom}_{\Lambda^G}(\Lambda_{\Lambda^G}, \Lambda^G)$ *is generated as a right Λ-module by $\sum_{g \in G} g$.*
c) *Λ is a faithful $\Lambda[G]$-module.*

PROOF: Statement a) that Λ is a finitely projective Λ^G-module implies that Λ is an $\mathrm{End}_{\Lambda^G}(\Lambda_{\Lambda^G})$-generator. Statement c) is equivalent to $\Lambda[G]$ being a subring of $\mathrm{End}_{\Lambda^G}(\Lambda_{\Lambda^G})$ by the natural map $\psi: \Lambda[G] \to \mathrm{End}_{\Lambda^G}(\Lambda_{\Lambda^G})$ given by $\psi(\lambda_g g)(\lambda) = \lambda_g g(\lambda)$. Statement b) together with a) implies that every element of $\mathrm{End}_{\Lambda^G}(\Lambda_{\Lambda^G})$ is represented by an element in $\Lambda[G]$. So ψ is surjective and therefore $\Lambda[G] \simeq \mathrm{End}_{\Lambda^G}(\Lambda_{\Lambda^G})$. Hence Λ is a $\Lambda[G]$-generator.

We now apply this lemma to the ring Λ^H with the action of the group G/H for a normal subgroup H of G, to prove the main result of this section.

THEOREM 2.8. *Let H be a normal subgroup of G. If (Λ, G) is pregalois, then $(\Lambda^H, G/H)$ is pregalois.*

PROOF. We first prove that Λ^H is a $\Lambda^H[G/H]$-generator by using Lemma 2.7. Assume that (Λ, G) is pregalois. Then by Corollary 2.4, Λ^H is a projective Λ^G-module. Hence, condition a) of Lemma 2.7 is satisfied for $(\Lambda^H, G/H)$. Using Lemma 2.5 we know that $\mathrm{Hom}_{\Lambda^G}(\Lambda^H, \Lambda^G)$ is generated by $\sum_{\tau \in G/H} \tau$, so condition b) of Lemma 2.7 is satisfied for the ring Λ^H and the induced action from the group G/H. Finally, Lemma 2.6 gives that Λ^H is a faithful $\Lambda^H[G/H]$-module, so the third hypothesis of Lemma 2.7 is satisfied. Therefore we conclude that Λ^H is a $\Lambda^H[G/H]$-generator by Lemma 2.7.

It remains to prove that Λ^H is a projective $\Lambda^H[G/H]$-module when (Λ, G) is pregalois. This follows from the next lemma which completes the proof.

LEMMA 2.9. *Let H be a normal subgroup of G. If Λ is a projective $\Lambda[G]$-module, then Λ^H is a projective $\Lambda^H[G/H]$-module.*

PROOF. Assume that Λ is a projective $\Lambda[G]$-module. Then there exists a λ in Λ such that $\sum_{g \in G} g(\lambda) = 1$. Since $\sum_{h \in H} h(\lambda)$ is in Λ^H and $\sum_{\tau \in G/H} \tau\left(\sum_{h \in H} h(\lambda)\right) = \sum_{g \in G} g(\lambda) = 1$, Λ^H is a projective $\Lambda^H[G/H]$-module.

It would be nice to have a ring theoretical charaterization of the fixpoint rings corresponding to subgroups H of G in the case (Λ, G) is pregalois.

3. Factor rings.

In this section we want to look into how the induced action of G on quotient rings of Λ by twosided G-ideals of Λ behaves with respect to the notion of pregalois, where by a twosided G-ideal of Λ we mean a twosided Λ-ideal which is also a $\Lambda[G]$-submodule of Λ. We first prove that if (Λ, G) is pregalois, then $(\Lambda/I, G)$ is pregalois for each G-ideal I in Λ. The converse of this also holds if we restrict to those G-ideals which are contained in the radical of Λ. More precisely, if $(\Lambda/\mathrm{rad}\,\Lambda, G)$ is pregalois, then (Λ, G) is pregalois.

In order to prove the first of these results we need a result connecting the rings $\Lambda[G]$ and $(\Lambda/I)[G]$ for a G-ideal I of Λ.

LEMMA 3.1. *Let I be a twosided G-ideal of Λ. Then $\alpha: \Lambda[G] \to (\Lambda/I)[G]$ defined by $\alpha\left(\sum_{g \in G} \lambda_g g\right) = \sum_{g \in G} \lambda'_g g$ induces an isomorphism $\Lambda[G]/(I \cdot \Lambda[G]) \simeq (\Lambda/I)[G]$ where λ' denotes the residue class of λ in Λ/I.*

PROOF. It is easy to see that α is a ring map and that the kernel of α is $I \cdot \Lambda[G]$, which completes the proof of the lemma.

PROPOSITION 3.2. *Let I be a twosided G-ideal in Λ. Then*
a) *Λ/I is a $(\Lambda/I)[G]$-generator if Λ is a $\Lambda[G]$-generator;*
b) *Λ/I is a projective $(\Lambda/I)[G]$-module if Λ is a projective $\Lambda[G]$-module;*
c) *$(\Lambda/I, G)$ is pregalois if (Λ, G) is pregalois.*

PROOF. Using the above lemma and the elementwise description of Λ being a $\Lambda[G]$-generator given in Corollary 1.5, statement a) is easily established.

In the same way, using the elementwise description of Proposition 1.7 of when Λ is a projective $\Lambda[G]$-module, statement b) follows.

The main result of this section is the following result.

THEOREM 3.3. *Let (Λ, G) be as usual and let* rad Λ *denote the radical of Λ. Then the following are true.*
a) *If $\Lambda/\text{rad}\,\Lambda$ is a $(\Lambda/\text{rad}\,\Lambda)[G]$-generator, then Λ is a $\Lambda[G]$-generator.*
b) *If $\Lambda/\text{rad}\,\Lambda$ is a projective $(\Lambda/\text{rad}\,\Lambda)[G]$-module, then Λ is a projective $\Lambda[G]$-module.*
c) *If the induced action of G on $\Lambda/\text{rad}\,\Lambda$ is pregalois, then (Λ, G) is pregalois.*

PROOF. Assume that $\Lambda/\text{rad}\,\Lambda$ is a $(\Lambda/\text{rad}\,\Lambda)[G]$-generator. Then there exist λ_i and μ_i in Λ such that $\sum_i \lambda_i'\left(\sum_{g\in G} g\right)\mu_i' = 1$, where λ_i' and μ_i' are the images of λ_i and μ_i by the natural map from Λ to $\Lambda/\text{rad}\,\Lambda$, respectively. Hence $u = 1 - \sum_i \lambda_i\left(\sum_{g\in G} g\right)\mu_i$ is in $(\text{rad}\,\Lambda)\cdot\Lambda[G] \subseteq \text{rad}\,(\Lambda[G])$. So $\sum_i \lambda_i\left(\sum_{g\in G} g\right)\mu_i$ is a unit in $\Lambda[G]$. Therefore the trace, $\tau_\Lambda(\Lambda[G])$, of Λ in $\Lambda[G]$ contains a unit (see Corollary 1.4) and must therefore be all of $\Lambda[G]$. So Λ is a $\Lambda[G]$-generator.

To prove statement b) assume that $\Lambda/\text{rad}\,\Lambda$ is a projective $(\Lambda/\text{rad}\,\Lambda)[G]$-module. Then there exists a λ in Λ such that $\sum_{g\in G} g(\lambda') = 1$ in $\Lambda/\text{rad}\,\Lambda$, where λ' is the image of λ by the natural map from Λ to $\Lambda/\text{rad}\,\Lambda$. So $1 - \sum_{g\in G} g(\lambda) \in \text{rad}\,\Lambda$ and therefore $\sum_{g\in G} g(\lambda)$ is a twosided unit with unique inverse μ. Since $\sum_{g\in G} g(\lambda)$ is in Λ^G, also μ is in Λ^G, but then $1 = \left(\sum_{g\in G} g(\lambda)\right)\mu = \sum_{g\in G} g(\lambda\mu)$. This shows that Λ is a projective $\Lambda[G]$-module, which completes the proof of statement b) as well as the proof of the whole theorem.

4. Galois actions.

In this section we introduce the notion of galois and extend the results from Section 2 and Section 3 by substituting pregalois by galois.

Let Λ and G be as usual. We say that (Λ, G) is left (right) galois if the following two conditions are satisfied:

a) (Λ, G) is pregalois,

b) $\Lambda^G/\operatorname{ann}_{\Lambda^G} S$ is a semisimple artinian ring for each simple left (right) Λ-module S. (Here $\operatorname{ann}_{\Lambda^G} S$ denotes the annihilator of S as a Λ^G-module.)

We say that (Λ, G) is galois if (Λ, G) is both left and right galois.

The notions of left galois and right galois are equivalent for the class of rings Λ such that $\Lambda/\operatorname{ann}_\Lambda S$ is a simple artinian ring for each simple left and right Λ-module S. This follows because for this class of rings there is a duality between the left and right simple Λ-modules. Polynomial identity rings satisfy this property as well as rings which are finitely generated as modules over their centers and rings which are semisimple artinian modulo their radicals.

We prove that the notion of galois and pregalois coincide for basic rings, where by a basic ring we mean a ring such that $\Lambda/\operatorname{ann}_\Lambda S$ is a division ring for each simple left and right Λ-module S. In particular the notion of galois and pregalois coincide for commutative rings as well as rings which are basic semisimple artinian modulo their radicals.

For simplicity, we will let simple and semisimple mean simple artinian and semisimple artinian respectively, both when used for modules and rings through the rest of this paper.

We first prove that the results from Section 3 hold if we substitute pregalois by left (right) galois.

PROPOSITION 4.1. *Let I be a G-ideal of Λ. If (Λ, G) is left (right) galois, then $(\Lambda/I, G)$ is left (right) galois.*

PROOF. From Proposition 3.2 we know that $(\Lambda/I, G)$ is pregalois. Therefore it is enough to prove that $(\Lambda/I)^G/\operatorname{ann}_{(\Lambda/I)^G} S$ is semisimple for each simple Λ/I-module S. Let S be a simple Λ/I-module. Then I is contained in $\operatorname{ann}_\Lambda S$, so $I \cap \Lambda^G \subseteq \operatorname{ann}_\Lambda S \cap \Lambda^G = \operatorname{ann}_{\Lambda^G} S$. However, (Λ, G) being galois implies that the fixpoint functor is exact, so

$$\Lambda^G/\operatorname{ann}_{\Lambda^G} S = (\Lambda^G/I \cap \Lambda^G)/(\operatorname{ann}_{\Lambda^G} S/I \cap \Lambda^G) = (\Lambda/I)^G/\operatorname{ann}_{(\Lambda/I)^G} S$$

since obviously $I^G = I \cap \Lambda^G$. Further, $\Lambda^G/\operatorname{ann}_{\Lambda^G} S$ is semisimple by assumption, which completes the proof of the proposition.

The analog of Theorem 3.3 is also valid when pregalois is replaced by left (right) galois.

THEOREM 4.2. *If the induced action of G on $\Lambda/\operatorname{rad}\Lambda$ is left (right) galois, then (Λ, G) is left (right) galois.*

PROOF. This follows from the fact that $\operatorname{rad}\Lambda$ is contained in the annihilator of all simple left and right Λ-modules.

COMMENT. The following is an easily verified consequence of Theorem 4.2. Suppose $\Lambda/\mathrm{rad}\,\Lambda$ is semisimple and Λ is a $\Lambda[G]$-generator. Then the following statements are equivalent. a) (Λ, G) is galois, b) $(\Lambda/\mathrm{rad}\,\Lambda)^G$ is semisimple, c) $(\Lambda/\mathrm{rad}\,\Lambda)[G]$ is semisimple.

We next want to give an example showing that the notions of galois and pregalois do not coincide even when the ring is a finite dimensional algebra.

EXAMPLE 4.3. Let k be a field of characteristic 2 and let $\Lambda = M_2(k)$, the ring of two by two matrices over k. Let ϕ be the inner automorphism of Λ obtained by conjugation by the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Using that char $k = 2$, one easily sees that the explicit action of ϕ is given by $\phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & b \\ a+b+c+d & b+d \end{pmatrix}$ and that ϕ has order two. Let $G = \{\mathrm{id}, \phi\}$. Then an easy calculation gives that $\Lambda^G = \left\{\begin{pmatrix} a & 0 \\ b & a \end{pmatrix} \middle| \, a, b \in k\right\}$. To prove that Λ is a projective $\Lambda[G]$-module, we just use the element $\lambda = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ in the elementwise description of when Λ is a projective $\Lambda[G]$-module, since we have that $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \phi\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Similarly to prove that Λ is a $\Lambda[G]$-generator, we use the elements $\lambda_1 = \mu_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $\lambda_2 = \mu_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ in the elementwise description of when Λ is a $\Lambda[G]$-generator, since $\lambda_1^2 + \lambda_2^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\lambda_1\phi(\lambda_1) + \lambda_2\phi(\lambda_2) = 0$. Therefore (Λ, G) is pregalois, however the simple Λ-module is not semisimple as a Λ^G-module.

In order to prove that the results from Section 2 hold for subgroups H of G when we replace pregalois by galois, we need some intermediate results.

PROPOSITION 4.4. *Assume that (Λ, G) is pregalois and that $\Lambda/\mathrm{ann}_\Lambda S$ is a simple ring for all simple left (right) Λ-modules S. Then the following hold.*

a) *If T is simple left (right) $\Lambda[G]$-module, then T is semisimple as a left (right) Λ-module.*

b) *If T is a simple left (right) Λ^G-module, then $\Lambda \otimes_{\Lambda^G} T$ $(T \otimes_{\Lambda^G} \Lambda)$ is a semisimple left (right) Λ-module.*

c) *$\Lambda[G]/\mathrm{ann}_{\Lambda[G]} T$ is a simple ring for each simple $\Lambda[G]$-module T.*

d) *$\Lambda^G/\mathrm{ann}_{\Lambda^G} T$ is a simple ring for each simple Λ^G-module T.*

PROOF. The proof of b) follows from that of a) by the equivalence $\Lambda \otimes_{\Lambda^G} -$: $\mathrm{Mod}\,\Lambda^G \to \mathrm{Mod}\,\Lambda[G]$, hence in order to establish the two first statements it suffices to prove a). Therefore let T be a simple $\Lambda[G]$-module. Since Λ is a $\Lambda[G]$-generator, $T \simeq \Lambda/I$ where I is a maximal left G-ideal of Λ. But then there exists a maximal left Λ-ideal J of Λ containing I. Consider $\bigcap_{g\in G} gJ$, which is a left G-ideal of Λ. Therefore $I = \bigcap_{g\in G} gJ$ by the maximality of I since obviously I is contained in $\bigcap_{g\in G} gJ$. Hence T is a semisimple Λ-module since each gJ is a maximal left Λ-ideal.

That the statements d) and c) are equivalent follows since Λ^G and $\Lambda[G]$ are Morita equivalent rings. Therefore it is enough to prove c). Let T be a simple $\Lambda[G]$-module. Then from a) we know that T is semisimple as a Λ-module. Hence, $\Lambda/\text{ann}_\Lambda T$ is a semisimple ring by the assumption. From this it follows that $(\Lambda/\text{ann}_\Lambda T)[G]$ is a module of finite length over Λ, hence also of finite length over $\Lambda[G]$. But this implies that $\Lambda[G]/\text{ann}_{\Lambda[G]} T$ is a simple ring since it is a quotient of $(\Lambda/\text{ann}_\Lambda T)[G]$.

LEMMA 4.5. *If* (Λ,G) *is left (right) galois, then* $\Lambda/\text{ann}_\Lambda S$ *is a simple ring for each simple left (right)* Λ*-module* S.

PROOF. To prove this let S be a simple left Λ-module. Then we know by the assumption that $\Lambda^G/\text{ann}_{\Lambda^G} S$ is a semisimple ring. Obviously, $\Lambda\cdot(\text{ann}_{\Lambda^G} S)\cdot\Lambda$ is contained in the annihilator of S as a Λ-module. So there exists a Λ-epimorphism from $\Lambda/\Lambda\cdot(\text{ann}_{\Lambda^G} S)\cdot\Lambda$ to $\Lambda/\text{ann}_\Lambda S$. But $\Lambda/\Lambda\cdot(\text{ann}_{\Lambda^G} S)\cdot\Lambda$ is an artinian Λ^G-module and therefore also artinian as a Λ-module. Hence $\Lambda/\text{ann}_\Lambda$ S is a simple ring.

We next want to give some other characterizations of (Λ, G) being left (right) galois.

PROPOSITION 4.6. *Assume that* (Λ, G) *is pregalois. Then the following statements are equivalent:*

i) (Λ, G) *is left (right) galois.*

ii) $\Lambda/\text{ann}_\Lambda S$ *is a simple ring for each simple left (right)* Λ*-module* S, *and* $(\Lambda/\text{ann}_\Lambda T)[G]$ *is a semisimple ring for each simple left (right)* $\Lambda[G]$*-module* T.

iii) $\Lambda/\text{ann}_\Lambda S$ *is a simple ring for each simple left (right)* Λ*-module* S, *and* $\Lambda[G] \otimes_\Lambda S$ *is a semisimple* $\Lambda[G]$*-module for each simple left (right)* Λ*-module* S.

iv) $\Lambda/\text{ann}_\Lambda S$ *is a simple ring for each simple left (right)* Λ*-module* S, *and each simple* Λ*-module* S *is semisimple as a* Λ^G*-module.*

PROOF. We prove this by proving that i) implies ii), that ii) implies iii), that iii) implies iv) and that iv) implies i). Assume therefore that i) holds. Lemma 4.5 then states that $\Lambda/\text{ann}_\Lambda S$ is a simple ring for each simple left Λ-module S. Therefore it only remains to prove the other half of ii). Let T be a simple left

$\Lambda[G]$-module. Then T is semisimple as a Λ-module by Proposition 4.4. By i) $\Lambda^G/\mathrm{ann}_{\Lambda^G} T$ is therefore a semisimple ring. But from Proposition 3.2 we know that $(\Lambda/\mathrm{ann}_\Lambda T, G)$ is pregalois since $\mathrm{ann}_\Lambda T$ is a G-ideal in Λ. Therefore $(\Lambda/\mathrm{ann}_\Lambda T)^G$ and $(\Lambda/\mathrm{ann}_\Lambda T)[G]$ are Morita equivalent rings. But since the fixpoint functor is exact when (Λ, G) is pregalois, it follows that $(\Lambda/\mathrm{ann}_\Lambda T)^G \simeq \Lambda^G/\mathrm{ann}_{\Lambda^G} T$. This proves that ii) follows from i).

The first parts of ii) and iii) are identical, so we prove that the second part of iii) follows from ii). Let S be a simple Λ-module. Since $\bigcap_{g\in G} g(\mathrm{ann}_\Lambda S)$ is a G-ideal of Λ, we know that S is a Λ-submodule of a $\Lambda[G]$-module of finite length, hence that S is a Λ-submodule of a simple $\Lambda[G]$-module T. Now $\Lambda[G] \otimes_\Lambda S$ is annihilated by $\mathrm{ann}_\Lambda T$ and is therefore a $(\Lambda/\mathrm{ann}_\Lambda T)[G]$-module. But from ii) this is a semisimple ring, so $\Lambda[G] \otimes_\Lambda S$ is a semisimple $\Lambda[G]$-module.

To prove that iii) implies iv) let S be a simple Λ-module. Then $\Lambda[G] \otimes_\Lambda S$ is semisimple as a $\Lambda[G]$-module. Consider $(\Lambda[G] \otimes_\Lambda S)^G$ which is equal to $\left(\sum_{g\in G} g\right)\cdot(\Lambda[G] \otimes_\Lambda S)$ by Corollary 1.12. But this is equal to $\left(\sum_{g\in G} g\right)\Lambda \otimes_\Lambda S$ which is isomorphic to S as a Λ^G-module. Hence, S is semisimple as a Λ^G-module.

That iv) implies i) is a direct consequence of Proposition 4.4d). This finishes the proof of the proposition.

Before we go on to consider the induced action of subgroups and factor groups of G, we include a result about when galois and pregalois coincide, based on the above characterization of galois.

PROPOSITION 4.7. *Assume $\Lambda/\mathrm{ann}_\Lambda S$ is a division ring for each simple left or right Λ-module S. Then (Λ, G) is pregalois if and only if (Λ, G) is galois.*

PROOF. In the proof of this we use the characterization of (Λ, G) being galois given in part iii) of the last proposition. Assume that (Λ, G) is pregalois and let T be a simple $\Lambda[G]$-module. Since $\Lambda/\mathrm{ann}_\Lambda T$ is a semisimple ring by Proposition 4.4, it follows from the hypothesis of the proposition that the sets of maximal left-, maximal right- and maximal twosided Λ-ideals coincide. Therefore $\Lambda/\mathrm{ann}_\Lambda T$ is isomorphic to T as a $\Lambda[G]$-module. Now $(\Lambda/\mathrm{ann}_\Lambda T, G)$ is pregalois by Proposition 3.2 because $\mathrm{ann}_\Lambda T$ is a G-ideal of Λ. Therefore $(\Lambda/\mathrm{ann}_\Lambda T)[G]$ is a simple ring since it has a simple generator. This obviously implies that iii) of the previous proposition is satisfied, hence (Λ, G) is galois.

We are now in a position where we can prove that the notion of left (right) galois behaves nicely with respect to subgroups and quotient groups.

THEOREM 4.8. *Let H be a subgroup of G. If (Λ, G) is left (right) galois, then (Λ, H) is left (right) galois.*

PROOF. From earlier results we know that (Λ, H) is pregalois. (See Proposition 2.3). Therefore by Proposition 4.6 it is enough to prove that $(\Lambda/\operatorname{ann}_\Lambda T)[H]$ is a semisimple ring for each simple $\Lambda[H]$-module T. Let T be a simple $\Lambda[H]$-module. Then T is a simple $\Lambda[H]$-submodule of a simple $\Lambda[G]$-module M. Further $\operatorname{ann}_\Lambda M \subseteq \operatorname{ann}_\Lambda T$ so $(\Lambda/\operatorname{ann}_\Lambda T)[H]$ is a quotient of $(\Lambda/\operatorname{ann}_\Lambda M)[H]$. Therefore it is enough to prove that $(\Lambda/\operatorname{ann}_\Lambda M)[H]$ is semisimple. Now $(\Lambda/\operatorname{ann}_\Lambda M)[G]$ contains $(\Lambda/\operatorname{ann}_\Lambda M)[H]$ as a twosided $(\Lambda/\operatorname{ann}_\Lambda M)[H]$-summand. From this it follows that any $(\Lambda/\operatorname{ann}_\Lambda M)[H]$-module N is a direct summand of $(\Lambda/\operatorname{ann}_\Lambda M)[G] \otimes_{(\Lambda/\operatorname{ann}_\Lambda M)[H]} N$. Since $(\Lambda/\operatorname{ann}_\Lambda M)[G]$ is a semisimple ring by assumption, $(\Lambda/\operatorname{ann}_\Lambda M)[G] \otimes_{(\Lambda/\operatorname{ann}_\Lambda M)[H]} N$ is a projective $(\Lambda/\operatorname{ann}_\Lambda M)[G]$-module. But $(\Lambda/\operatorname{ann}_\Lambda M)[G]$ is also free as a $(\Lambda/\operatorname{ann}_\Lambda M)[H]$-module, which then gives that all $(\Lambda/\operatorname{ann}_\Lambda M)[H]$-modules are projective, showing that $(\Lambda/\operatorname{ann}_\Lambda M)[H]$ is a semisimple ring. This completes the proof of the theorem.

THEOREM 4.9. *Let H be a normal subgroup of G. If (Λ, G) is left (right) galois, then $(\Lambda^H, G/H)$ is left (right) galois.*

PROOF. Let S be a simple Λ^H-module. Then $\Lambda \otimes_{\Lambda^H} S$ is a simple $\Lambda[H]$-module since (Λ, H) is pregalois. But then $\Lambda \otimes_{\Lambda^H} S$ is semisimple as a Λ-module by Proposition 4.4. By assumption $\Lambda^G/\operatorname{ann}_{\Lambda^G}(\Lambda \otimes_{\Lambda^H} S)$ is therefore a semisimple ring. However S is a submodule of $\Lambda \otimes_{\Lambda^H} S$ as a Λ^H-module, hence also as a Λ^G-module. Therefore S is semisimple as a Λ^G-module which completes the proof of the theorem.

5. Free actions.

The main aim of this section is to relate the notion of galois to the induced action of G on the isomorphism classes of simple Λ-modules.

We first prove that if $\Lambda/\operatorname{rad}\Lambda$ is a semisimple ring and the induced action of G on the isomorphism classes of the simple Λ-modules is free, then (Λ, G) is galois. After this we consider the situation where Λ is an R-algebra where R is a commutative ring such that $\Lambda_{\underline{m}}/\operatorname{rad}\Lambda_{\underline{m}}$ is a finite dimensional $R/\underline{m}$-algebra for each maximal ideal $\underline{m}$ of R, and where G acts a group of R-automorphisms. Then using the above local result, we prove that if the induced action of G on the isomorphism classes of simple Λ-modules is free, then (Λ, G) is galois.

We next consider the situation where $R/\underline{m}$ is an algebraically closed field and that $\Lambda_{\underline{m}}/\operatorname{rad}\Lambda_{\underline{m}}$ is a product of a finite number of copies of the field $R/\underline{m}$, for each maximal ideal $\underline{m}$ of R. In this situation we prove that (Λ, G) being galois implies that the induced action of G on the isomorphism classes of simple Λ-modules is free. If R is a finitely generated algebra over an algebraically closed field k and Λ/I is a one dimensional vectorspace over k for each maximal ideal I of Λ, then Λ satisfies the conditions above. Hence for this class of rings the notions of pregalois and galois are equivalent to that the induced action of G on the isomorphism classes of simple Λ-modules is free.

PROPOSITION 5.1. *Assume that $\Lambda/\mathrm{rad}\,\Lambda$ is a semisimple ring. If the induced action of G on the isomorphism classes of simple Λ-modules is free, then (Λ, G) is galois.*

PROOF. We first prove that $\Lambda/\mathrm{rad}\ \Lambda$ is a projective $(\Lambda/\mathrm{rad}\,\Lambda)[G]$-module by using the elementwise description of this. Let $e_1, e_2, \ldots, e_n$ be the central primitive idempotents of $\Lambda/\mathrm{rad}\,\Lambda$. From the assumptions it follows that the induced action of G on the central primitive idempotents e_i of $\Lambda/\mathrm{rad}\,\Lambda$ is free. Now choose one idempotent $e_{i_j}; j = 1, \ldots, m$ from each of these orbits. It is then clear that $\sum_{g \in G} g\left(\sum_{j=1}^{n} e_{i_j}\right) = \sum_{i=1}^{n} e_i = 1$ which shows that $\Lambda/\mathrm{rad}\,\Lambda$ is a projective $(\Lambda/\mathrm{rad}\ \Lambda)[G]$-module.

Next we ue the elementwise description of $\Lambda/\mathrm{rad}\,\Lambda$ being a $(\Lambda/\mathrm{rad}\,\Lambda)[G]$-generator from Corollary 1.5 to complete the proof that $(\Lambda/\mathrm{rad}\ \Lambda, G)$ is pregalois. Let $\lambda_i = \mu_i = e_i$. Then we have that $\sum_{i=1}^{n} \lambda_i \cdot \left(\sum_{g \in G} g\right) \mu_i = 1$. Hence $(\Lambda/\mathrm{rad}\ \Lambda, G)$ is pregalois. But then (Λ, G) is pregalois by Theorem 3.3.

In order to complete the proof that (Λ, G) is galois, let S be a simple Λ-module. For each g in G, let S^g denote the Λ-module obtained from S by using the opration of Λ on S defined by $\lambda \cdot s = g(\lambda)s$. Then $\coprod_{g \in G} S^g$ is a $\Lambda[G]$-module by the action $\lambda_h h \cdot (s_g) = ((g(\lambda_h)s_{h^{-1}g})_g)$. It is easy to see that this $\Lambda[G]$-module is simple. Hence, $\left(\coprod_{g \in G} S^g\right)^G$ is a simple Λ^G-module. However, the fixpoint set is easily seen to be $\{(s_g) \mid s_g = s, \forall g \in G, \text{ where } s \in S\}$. Now $\left(\coprod_{g \in G} S^g\right)^G$ is a Λ^G-submodule of $\left(\coprod_{g \in G} S^g\right)$ and the projection from $\left(\coprod_{g \in G} S^g\right)$ onto S is a Λ^G-morphism mapping $\left(\coprod_{g \in G} S^g\right)^G$ onto S. Hence S is a simple Λ^G-module and therefore $\Lambda^G/\mathrm{ann}_{\Lambda^G} S$ is a simple ring according to Proposition 4.4.

Now let R be a commutative ring, Λ an R-algebra such the $\Lambda_{\underline{m}}/\mathrm{rad}\,\Lambda_{\underline{m}}$ is a finite dimensional $R/\underline{m}$-algebra for each maximal ideal $\underline{m}$ in R, and assume that the induced action of G on the isomorphism classes of simple Λ-modules is free. Then we have a global version of the previous result.

PROPOSITION 5.2. *Let R, Λ and G be as above. If the induced action of G on the isomorphism classes of simple Λ-modules is free, then (Λ, G) is galois.*

PROOF. We first show that (Λ, G) is pregalois by showing that $\tau_\Lambda \Lambda[G] = \Lambda[G]$

and that the map $\sigma : \Lambda \to \Lambda^G$ given by $\sigma(\lambda) = \sum_{g \in G} g(\lambda)$ is surjective. The first of these claims follows by the previous proposition since it implies that $(\Lambda[G]/\tau_\Lambda \Lambda[G])_{\underline{m}} = 0$ for each maximal ideal $\underline{m}$ of R. The second claim follows in the same way since the previous proposition implies that $\sigma_{\underline{m}}$ is an epimorphism for each maximal ideal $\underline{m}$ *of* R. Therefore σ is an epimorphism, which implies that Λ is a projective $\Lambda[G]$-module.

Next let S be a simple Λ-module. Then $S_{\underline{m}}$ is nonzero for some maximal ideal $\underline{m}$ of R. Hence $S_{\underline{m}}$ is a $\Lambda_{\underline{m}}/\operatorname{rad}\Lambda_{\underline{m}}$-module. But then S is a simple $\Lambda_{\underline{m}}/\operatorname{rad}\Lambda_{\underline{m}}$-module, which by the local version in the previous proposition implies that S is semisimple as a $(\Lambda_{\underline{m}}/\operatorname{rad}\Lambda_{\underline{m}})^G$-module. Hence, S is semisimple as a Λ^G-module.

We next consider the situation where $\Lambda/\operatorname{rad}\Lambda$ is a finite dimensional basic k-algebra, where k is algebraically closed and where G operates as a group of k-automorphisms. Under these assumptions we have the following.

PROPOSITION 5.3. *Let Λ and G be as above. Then (Λ, G) is galois if and only if the induced action of G on the isomorphism classes of simple Λ-modules is free.*

PROOF. From the previous proposition we only have to prove that if (Λ, G) is galois, then the induced action of G on the isomorphism classes of simple Λ-modules is free. This will be done by using a dimension argument. From the hypothesis $\Lambda/\operatorname{rad}\Lambda$ is isomorphic to a product of copies of k with the induced action of G as a group of k-automorphisms. By dividing the algebra summands of $\Lambda/\operatorname{rad}\Lambda$ into orbits, one can treat one orbit at a time. Then we have that $(\Lambda/\operatorname{rad}\Lambda)^G$ is equal to a product of copies of k, one for each orbit. Now using the isomorphism $(\Lambda/\operatorname{rad}\Lambda)[G] \simeq \operatorname{End}_{(\Lambda/\operatorname{rad}\Lambda)^G}(\Lambda/\operatorname{rad}\Lambda)$ and counting dimensions, we get that the dimension on the right hand side of the isomorphism sign is equal to the sum of the squares of the size of the orbits. However, this is strictly less than $|G| \cdot \dim_k(\Lambda/\operatorname{rad}\Lambda)$ unless all the orbits have $|G|$ elements. This completes the proof of the proposition.

We will now use this result to prove the main result of this section.

Let k be an algebraically closed field and let Λ be an R-algebra with R a commutative k-algebra. Assume further that G operates as a group of R-algebra automorphisms and that $\Lambda_{\underline{m}}/\operatorname{rad}\Lambda_{\underline{m}}$ is a basic finite dimensional k-algebra for each maximal ideal $\underline{m}$ of R.

THEOREM 5.4. *Let Λ and G be as above. Then (Λ, G) is galois if and only if the induced action of G on the isomorphism classes of the simple Λ-modules is free.*

PROOF. The proof of this follows by standard localization techniques using the maximal ideals of R.

6. Galois actions and galois coverings.

In this section we want to compare the notion of galois action developed in this paper with the notion of galois covering for finite dimensional algebras introduced by K. Bongartz, P. Gabriel and C. Riedtmann. (See [BG, G, R].) The main aim of this section is to show that finite galois coverings for finite dimensional algebras correspond to galois extensions developed in this paper.

Before doing this, we recall the notion of a k-category for a field k and the definition of a covering functor.

Let k be an algebraically closed field. A k-category $\underline{\Gamma}$ is a preadditive category in which the morphism sets are k-vectorspaces, and the compositions are k-bilinear. We will let $\operatorname{Hom}(x, y)$ and $\operatorname{End}(x)$ denote the space of morphisms between two objects x and y and the ring of endomorphisms of an object x respectively when it is clear to which category these belong. The k-categories mostly studied in the representation theory of finite dimensional algebras satisfy the following additional properties; a) for each object x in $\underline{\Gamma}$, $\operatorname{End}(x)$ is a local ring, b) for each pair of objects x and y in $\underline{\Gamma}$, $\dim_k(\operatorname{Hom}(x, y)) < \infty$, c) distinct objects of $\underline{\Gamma}$ are nonisomorphic, and d) for each x in $\underline{\Gamma}$ there are only a finite number of objects y in $\underline{\Gamma}$ such that $\operatorname{Hom}(x, y) \neq 0$ or $\operatorname{Hom}(y, x) \neq 0$. A k-category satisfying a), b), c) and d) above, is called a locally bounded k-category.

If $\underline{\Gamma}$ and $\underline{\Lambda}$ are locally bounded k-categories, then a k-linear functor $\underline{F}: \underline{\Gamma} \to \underline{\Lambda}$ is called a covering functor if, a) $\underline{F}$ is surjective on objects, b) for each x in $\underline{\Gamma}$ and a in $\underline{\Lambda}$, $\underline{F}$ induces isomorphisms

$$\coprod_{y \in \underline{F}^{-1}(a)} \operatorname{Hom}(y, x) \to \operatorname{Hom}(a, \underline{F}(x))$$

and

$$\coprod_{y \in \underline{F}^{-1}(a)} \operatorname{Hom}(x, y) \to \operatorname{Hom}(\underline{F}(x), a).$$

We now consider covering functors where $\underline{\Gamma}$ has only a finite number of objects, and hence $\underline{\Lambda}$ also has only a finite number of objects. Then we associate to $\underline{\Gamma}$ and $\underline{\Lambda}$ the following rings

$$\Gamma = \coprod_{x, y \in \underline{\Gamma}} \operatorname{Hom}(x, y)$$

and

$$\Lambda = \coprod_{x, y \in \underline{\Lambda}} \operatorname{Hom}(x, y),$$

where the addition is componentwise and the multiplication is usual matrix multiplication using composition in $\underline{\Gamma}$ and $\underline{\Lambda}$ respectively, i.e. $(f_{x,y})_{x,y \in \underline{\Gamma}} \cdot (f'_{x,y})_{x,y \in \underline{\Gamma}} = (f''_{x,y})_{x,y \in \underline{\Gamma}}$, where $f''_{x,y} = \sum_{z \in \underline{\Gamma}} f_{x,z} \circ f'_{z,y}$. Then it is well known that Γ and Λ are basic finite dimensional k-algebras. $\underline{F}$ induces a k-linear map $F:\Gamma \to \Lambda$. Further, the two isomorphisms above will in general induce two maps $\phi_{\alpha,\underline{F}}:\Lambda \to \Gamma$ and $\phi_{\omega,\underline{F}}:\Lambda \to \Gamma$ defined by $(\phi_{\alpha,\underline{F}}(f_{a,b}))_{x,y} = g_{x,y}$, where $\sum_{x\in\underline{F}^{-1}(a)} \underline{F}(g_{x,y}) = f_{a,\underline{F}(y)}$ and $(\phi_{\omega,\underline{F}}(f_{a,b}))_{x,y} = h_{x,y}$, where $\sum_{y\in\underline{F}^{-1}(b)} \underline{F}(h_{x,y}) = f_{\underline{F}(x),b}$. Obviously, $\phi_{\alpha,\underline{F}}$ and $\phi_{\omega,\underline{F}}$ are k-linear and they coincide if and only if for each $a,b\in\underline{\Lambda}$, $x_0\in\underline{F}^{-1}(a)$ and $y_0\in\underline{F}^{-1}(b)$, the diagram below commutes, where p_{x_0} and p_{y_0} are the natural projections.

$$\begin{array}{ccc}
\operatorname{Hom}(a,b) & \xrightarrow{\underline{F}^{-1}} & \coprod_{y\in\underline{F}^{-1}(b)} \operatorname{Hom}(x_0,y) \\
\downarrow{\scriptstyle \underline{F}^{-1}} & & \downarrow{\scriptstyle p_{y_0}} \\
\coprod_{x\in\underline{F}^{-1}(a)} \operatorname{Hom}(x,y_0) & \xrightarrow{p_{x_0}} & \operatorname{Hom}(x_0,y_0)
\end{array}$$

Proposition 6.1. *The morphisms $\phi_{\alpha,\underline{F}}$ and $\phi_{\omega,\underline{F}}$ are both k-algebra morphisms and the induced Λ-bimodule structure on Γ by $\phi_{\omega,\underline{F}}$ on the left and by $\phi_{\alpha,\underline{F}}$ on the right, makes F a Λ-bimodule map.*

Proof. We first prove that $\phi_{\alpha,\underline{F}}$ and $\phi_{\omega,\underline{F}}$ are k-algebra inclusions. Clearly, both $\phi_{\alpha,\underline{F}}$ and $\phi_{\omega,\underline{F}}$ are k-linear inclusions and take the identity to the identity. Therefore it suffices to prove that they respect the multiplication. We carry out the calculation for $\phi_{\alpha,F}$. Let $(f_{a,b})$ and $(f'_{a,b})$ be elements of Λ. Then $(f_{a,b})\cdot(f'_{a,b}) = (f''_{a,b})$, where $f''_{a,b} = \sum_{c\in\underline{\Lambda}} f_{a,c}\cdot f'_{c,b}$. Now consider $\phi_{\alpha,\underline{F}}(f_{a,b})\cdot\phi_{\alpha,\underline{F}}(f'_{a,b}) = (g_{x,y})\cdot(g'_{x,y}) = (g''_{x,y})$, where $g''_{x,y} = \sum_{z\in\underline{\Gamma}} g_{x,z}g'_{z,y}$. The formulas connecting these expressions are given by $\sum_{x\in\underline{F}^{-1}(a)} \underline{F}(g_{x,y}) = f_{a,\underline{F}(y)}$ and $\sum_{x\in\underline{F}^{-1}(a)} \underline{F}(g'_{x,y}) = f'_{a,F(y)}$ for all a in $\underline{\Lambda}$ and y in $\underline{\Gamma}$. We have to prove that the same relation holds between g'' and f''. Let a in $\underline{\Lambda}$ and y in $\underline{\Gamma}$ be fixed. Then

$$\sum_{x\in\underline{F}^{-1}(a)} \underline{F}(g''_{x,y}) = \sum_{x\in\underline{F}^{-1}(a)} \underline{F}\Big(\sum_{z\in\underline{\Gamma}}(g_{x,z}g_{z,y})\Big) = \sum_{x\in\underline{F}^{-1}(a)}\sum_{z\in\underline{\Gamma}} \underline{F}(g_{x,z})\underline{F}(g'_{z,y})$$

$$= \sum_{z\in\underline{\Gamma}}\Big(\sum_{x\in\underline{F}^{-1}(a)} \underline{F}(g_{x,z})\Big)\underline{F}(g'_{z,y}) = \sum_{x\in\underline{\Gamma}} f_{a,\underline{F}(z)}\underline{F}(g'_{z,y}) = \sum_{c\in\underline{\Lambda}}\sum_{z\in\underline{F}^{-1}(c)} f_{a,F(z)}\underline{F}(q'_{z,y})('_{z,y})$$

$$= \sum_{c\in\underline{\Lambda}} f_{a,c} \sum_{z\in\underline{F}^{-1}(c)} \underline{F}(g'_{z,y}) = \sum_{c\in\underline{\Lambda}} f_{a,c} f'_{c,F(y)} = f''_{a,F(y)}.$$

Hence $\phi_{\alpha,\underline{F}}$ is a ring morphism.

Let $(h_{x,y})$ be in Γ, and $(f_{a,b})$ and $(f'_{a,b})$ be in Λ. Then a straightforward calculation shows that

$$F(\phi_{\omega,\underline{F}}(f_{a,b})(h_{x,y})\phi_{\alpha,\underline{F}}(f'_{a,b})) = (f_{a,b})F(h_{x,y})(f'_{a,b}).$$

Therefore F is a Λ-bimodule map when $\phi_{\omega,\underline{F}}$ is used to define the left Λ-structure on Γ, and $\phi_{\alpha,\underline{F}}$ is used to define the right Λ-structure on Γ. This completes the proof of the proposition.

In the rest of this section we will only consider finite galois coverings.

Let $\underline{\Gamma}$ be a finite locally bounded k-category and assume that G is a finite group of k-automorphisms on $\underline{\Gamma}$ such that the induced action on the objects is free. Then by [G, Prop. 3.1], the quotient category $\underline{\Gamma}/G$ exists and the canonical projection $\underline{\Gamma} \to \underline{\Gamma}/G$ is a covering functor. The objects of $\underline{\Gamma}/G$ are the orbits of the objects of $\underline{\Gamma}$ under the group action, and a morphism $f: a \to b$ in $\underline{\Gamma}/G$ is a family $({}_yf_x) \in \prod_{x\in a, y\in b} \underline{\Gamma}(x, y)$ such that $g({}_yf_x) = {}_{g(y)}f_{g(x)}$ for all g in G. Because the action of G on the objects is assumed to be free, the induced covering functor makes the following diagram commute for each y_0 in b and x_0 in a.

$$\begin{array}{ccc} \operatorname{Hom}(a,b) & \xrightarrow{\underline{F}^{-1}} & \coprod_{g\in G} \operatorname{Hom}(x_0, g(y_0)) \\ \downarrow{\scriptstyle \underline{F}^{-1}} & & \downarrow{\scriptstyle p_{y_0}} \\ \coprod_{x\in G} \operatorname{Hom}(g(x_0), y_0) & \xrightarrow{p_{x_0}} & \operatorname{Hom}(x_0, y_0) \end{array}$$

Hence for finite galois coverings $\phi_{\alpha,\underline{F}}$ and $\phi_{\omega,\underline{F}}$ coincide. Further the action of G on $\underline{\Gamma}$ induces an action of G on the associated ring Γ of $\underline{\Gamma}$ by $g((f_{x,y})) = ((g^{-1}f_{g(x),g(y)})_{x,y})$. The fixpoint ring of Γ under this action is the set of morphisms $(f_{x,y})$ such that $f_{x,y} = g^{-1}f_{g(x),g(y)}$. This may be reformulated as the set of morphisms $(f_{x,y})$ such that $g(f_{x,y}) = f_{g(x),g(y)}$. But this is also the image of Λ, the ring of $\underline{\Gamma}/G$ by the map $\phi_{\underline{F}}$.

From this discussion it follows that a finite galois cover $\underline{F}: \underline{\Gamma} \to \underline{\Lambda}$ with galois group G gives rise to an action of the finite group G on the ring Γ, and that the ring of fixpoints, Γ^G, corresponds to the ring Λ.

We will now show that (Γ, G) is galois in the sense of this paper.

THEOREM 6.2. *Let $\underline{\Gamma}$ be a finite locally bounded k-category, and let $\underline{F}: \underline{\Gamma} \to \underline{\Lambda}$ be a galois covering with galois group G. Then,*

a) *(Γ, G) is galois, where Γ and the action of G on Γ is as described above,*

b) *the induced action of G on the isomorphism classes of simple Γ-modules is free,*

c) *the ring Λ of $\underline{\Lambda}$ is identified with Γ^G by means of the ring injection $\phi_{\underline{F}}: \Lambda \to \Gamma$.*

PROOF. Obviously $\Gamma/\text{rad}\,\Gamma$ is a semisimple basic k-algebra and the induced action of G on the isomorphism classes of simple Γ-modules corresponds to the action of G on the objects of $\underline{\Gamma}$, which by assumption is free. Therefore we can conclude that (Γ, G) is galois by Proposition 5.3.

To verify statement c) is straightforward by elementary calculations and the discussion before the theorem.

We will now show a converse of this theorem.

Let k be an algebraically closed field, let Γ be a basic finite dimensional k-algebra, and let G be a group of k-automorphisms of Γ such that the induced action of G on the isomorphism classes of simple Γ-modules is free.

THEOREM 6.3. *Let Γ and G be as above. Then there exists a finite locally bounded k-category $\underline{\Gamma}$ and an action of G on $\underline{\Gamma}$ as a group of k-automorphisms such that the action of G on the objects of $\underline{\Gamma}$ is free. Further, the ring of $\underline{\Gamma}$ is identified with Γ and the action of G on Γ induced from the action of G on $\underline{\Gamma}$ is the same as the original action.*

PROOF. Since by assumption the induced action of G on the isomorphism classes of simple Γ-modules is free, G acts as a group of permutations on the central primitive idempotents $\{e_1, e_2, \ldots, e_n\}$ of $\Gamma/\text{rad}\,\Gamma$ with the action being free. In the next lemma we will prove that the idempotents $\{e_1, e_2, \ldots, e_n\}$ can be lifted to a complete set of orthogonal idempotents $\{E_1, E_2, \ldots, E_n\}$ of Γ such that the action of G permutes this set of idempotents. We can then form the k-category $\underline{\Gamma}$ with objects $E_1, E_2, \ldots, E_n$ and with morphism set $(E_i, E_j) = E_j \Gamma E_i$. There is an obvious action of G on this k-category and it is easy to see that Γ is the ring associated to $\underline{\Gamma}$ and that the action of G is not altered by going back and forth.

LEMMA 6.4. *Let (Λ, G) be as before and let I be a G-ideal of Λ with $I^2 = 0$. If $\{e_1, e_2, \ldots, e_n\}$ is a set of orthogonal idempotents in Λ/I which are permuted by the induced action of G on Λ/I with the action being free, then $\{e_1, e_2, \ldots, e_n\}$ can be lifted to a set of orthogonal idempotents $\{E_1, E_2, \ldots, E_n\}$ in Λ such that G acts as a group of permutations on $\{E_1, E_2, \ldots, E_n\}$ with the action being free.*

PROOF. Let $\{e_1, e_2, \ldots, e_n\}$ be the idempotents of Λ/I and assume they are divided into orbits $\{e_1, e_2, \ldots, _{|G|}; e_{|G|+1}, \ldots, e_{2|G|}; \ldots; e_{n-|G|+1}, \ldots, e_n\}$. Lift e_1 to an idempotent E_1' and consider the element $E_1 = E_1' - \sum_{g \in G \setminus \{1\}} E_1' g(E_1')$. Then elementary calculations show that E_1 is an idempotent and that $\{g(E_1) \mid g \in G\}$ is in fact a set of orthogonal idempotents mapped onto the orbit of e_1 in $\{e_1, e_2, \ldots, e_n\}$ by the natural map from Λ to Λ/I. Now consider $e_{|G|+1}$. Lift this to an idempotent $E''_{|G|+1}$ and let $E'_{|G|+1} = E''_{|G|+1} - \sum_{g \in G} E''_{|G|+1}\, g(E_1) - \sum_{g \in G} g(E_1) E''_{|G|+1}$.

Then $E'_{|G|+1}$ is an idempotent orthogonal to the set $\{g(E_1) \mid g \in G\}$. Hence $g(E'_{|G|+1})$ is orthogonal to $\{g(E_1) \mid g \in G\}$ for each $g \in G$. Now produce the idempotents $E_{|G|+1}$ in the same way as E_1 by letting $E_{|G|+1} = E'_{|G|+1} - \sum_{g \in G \setminus \{1\}} E'_{|G|+1} g(E'_{|G|+1})$. By continuing this process, one obtains a complete set of orthogonal idempotents in Λ which are permuted by G.

COROLLARY 6.5. *Let Λ and G be as before and assume that Λ be a basic artinian algebra such that the induced action of G on the isomorphism classes of simple Λ-modules is free. Then there exists a complete set of orthogonal idempotents in Λ permuted by G. In particular, this happens if Λ is a finite dimensional algebra over a field.*

PROOF. The proof goes by induction on the Loewy-length of Λ.

7. Split epimorphisms and derivations.

We have seen in section 1 that if Λ is a $\Lambda[G]$-generator, then the natural map $\Lambda \otimes_{\Lambda^G} \Lambda \to \Lambda$ induced by multiplication is a split epimorphism as Λ-bimodules. Even when (Λ, G) is pregalois, the natural map $\Lambda[G] \otimes_\Lambda \Lambda[G] \to \Lambda[G]$ is not necessarily a split epimorphism as $\Lambda[G]$-bimodules, and we give necessary and sufficient conditions for this to be the case. The interest in these questions has its origin in the fact that the following three properties are equivalent for a pair of rings $R \subseteq S$;

i) the map $S \otimes_R S \to S$ induced by the multiplication splits as a S-bimodule map,

ii) there is a functorial splitting of the natural map $S \otimes_R M \to M$ for each left S-module M,

iii) for each S-bimodule A, every derivation $d: S \to A$ vanishing on R is inner.

Using this we may conclude that if Λ is a $\Lambda[G]$-generator, then every Λ-module M is a summand of a module induced from Λ^G, and if A is a Λ-bimodule, then every derivation $d: \Lambda \to A$ vanishing on Λ^G is an inner derivation.

We now turn to the map from $\Lambda[G] \otimes_\Lambda \Lambda[G]$ to $\Lambda[G]$ induced by the multiplication and give necessary and sufficient conditions for this map to split as a $\Lambda[G]$-bimodule map, when (Λ, G) is pregalois.

PROPOSITION 7.1. a) *The following are equivalent, if (Λ, G) is pregalois:*

i) *The map $m: \Lambda[G] \otimes_\Lambda \Lambda[G] \to \Lambda[G]$ induced by the multiplication is a split surjection as a $\Lambda[G]$-bimodule map.*

ii) *The left augmentation map $\varepsilon_l: \Lambda[G] \to \Lambda$ given by $\varepsilon_l\left(\sum_{g \in G} \lambda_g g\right) = \sum_{g \in G} \lambda_g$ is a split surjection as a $\Lambda[G]$-Λ^G-bimodule map.*

iii) *The right augmentation map* $\varepsilon_r: \Lambda[G] \to \Lambda$ *given by* $\varepsilon_r\left(\sum_{g\in G} \lambda_g g\right) = \sum_{g\in G} g^{-1}(\lambda_g)$ *is a split surjection as a* Λ^G-$\Lambda[G]$*-bimodule map.*

iv) *There exists a* λ *in the centralizer of* Λ^G *in* Λ *such that* $\sum_{g\in G} g(\lambda) = 1$.

v) *The* Λ^G*-bimodule map* $\sigma: \Lambda \to \Lambda^G$ *given by* $\sigma(\lambda) = \sum_{g\in G} g(\lambda)$ *is a split surjection as a* Λ^G*-bimodule map.*

b) *It the above equivalent conditions are statisfied, then* Λ^G *has a twosided* Λ^G*-complement in* Λ.

Before giving the proof of this proposition, we make some remarks.

In Section 1 we gave an example where (Λ, G) was pregalois, but where Λ^G had no twosided Λ^G-complement in Λ (see Example 1.11). Hence this is an example where the equivalent conditions of a) are not satisfied. A closer study of this example shows that each Λ[G]-module is a summand of some induced module from Λ, but that the splitting is not done in a functorial way. An example of a ring Λ with a galois action of a group G such that not every $\Lambda[G]$-module is a summand of an induced module can be constructed.

EXAMPLE 7.2. Let k be a field of characteristic 2 and let Λ be the subring of the 4 by 4 matrix ring over k described by .

$$\Lambda = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ c & d & a & 0 \\ e & f & 0 & b \end{pmatrix} \mid a, b, c, d, e, f \in k \right\}.$$

Let $G = \{1, \phi\}$, where ϕ is conjugation by the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then some elementary calculations on the natural four dimensional representation M of $\Lambda[G]$ shows that it is indecomposable as a Λ^G-module, hence $\Lambda[G] \otimes_\Lambda M$ is indecomposable. This shows that M can not be a summand of the induced module and therefore not a summand of any induced module.

Before returning to the proof of the proposition, we also point out that if the order $|G|$ of the group G is invertible in Λ, one may use the map $h: \Lambda[G] \to \Lambda[G] \otimes_\Lambda \Lambda[G]$ defined by $h(\lambda) = \lambda \frac{1}{|G|} \sum_{g\in G} (g \otimes g^{-1})$ to define a right inverse of m, the map induced by the multiplication.

We will now give the proof of the proposition.

PROOF. Part b) is just a restatement of the last part of Proposition 1.9, so it remains to prove part a). This will be done by establishing the equivalences between i) and iii), iii) and iv) and iv) and v). To prove that ii) and iv) are equivalent may be done by copying the proof of the equivalence of iii) and iv). We prove the equivalence of i) and iii) by using that the functors ${}_{\Lambda^G}\Lambda_{\Lambda[G]} \otimes_{\Lambda[G]} -$ and $\operatorname{Hom}_{\Lambda^G}({}_{\Lambda^G}\Lambda_{\Lambda[G]}, -)$ are inverse equivalences between $\operatorname{Mod} \Lambda[G]$ and $\operatorname{Mod} \Lambda^G$. Let m be the map induced by the multiplication. Then the composed map

$$\Lambda[G] \xrightarrow{1 \otimes \Lambda[G]} \Lambda \otimes_\Lambda \Lambda[G] \xrightarrow{(\Lambda \otimes 1) \otimes \Lambda[G]} \Lambda \otimes_{\Lambda[G]} \Lambda[G] \xrightarrow{\Lambda \otimes m} \Lambda \otimes_{\Lambda[G]} \Lambda[G] \to \Lambda$$

is easily seen to be the right augmentation map. If $f: \Lambda[G] \to \Lambda[G] \otimes_\Lambda \Lambda[G]$ is a $\Lambda[G]$-bimodule map, then $\Lambda \otimes f: \Lambda \otimes_{\Lambda[G]} \Lambda[G] \to \Lambda \otimes_{\Lambda[G]} \Lambda[G] \otimes_\Lambda \Lambda[G]$ is a Λ^G-$\Lambda[G]$-bimodule map. Conversely, if $h: \Lambda \otimes_{\Lambda[G]} \Lambda[G] \to \Lambda \otimes_{\Lambda[G]} \Lambda[G] \otimes_\Lambda \Lambda[G]$ is a Λ^G-$\Lambda[G]$-bimodule map, then $\operatorname{Hom}_{\Lambda^G}(\Lambda, h)$ is a $\Lambda[G]$-bimodule map. Hence there is a $\Lambda[G]$-bimodule map $f: \Lambda[G] \to \Lambda[G] \otimes_\Lambda \Lambda[G]$ with $m \circ f = \mathrm{id}$ if and only if there is a Λ^G-$\Lambda[G]$-bimodule map $h: \Lambda \to \Lambda[G]$ such that $\varepsilon_r \circ h = \mathrm{id}$, where ε_r is the right augmentation map. This proves that i) and iii) are equivalent.

In order to prove the equivalence of iii) and iv) we first prove that iv) follows from iii). Assume therefore that $\varepsilon_r: \Lambda[G] \to \Lambda$ splits as a Λ^G-$\Lambda[G]$-bimodule map and let f be such a splitting. Then $f(1) = \lambda \sum_{g \in G} g$ for a λ in Λ (see Proposition 1.3). Now $1 = (\varepsilon_r \circ f)(1) = \sum_{g \in G} g^{-1}(\lambda) = \sum_{g \in G} g(\lambda)$. Further, since f is assumed to be a left Λ^G-morphism, we have that $f(x) = x \cdot f(1) = x\lambda \sum_{g \in G} g$. On the other hand since f is a right $\Lambda[G]$-morphism $f(x) = \left(\lambda \sum_{g \in G} g\right) x = \lambda x \sum_{g \in G} g$ using that $x \in \Lambda^G$. Hence, $x\lambda = \lambda x$ for all $x \in \Lambda^G$, which shows that λ is in the centralizer of Λ^G in Λ. So iv) follows form iii). To prove that iii) follows from iv), let λ be an element of the centralizer of Λ^G in Λ such that $\sum_{g \in G} g(\lambda) = 1$. Define $f: \Lambda \to \Lambda[G]$ by $f(x) = \lambda\left(\sum_{g \in G} g\right) x$. Then it is easily checked that f is a Λ^G-$\Lambda[G]$-bomidule map and that $\varepsilon_r \circ f = \mathrm{id}$. Hence we have established the equivalence of iii) and iv).

To prove that iv) and v) are equivalent we first observe that the Λ^G-bimodule map from Λ^G to Λ corresponds to the elements of the centralizer of Λ^G in Λ. So one immediately sees that $\sigma : \Lambda \to \Lambda^G$ given by $\sigma(\lambda) = \sum_{g \in G} g(\lambda)$ split as a Λ^G-bimodule map if and only if there exists a λ in the centralizer of Λ^G in Λ such that $\sum_{g \in G} g(\lambda) = 1$. This ends the proof of the proposition.

REFERENCES

[AF] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Graduate Text in Mathematics, Springer Verlag, N.Y. 1974.

[AG1] M. Auslander and O. Goldman, *Maximal orders*, Trans. Amer. Math. Soc. 97 (1960), 1–24.

[AG2] M. Auslander and O. Goldman, *The Brauer Group of a Commutative Ring*, Trans. Amer. Math. Soc. 97 (1960), 367–409.

[BG] K. Bongartz, and P. Gabriel, *Covering spaces in representation theory*, Invent. Math. 65 (1982), 331–378.

[CE] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton, 1956.

[G] P. Gabriel, *The universal cover of a representation-finite algebra*, in: *Representations of Algebras*; Proc. of the third int. Conf. on Rep. of Algebras, Puebla, Mexico, 1980. Lecture Notes 903, Springer (1981), 68–105.

[RR] I. Reiten and Chr. Riedtmann, *Skew group algebras in the representation theory of Artin algebras*, J. Algebra 92 (1985), 224–282.

[R] Chr. Riedtmann, *Algebren, Darstellungsköcher, Überlagerungen und zurück*, Comment Math. Helv. 55 (1980), 199–224.

[V] M. Van den Bergh, *A duality theorem for Hopf algebras, Methods in ring theory*, Reidel Publishing company (1984), 517–522.

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