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LOCAL MODULI FOR PLANE CURVE SINGULARITIES, THE DIMENSION OF THE τ-CONSTANT STRATUM

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1. Introduction and Generalities

Consider the plane curve singularity defined by $f = x_1^p + x_2^q$, and the set of μ-constant deformations of f with minimal Tjurina number. The set $T_{p,q}$ of isomorphism classes of such deformations, has a natural scheme structure, see [L-M-P]. Zariski, [Z], gave a formula for the dimension of $T_{p,q}$ when $q = p + 1$, and in [D], Delorme proves a formula for the case $\gcd(p, q) = 1$. In the general case there are recursion formulas, see [L-M-P], best to my knowledge, no other closed formulas are known.

The aim of this paper is to give such a closed formula for the dimension of $T_{p,q}$ when $2 \mid \gcd(p, q)$.

Let k be any field, and consider a polynomial $f \in k[x_1, x_2]$. Put $\underline{x}^{\underline{\alpha}} := x_1^{\alpha_1} x_2^{\alpha_2}$ for $\underline{\alpha} = (\alpha_1, \alpha_2)$, and let $\{\underline{x}^{\underline{\alpha}}\}_{\underline{\alpha} \in I}$ be a monomial basis for $H^1(f) := k[x_1, x_2]/(f, \partial f/\partial x_1, \partial f/\partial x_2)$. Put

$$\tau(f) = \dim_k H^1(f),$$

$$\mu(f) = \dim_k k[x_1, x_2]/(\partial f/\partial x_1, \partial f/\partial x_2).$$

When $f = x_1^p + x_2^q$, $I = \{(\alpha_1, \alpha_2) \mid 0 \leqq \alpha_1 \leqq p - 2, 0 \leqq \alpha_2 \leqq q - 2\}$.
Moreover, putting $I_\mu = \{(\alpha_1, \alpha_2) \in I \mid \alpha_1/p + \alpha_2/q \geqq 1\}$, one knows that any μ-constant deformation of f is isomorphic to one in the family $F_\mu = x_1^p + x_2^q + \sum_{\underline{\alpha} \in I_\mu} t_{\underline{\alpha}} x_1^{\alpha_1} x_2^{\alpha_2}$. Put $H_\mu = k[t_{\underline{\alpha}}]_{\underline{\alpha} \in I_\mu}$, $\underline{H}_\mu = \operatorname{Spec}(H_\mu)$.

The moduli space $T_{p,q}$, parametrizing isomorphism classes of μ-constant deformations of the singularity f with minimal τ, is a quotient scheme $(\underline{S}/V_\mu)/G$, where $\underline{S}$ is an open subscheme of $\operatorname{Spec}(H_\mu)$ and V_μ is the kernel of the Kodaira-Spencer map associated to the family F_μ. Recall, see [*L-M-P*], that V_μ is

* This paper contains the results of my cand. scient. thesis at the University of Oslo 1985, see [HOH].

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a graded Lie-algebra generated as H_μ-module by a finite dimensional Lie-algebra V_0, acting rationally on $\underline{S}$, such that $\underline{S}/V_\mu = \underline{S}/\exp V_0$. Finally G is a finite group acting rationally on $(\underline{S}/V_\mu)$.

Let $H^1_\mu(f)$ be the subspace of $H^1(f)$ generated by $\{\underline{x}^{\underline{\alpha}}\}_{\underline{\alpha}\in I_\mu}$. Then $H^1_\mu(f)$ is the tangent space of $\underline{H}_\mu$ at 0. In the paper [*L-P*], Laudal and Pfister consider the action σ of $\mathrm{Der}_k(k[\underline{x}]/(f))$ on $H^1(f)$ defined as follows. Let $\bar{D}$ be a derivation of $k[\underline{x}]$ representing the derivation $D \in \mathrm{Der}_k(k[\underline{x}]/(f))$. Then $\bar{D}(f) = q \cdot f$. Let $\bar{\xi} \in k[\underline{x}]$ represent the element $\xi \in H^1(f) = k[\underline{x}]/(f, \partial f/\partial x_1, \partial f/\partial x_2)$, then $\sigma(D)$ is the class of $\bar{D}(\bar{\xi}) - q \cdot \bar{\xi}$ in $H^1(f)$.

It is easy to see that $H^1_\mu(f)$ is invariant under σ. Let for $\xi \in H^1_\mu(f)$, $o(\xi) \subseteq H^1(f)$ be the orbit of ξ under $\mathrm{Der}_k(k[\underline{x}]/(f))$, i.e. $o(\xi) = \{\sigma(D)\cdot\xi \mid D \in \mathrm{Der}_k(k[\underline{x}]/(f))\}$, then it follows from [*L-P*], that we have the following results.

PROPOSITION 1. *Let* $f = x_1^p + x_2^q$, *then* $\dim T_{p,q} = \dim_k H^1_\mu(f) - \max_{\xi\in H^1_\mu(f)} \dim o(\xi)$.

PROOF. see [*L-P*] (4.6) (ii), (4.7) and remarks following (4.7) together with (5.7). See also remarks preceeding (5.12).

PROPOSITION 2. (i) *Let* $\xi \in H^1_\mu(f)$ *be represented by* $\underline{x}^{\underline{\alpha}}$, *then* $o(\xi)$ *is the subspace of* $H^1_\mu(f)$ *generated by the classes of*

$$\{(\alpha_1/p + \alpha_2/q - 1)\,\underline{x}^{\underline{\alpha}} \cdot s \mid s \in k[\underline{x}]\}$$

(ii) *Let* $H^1_+(f)$ *be the subspace of* $H^1(f)$ *generated by*

$$I_+ = \{\underline{x}^{\underline{\alpha}} \mid \underline{\alpha} \in I_\mu, \alpha_1/p + \alpha_2/q > 1\},$$

then

$$\max_{\xi\in H^1_\mu(f)} \dim o(\xi) = \max_{\xi\in H^1_+(f)} \dim_k O(\xi),$$

where $O(\xi)$ *is the subspace of* $H^1_+(f)$ *generated by* $\{\xi \cdot s \mid s \in k[\underline{x}]\}$.

PROOF. See [*L-P*] (4.6) (iii).

2. Dimension of the generic Component.

The aim of this part is the calculation of the dimension of the maximal orbit of the action σ on $H^1_+(f)$, which according to proposition 2 above enables us to calculate $\dim T_{p,q}$. The main result, which will be proved at the end of this paper, is

THEOREM 1. *Let* $f = x_1^p + x_2^q$ *and suppose* $2|\gcd(p,q)$. *The maximal orbit dimension of the action* σ *on* $H^1_+(f)$ *is then*

$$\mathrm{maxorbdim} = \left(\frac{p}{2} - 1\right)\left(\frac{q}{2} - 1\right) - \gcd\left(\frac{p}{2}, \frac{q}{2}\right) + \begin{cases} 1 & \text{if } p|q \text{ or } q|p \\ 0 & \text{otherwise} \end{cases}$$

i) Let $h_{\text{gen}} = \sum_{\underline{\alpha}\in I_+} t_{\underline{\alpha}} \underline{x}^{\underline{\alpha}}$ where the $t_{\underline{\alpha}}$ are variables over the field k, i.e. $h_{\text{gen}} \in k[t_{\underline{\alpha}}][\underline{x}]/(x_1^{p-1}, x_2^{q-1})$

ii) Let $h \in H^1_+(f)$, then $h = \sum_{\underline{\alpha}\in I_+} c_{\underline{\alpha}} \underline{x}^{\underline{\alpha}}, c_{\underline{\alpha}} \in k$. Define Support$(h) = S(h) = \{\underline{\alpha} \in I_+ \mid c_{\underline{\alpha}} \neq 0\}$, $S(\underline{\alpha}) = S(\underline{x}^{\underline{\alpha}} h_{\text{gen}})$. Then

LEMMA 2. $S(\underline{\alpha}) = \{\underline{\alpha}' \in I_+ \mid \alpha'_1/p + \alpha'_2/q > 1 + \alpha_1/p + \alpha_2/q\}$

PROOF. Follows directly from i) and ii).

iii) Lemma 2 shows that the set $\{\text{Support}(\underline{\alpha}) \mid 0 \leqq \alpha_1 \leqq p-2, 0 \leqq \alpha_2 \leqq q-2\}$ is linearly ordered under inclusion.

Let $S(\underline{\alpha}_M) \subsetneq \ldots \subsetneq S(\underline{\alpha}_0) = I_+$ be a maximal chain of proper inclusions. We define a subdivision of I_+ as follows:

$I_M = S(\underline{\alpha}_M)$, $I_m = S(\underline{\alpha}_m) \backslash S(\underline{\alpha}_{m+1})$ for $0 \leqq m < M$.

Set $I_a^b = \bigcup_{m=a}^{b} I_m$.

iv) Define Set$(m) = \{\underline{x}^{\underline{\alpha}} h_{\text{gen}} \mid S(\underline{\alpha}) \subseteq I_m^M, S(\underline{\alpha}) \nsubseteq I_{m+1}^M\}$. We observe that for fixed m, every element of Set(m) has the same support and that for any $h \in H^1_\mu(f)$ a basis for the orbit of h can be injectively embedded in $\bigcup_{m=0}^{M}$ Set(m).

v) For every finite set X, let $\#X$ denote the number of elements of X.

We are going to show that there exists a $w \in \mathbb{N}$, such that

PROPOSITION 3.
$$\begin{aligned} &\#\text{Set}(m) &&\leqq \#I_m, && 0 \leqq m \leqq w-2 \\ &\#\text{Set}(w-1) &&= \#I_{w-1} + 1 && \\ &\#\text{Set}(w) &&= \#I_w \quad - 1 && \\ &\#\text{Set}(m) &&\geqq \#I_m, && w+1 \leqq m \leqq M \end{aligned}$$

Accepting this, we can prove

PROPOSITION 4. *The dimension of the maximal orbit of the action σ on $H^1_+(f)$ is*

$$\max_{\xi \in H^1_+(f)} \dim o(\xi) = \sum_{m=0}^{M} \min(\#I_m, \#\text{Set}(m)) + 1.$$

PROOF. The inequality $\leqq$ follows using Proposition 3:

$$\sum_{m=0}^{M} \min(\#I_m, \#\text{Set}(m)) + 1 = \sum_{m=0}^{w-2} \#\text{Set}(m) + \#I_{w-1}^{M} \geqq \max_{\xi \in H^1_+(f)} \dim o(\xi).$$

The other direction $\geqq$ can be proved as follows:

i) Define a new subdivision of I_+ by fixing an element $\underline{\alpha} \in I_w$, and putting

$$\begin{aligned} J_{w-1} &= I_{w-1} \cup \{\underline{\alpha}\} \\ J_w &= I_w \setminus \{\underline{\alpha}\} \\ J_m &= I_m \qquad \text{otherwise.} \end{aligned}$$

ii) For $\#\mathrm{Set}(m) \leqq \# J_m$ choose $\#\mathrm{Set}(m)$ points from J_m. Enumerate the polynomials of $\mathrm{Set}(m)$ and the chosen points of J_m from 1 to $\#\mathrm{Set}(m)$. For $\#\mathrm{Set}(m) > \# J_m$ choose $\# J_m$ polynomials from $\mathrm{Set}(m)$. Enumerate the points of J_m and the chosen polynomials of $\mathrm{Set}(m)$ from 1 to $\# J_m$.

iii) We then construct the square matrixes C_m, $m = 0, \ldots, M$ by setting c_i^j in C_m equal to the coefficient of the monomial of polynomial i in $\mathrm{Set}(m)$ corresponding to point j in J_m, (i.e. the monomial $\underline{x}^{\underline{\alpha}}$ corresponds to the point $\underline{\alpha}$). We obtain $0 \neq \det C_m \in k[t_{\underline{\alpha}}]_{\underline{\alpha} \in I_+}$, $m = 0, \ldots, M$, immediately from the fact that no column contains the same $t_{\underline{\alpha}}$ twice. Setting $\{t_{\underline{\alpha}}\}_{\underline{\alpha} \in I_+}$ equal to a closed point of $\mathrm{Spec}(k[t_{\underline{\alpha}}]_{\underline{\alpha} \in I_+}/(1 - \prod_{m=0}^{M} \det C_m))$ and counting, (using i), ii)) gives the wanted inequality.

We shall now prove that there is a duality between I_{M-m} and $\mathrm{Set}(m)$ and later on we shall actually compute I_m and therefore $\mathrm{Set}(m)$.

PROPOSITION 5. $\#\mathrm{Set}(m) = \# I_{M-m}$.

PROOF. The 1-1 pairing of the two sets is given by associating to $x_1^{\alpha_1} x_2^{\alpha_2} h_{\mathrm{gen}} \in \mathrm{Set}(m)$ the element $(p - 2 - \alpha_1, q - 2 - \alpha_2) \in I_{M-m}$.

i) Different $\mathrm{Set}(i)$ are sent into different I_j: Let $\underline{x}^{\underline{\alpha}} h_{\mathrm{gen}} \in \mathrm{Set}(m)$ and $\underline{x}^{\underline{\alpha}'} h_{\mathrm{gen}} \in \mathrm{Set}(m')$ where $m < m'$. Choose an element $\underline{\alpha}''$ of I_m. Then, using Lemma 2, $1 + \alpha'_1/p + \alpha'_2/q \geqq \alpha''_1/p + \alpha''_2/q > 1 + \alpha_1/p + \alpha_2/q$ or rearranging $(p - 2 - \alpha_1)/p + (q - 2 - \alpha_2)/q > 1 + (p - 2 - \alpha''_1)/p + (q - 2 - \alpha''_2)/q \geqq (p - 2 - \alpha'_1)/p + (q - 2 - \alpha'_2)/q$ which means that (see Lemma 2 and iv) above) $(p - 2 - \alpha_1, q - 2 - \alpha_2)$ and $(p - 2 - \alpha'_1, q - 2 - \alpha'_2)$ belong to different I_j.

ii) Different I_j are sent into different $\mathrm{Set}(i)$: Let $\underline{\alpha} \in I_m$, $\underline{\alpha}' \in I_{m'}$, where $m < m'$. Then there exists $\underline{\alpha}''$, such that (see Lemma 2 and iv) above) $\alpha'_1/p + \alpha'_2/q > 1 + \alpha''_1/p + \alpha''_2/q \geqq \alpha_1/p + \alpha_2/q$. Rearranging as in i) and using once more Lemma 2 and (v) above, we reach the desired conclusion.

Applying Proposition 3 we get

COROLLARY 6, $\max_{\xi \in H^1_+(f)} \dim o(\xi) = 2\# I_w^M - 1$

PROOF. Proposition 4 gives $\max_{\xi \in H^1_+(f)} \dim o(\xi) = \sum_{m=0}^{M} \min(\# I_m, \#\mathrm{Set}(m)) + 1$

$= \sum_{m=0}^{w-1} \#I_{M-m} + \sum_{m=w}^{M} I_m - 1$, using Propositions 3 and 5. Now these imply $M = 2w - 1$, i.e.

$$\max_{\xi \in H^1_+(f)} \dim o(\xi) = 2 \sum_{m=w}^{M} \#I_m - 1.$$

Set $r = p/\gcd(p,q), s = q/\gcd(p,q)$ and assume, as we may, $r \geqq s$. We call $\{(x,y) \in I_m^M \mid y = n\}$ a line in I_m^M.

PROPOSITION 7. *Suppose* $\underline{x}^{\underline{\alpha}} h_{\text{gen}} \in \text{Set}(m)$. *Then*

$$\#I_m \geqq \#\{\text{lines in } I_m^M \mid r(y - \alpha_2) \equiv 1 (\text{mod } s)\}$$

with equality if there exists an $\underline{\alpha}'$ *with*

$$(*) \qquad s\alpha_1' + r\alpha_2' = 1 \text{ and } \alpha_1 + \alpha_1' \geqq 0, \alpha_2 + \alpha_2' \geqq 0.$$

PROOF. Lines with $r(y - \alpha_2) \equiv 1 (\text{mod } s)$ represent the points in I_m^M minimalizing the expression $x/p + y/q$. $(*)$ implies the existence of an $\underline{\alpha}''$ with support $(\underline{\alpha}'')$ excluding exactly these points of I_m^M.

We say that m satisfies (C1) if there exists $\underline{x}^{\underline{\alpha}} h_{\text{gen}} \in \text{Set}(m)$ and $\underline{\alpha}'$, such that $(*)$ holds. Denote by $[x]$ $\max\{z \in \mathbb{Z} \mid z \leqq x\}$. Lemma 2 implies that one can find an element $\underline{x}^{\underline{\alpha}} h_{\text{gen}} \in \text{Set}(m)$ with $\underline{\alpha} = (\alpha r + \delta, \alpha_2)$ where $0 \leqq r, 0 \leqq \alpha_2 < s$.

COROLLARY 8. *Let* $\underline{x}^{\underline{\alpha}} h_{\text{gen}} \in \text{Set}(m)$ *be of the above mentioned type and suppose that m satisfies* (C1). *Then*

$$\#I_m = \gcd(p,q) - \alpha - \begin{cases} 2 & \text{if } \delta \geqq \left[r^{-1} \frac{r}{s} \right] - 1 \text{ and } \alpha_2 \geqq s - (r^{-1} + 1) \\ 1 & \text{if } \delta \geqq \left[r^{-1} \frac{r}{s} \right] - 1 \quad \text{or } \alpha_2 \geqq s - (r^{-1} + 1) \\ 0 & \text{otherwise} \end{cases}$$

where $rr^{-1} \equiv 1 (\text{mod } s), 0 < r^{-1} < s$.

PROOF. follows from Proposition 7.

PROPOSITION 9. *With the above notations* $\#\text{Set}(m) \geqq \alpha + 1$, *with equality hoding if* I_m^M *contains a point of the type* (x,y) *where* $r(y - \alpha_2) \equiv 1 (\text{mod } s)$.

PROOF. $\underline{x}^{\underline{\alpha}'} h_{\text{gen}}$, where $(\alpha_1', \alpha_2') = ((\alpha - n)r + \delta, \alpha_2 + ns), n = 0, \ldots, \alpha$, all belong to Set$(m)$. These are the only elements of Set(m) with support not excluding points of the mentioned type.

We call the condition in Proposition 9 (C2).

COROLLARY 10. *Suppose $m < m'$. If m satisfies* (C2) *then*

$$\#\mathrm{Set}(m') - \#\mathrm{Set}(m) \geqq -1.$$

PROOF. Immediate.

The corresponding result for I_m follows from Proposition 7, and we state it as

COROLLARY 11. *Suppose $m < m'$. If m' satisfies* (C1) *then*

$$\#I_m - \#I_{m'} \geqq -1.$$

PROOF. Immediate.

Consider the following conditions, the first implying (C1), the second implying (C2):

(C1′) There exists $\underline{x}^{\underline{\alpha}} h_{\mathrm{gen}} \in \mathrm{Set}(m)$ with $\alpha_1 \geqq r$ or $\alpha_2 \geqq s$.
(C2′) I_m^M contains at least s nonempty different lines.

The advantage of this reformulation of (C1) and (C2) is that if $m < m'$ and m satisfies (C1′) then m' also satisfies (C1′) and if $m < m'$ and m' satisfies (C2′) then m also satisfies (C2′).

Set $w_{\min} = \min \{m \mid \#\mathrm{Set}(m) > \#I_m\}$,
$w_{\max} = \max \{m \mid \#\mathrm{Set}(m) < \#I_m\}$.
We can then reformulate Proposition 3:

PROPOSITION 3. $w_{\min} = w_{\max} - 1$.

Now we prove

PROPOSITION 12. *Set* $w'_{\min} = \min \{m \mid \#\mathrm{Set}(m) = \#I_m + 1\}$,
$w'_{\max} = \max \{m \mid \#\mathrm{Set}(m) = \#I_m - 1\}$.
Suppose that $w'_{\min}$ satisfies (C2′) *and that $w'_{\max}$ satisfies* (C1′). *If $w'_{\min} < w'_{\max}$ then $w_{\min} = w'_{\min}$ and $w_{\max} = w'_{\max}$.*

PROOF. $w_{\min} \leqq w'_{\min} < w'_{\max}$ means, using Corollaries 10 and 11 that

$$(\#\mathrm{Set}(w'_{\max}) - \mathrm{Set}(w_{\min})) + (\#I_{w_{\min}} - \#I_{w'_{max}}) =$$

$$(\#\mathrm{Set}(w'_{\max}) - \#I_{w'_{\max}}) + (\#I_{w_{\min}} - \#\mathrm{Set}(w_{\min})) \geqq -2.$$

The definition of $w'_{\max}$ then implies $(\#I_{w_{\min}} - \#\mathrm{Set}(w_{\min})) \geqq -1$, i.e. $w_{\min} = w'_{\min}$, and $w_{\max} = w'_{\max}$ is proved the same way.

i) We first find $w'_{\min}$.
Let $\underline{x}^{\underline{\alpha}} h_{\mathrm{gen}}$ correspond to $w'_{\min}$, where $\underline{\alpha} = (\alpha r + \delta, \alpha_2)$ with $0 \leqq \delta < r$,

$0 \leqq \alpha_2 < s$. Using Propositions 7 and 9 we have $\alpha + 1 = \gcd(p, q) - \alpha - (0$ or 1 or $2) + 1$. Since $2 \mid \gcd(p, q)$ we have two possibilities

1) $\alpha = \gcd(p, q)/2$, $\delta = \alpha_2 = 0$ or

2) $\alpha = \gcd(p, q)/2 - 1, \delta = \left[r^{-1}\frac{r}{s}\right] - 1, \alpha_2 = s - (r^{-1} + 1)$.

But

$$(\gcd(p, q)/2)/p > (\gcd(p, q)/2 - 1)r + \left[r^{-1}\frac{r}{s}\right] - 1)/p + (s - (r^{-1} + 1))/q$$

shows that $w'_{\min}$ corresponds to the second possibility.

ii) Let $\underline{x}^{\underline{\alpha}'}h_{\text{gen}}$ correspond to $w'_{\max}$, $\underline{\alpha}'$ as in i). We have $\alpha' + 1 = \gcd(p, q) - \alpha' - (0$ or 1 or $2) - 1$, giving the possibilities

1) $\alpha' = \gcd(p, q)/2 - 1, \delta' = \left[r^{-1}\frac{r}{s}\right] - 2, \alpha'_2 = s - (r^{-1} + 1) - 1$

2) $\alpha' = \gcd(p, q)/2 - 2, \delta' = r - 1, \alpha'_2 = s - 1$.

Comparing as in i), it turns out that 1) is impossible.

iii) We have

$$((\alpha' - \alpha)r + (\delta' - \delta))/p + (\alpha'_2 - \alpha_2)/q = \gcd(p, q)/pq,$$

therefore $w'_{\max} = w'_{\min} + 1$.

iv) That $w'_{\min}$ satisfies (C2′) is equivalent to

$$(p - 2)/p + (q - 2 - s)/q > (\alpha r + \delta)/p + (\alpha_2)/q$$

where we suppose that $q - 2 - s \geqq 2$, which are both satisfied if $s > 1$ and $\gcd(p, q) > 2$. These two cases must be considered separately.

For $w'_{\max}$ to satisfy (C1′) it is sufficient to require that

$$(\gcd(p, q)/2 - 1)r - 1 \geq r - 1 \text{ or } s - 1 \geqq s - 1$$

which is obvious.

i)-iv) together with Proposition 12 proves Proposition 3.

From Corollary 6 we know that $\max_{\xi \in H^1_+(f)} \dim o(\xi) = 2\# I^M_w - 1$ and using the fact that $w = w'_{\max}$ we get

$$\max_{\xi \in H^1_+(f)} \dim o(\xi) =$$

$$= 2\# \{(\alpha_1, \alpha_2) \mid \alpha_1/p + \alpha_2/q > 1 + (p/2 - r - 1)/p + (s - 1)/q,$$
$$0 \leqq \alpha_1 \leqq p - 2, 0 \leqq \alpha_2 \leqq q - 2\} - 1$$

$$= \left(\frac{p}{2} - 1\right)\left(\frac{q}{2} - 1\right) - \gcd\left(\frac{p}{2}, \frac{q}{2}\right),$$

thus proving Theorem 1 under the assumptions $s > 1$ and $\gcd(p, q) > 2$. The -1 in Theorem 1 occurs for $s = 1$. The remaining cases are easy to check.

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