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QUASIHYPERBOLIC GEODESICS IN JOHN DOMAINS

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1. Introduction.

Suppose that D is a proper subdomain of euclidean n-space R^n. The *quasihyperbolic length* of an arc γ in D is defined as

$$(1.1) \qquad k_D(\gamma) = \int_\gamma d(x, \partial D)^{-1}\, ds,$$

where $d(x, \partial D)$ denotes the euclidean distance from x to ∂D. Next the *quasihyperbolic distance* between two points x_1, x_2 in D is given by

$$(1.2) \qquad k_D(x_1, x_2) = \inf_\gamma k_D(\gamma),$$

where the infimum is taken over all rectifiable arcs γ joining x_1 to x_2 in D. A *quasihyperbolic geodesic* is an arc γ for which the infimum in (1.2) is attained; see [GO], [GP] and [M].

Suppose that $x_0, x_1 \in D$ and that $b \geqq 1$. A rectifiable arc γ is said to be a *b-cone arc* from x_1 to x_0 if γ joins x_1 to x_0 in D and if

$$(1.3) \qquad l(\gamma(x_1, x)) \leqq b\, d(x, \partial D)$$

for all $x \in \gamma$; here $\gamma(x_1, x)$ denotes the subarc of γ between x_1 and x and $l(\alpha)$ the euclidean length of an arc α. The domain D is then said to be a *b-John domain* with center x_0 if for each $x_1 \in D$ there is a b-cone arc from x_1 to x_0. Inequality (1.3) implies that D contains the (*curvilinear*) *b-cone*

$$(1.4) \qquad \mathrm{Cone}(\gamma, b; x_0) = \bigcup_{x \in \gamma} B\left(x, \frac{1}{b} l(\gamma(x_1, x))\right),$$

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with *axis* γ, *vertex* x_1 and *center* x_0; here $B(x,r)$ denotes the open n-ball with center x and radius r. If γ is the closed segment $[x_1, x_0]$, then $\operatorname{Cone}(\gamma, b; x_0)$ is the union of a finite euclidean cone with vertex angle $\theta = \arcsin\left(\frac{1}{b}\right)$ at x_1 and a ball about x_0.

A rectifiable arc γ is said to be a *double b-cone arc* from x_1 to x_2 if γ joins x_1 to x_2 in D and if

$$(1.5) \qquad \begin{cases} l(\gamma) \leqq b|x_1 - x_2|, \\ \min(l(\gamma(x_1, x)), l(\gamma(x, x_2))) \leqq b\, d(x, \partial D) \end{cases}$$

for all $x \in \gamma$. The domain D is said to be *b-uniform* if for each $x_1, x_2 \in D$ there exists a double b-cone arc from x_1 to x_2. Inequality (1.5) implies that D contains the double cone

$$\operatorname{Cone}(\gamma_1, b; x_0) \cup \operatorname{Cone}(\gamma_2, b; x_0)$$

where x_0 denotes the midpoint of γ and $\gamma_j = \gamma(x_j, x_0)$ for $j = 1, 2$.

The classes of John and uniform domains described above are closely related. For example, D is a b-John domain if and only if *all* of its points are the vertices of b-cones in D with a common center; D is b-uniform if an only if each *pair* of its points are the vertices of two b-cones in D with a common center for which the axis length sum does not exceed b times the distance between the vertices. In particular, if D is b-uniform, then each pair of its points lie in the closure of a b-John subdomain of D. Moreover, every bounded uniform domain is a John domain [GM].

If D is c-uniform and if γ is a quasihyperbolic geodesic which joins x_1 and x_2 in D, then γ is a double cone arc with $b = b(c)$ [GO]. It is natural to ask if this result has a counterpart for John domains. In particular, suppose that D is a c-John domain with center x_0 and that γ is a quasihyperbolic geodesic which joins x_1 to x_0. Is γ a b-cone arc for some $b = b(c)$? The purpose of this paper is to show that the answer is yes when $n = 2$ and D is simply connected, and in general no when $n > 2$ or D is multiply connected. We establish these assertions in Sections 4 and 5. Section 4 also contains a new characterization for simply connected John domains in R^2. In Section 3 we exhibit two criteria which are necessary and sufficient for a quasihyperbolic geodesic γ to satisfy the cone condition (1.3). Section 2 contains estimates for the quasihyperbolic distance and a key lemma on the location of a quasihyperbolic geodesic in a simply connected plane domain.

2. Estimates for the quasihyperbolic distance.

We derive here three estimates for the quasihyperbolic distance in a proper subdomain D of R^n which will be needed in the remainder of this paper.

2.1 LEMMA. *Suppose that* x_1, x_2 *are points in* D *and that* $d_1 = d(x_1, \partial D)$, $d_2 = d(x_2, \partial D)$, $t = |x_1 - x_2|$. *If* $t < d_1 + d_2$, *then*

$$k_D(x_1, x_2) \leqq \log \frac{d_1 + d_2 + t}{d_1 + d_2 - t}. \tag{2.2}$$

This bound is sharp. If $t \leqq d_2$, *then*

$$k_D(x_1, x_2) \leqq \log\left(1 + \frac{2t}{d_1}\right). \tag{2.3}$$

PROOF. Let $\alpha = [x_1, x_2]$ and $B_j = B(x_j, d_j)$ for $j = 1, 2$. The triangle inequality implies that $d_1 \leqq d_2 + t$ and $d_2 \leqq d_1 + t$. Then by making a preliminary change of variables, we may assume that $0, x_1, x_2$ lie in a line λ and that

$$d_1^2 - |x_1|^2 = d_2^2 - |x_2|^2 = d^2. \tag{2.4}$$

Since $B_1 \cup B_2 \subset D$,

$$d(x, \partial D)^2 \geqq d(x, \partial(B_1 \cup B_2))^2 = d^2 + |x|^2 \tag{2.5}$$

for $x \in \alpha$.

Suppose that λ is parametrized with respect to arclength s with $\lambda(0) = 0$, $\lambda(s_j) = x_j$ for $j = 1, 2$ and $s_2 > 0$; by relabeling we may assume that $s_1 < s_2$. Then $t = s_2 - s_1$ and we obtain

$$\begin{aligned} k_D(x_1, x_2) &\leqq \int_\alpha (d^2 + |x|^2)^{-1/2} ds \\ &= \log \frac{d_2 + s_2}{d_1 + s_1} \\ &= \log \frac{d_1 + d_2 + t}{d_1 + d_2 - t} \end{aligned}$$

from integration and (2.4).

Next if $D = B_1 \cup B_2$ and if γ is any arc joining x_1 and x_2 in D, then

$$d(x, \partial D)^2 \leqq d^2 + |x|^2$$

for $x \in \gamma$ and we obtain equality in (2.2). Finally (2.2) implies (2.3) whenever $t \leqq d_2$.

2.6. LEMMA. *Suppose that* γ *is an arc which joins points* x_1, x_2 *in* D *and that* $d_1 = d(x_1, \partial D)$, $d_2 = d(x_2, \partial D)$, $l = l(\gamma)$. *Then*

$$k_D(\gamma) \geqq \log \frac{(d_1 + d_2 + l)^2}{4 d_1 d_2}. \tag{2.7}$$

This bound is sharp. In particular,

$$k_D(\gamma) \geqq \log\left(1 + \frac{l}{d_1}\right). \tag{2.8}$$

PROOF. If γ is parametrized by arclength s with $\gamma(0) = x_1$, then

$$d(x, \partial D) \leqq d_1 + s, \quad d(x, \partial D) \leqq d_2 + l - s$$

for $x \in \gamma$. Hence $r = \frac{1}{2}(l + d_2 - d_1) \in [0, l]$ and we obtain (2.7) from

$$\begin{aligned} k_D(\gamma) &= \int_\gamma d(x, \partial D)^{-1}\, ds \\ &\geqq \int_0^r (d_1 + s)^{-1}\, ds + \int_r^l (d_2 + l - s)^{-1}\, ds \\ &= \log \frac{(d_1 + d_2 + l)^2}{4 d_1 d_2}. \end{aligned}$$

Equality holds if x_1 and x_2 are points in an open subinterval β of a line λ, $\gamma = [x_1, x_2]$ and $D = (\mathsf{R}^n \setminus \lambda) \cup \beta$. Finally (2.8) follows from (2.7) and the fact that $d_2 \leqq d_1 + l$.

Our third estimate concerns the location of an arc which is a geodesic for either the quasihyperbolic or hyperbolic metric in a simply connected proper subdomain D of R^2. For each $x \in \mathsf{R}^2$ we let $C(x, r)$ denote the circle with center x and radius r.

2.9. LEMMA. *Suppose that D is a simply connected proper subdomain of R^2, that γ is a quasihyperbolic or hyperbolic geodesic in D and that x_1, x_0, x_2 is an ordered triple of points in γ with $|x_1 - x_0| = |x_2 - x_0| = r$. If D contains a component of $C(x_0, r) \setminus \{x_1, x_2\}$, then*

$$r \leqq a\, d(x_0, \partial D) \tag{2.10}$$

where a is an absolute constant.

PROOF OF LEMMA 2.9 FOR THE QUASIHYPERBOLIC CASE. Suppose that γ is a quasihyperbolic geodesic in D. By performing a preliminary similarity mapping we may assume that $x_0 = 0$ and that $d(0, \partial D) = 1$. Next by hypothesis, $C(0, r) \setminus \{x_1, x_2\}$ has a component C which joins x_1 and x_2 in D; by replacing γ and C by subarcs if necessary, we may assume that γ and $\bar{C}$ meet just at the points x_1 and x_2 and hence bound a Jordan domain G which lies in D.

Let $\gamma_j = \gamma(x_j, 0)$ for $j = 1, 2$. Then $C\left(0, \frac{3r}{4}\right) \cap G$ has a component $\tilde{C}$ which joins $y_1 \in \gamma_1$ to $y_2 \in \gamma_2$ in G. Let

$$\begin{aligned} E_1 &= \left\{x \in \tilde{C}\colon\ d(x,\gamma_1) \leqq \min\left(\frac{r}{4}, d(x,\gamma_2)\right)\right\}, \\ E_2 &= \left\{x \in \tilde{C}\colon\ d(x,\gamma_2) \leqq \min\left(\frac{r}{4}, d(x,\gamma_1)\right)\right\}. \end{aligned} \tag{2.11}$$

Then E_1 and E_2 are relatively closed subsets of the open arc $\tilde{C}$ with $y_1 \in \bar{E}_1 \setminus \bar{E}_2$ and $y_2 \in \bar{E}_2 \setminus \bar{E}_1$. Suppose that $x \in E_1 \cap E_2$. Then (2.11) implies that

$$d = d(x,\gamma_1) = d(x,\gamma_2) \leqq \frac{r}{4}$$

and since $|x| = \dfrac{3r}{4}$, the disk $\bar{B}(x,d)$ lies in D, meets both γ_1 and γ_2 but does not contain 0. Hence $\bar{B}(x,d) \cap \gamma$ is not connected and we have a contradiction to Theorem 2.2 in [M]. Thus $E_1 \cap E_2 = \varnothing$ and it follows that $\tilde{C} \setminus (E_1 \cup E_2)$ contains an open subarc α with endpoints $z_1 \in E_1$ and $z_2 \in E_2$. Moreover, we see from (2.11) that

$$d(x,\gamma_1 \cup \gamma_2) \geqq \frac{r}{4}, \quad d(x,\partial D) \geqq d(x,\partial G) \geqq \frac{r}{4} \tag{2.12}$$

for $x \in \bar{\alpha}$ and that $d(z_1,\gamma_1) = d(z_2,\gamma_2) = \dfrac{r}{4}$. Thus we can choose points $w_1 \in \gamma_1$ and $w_2 \in \gamma_2$ such that

$$|z_1 - w_1| = |z_2 - w_2| = \frac{r}{4}. \tag{2.13}$$

We now apply Lemmas 2.1 and 2.6 to obtain upper and lower bounds for $k_D(w_1, w_2)$ involving r. Let $d_j = d(w_j, \partial D)$ for $j = 1, 2$. Since

$$d(z_j, \partial D) \geqq \frac{r}{4},$$

(2.13) and Lemma 2.1 imply that

$$k_D(w_j, z_j) \leqq \log\left(1 + \frac{r}{2d_j}\right)$$

and hence with (2.12) that

$$\begin{aligned} k_D(w_1,w_2) &\leqq k_D(w_1,z_1) + k_D(w_2,z_2) + k_D(z_1,z_2) \\ &\leqq \log\left(1 + \frac{r}{2d_1}\right) + \log\left(1 + \frac{r}{2d_2}\right) + 6\pi. \end{aligned} \tag{2.14}$$

Next $d(0, \partial D) = 1$ and

$$l_j = l(\gamma(w_j, 0)) \geqq |w_j| \geqq |z_j| - |w_j - z_j| = \frac{r}{2}$$

for $j = 1, 2$. Since γ is a quasihyperbolic geodesic,

$$k_D(w_j, 0) \geqq \log \frac{(d_j + 1 + l_j)^2}{4d_j} \geqq \log\left(1 + \frac{r}{2d_j}\right) + \log \frac{r}{8}$$

by Lemma 2.6 and we obtain

$$k_D(w_1, w_2) = k_D(w_1, 0) + k_D(w_2, 0)$$

$$\geqq \log\left(1 + \frac{r}{2d_1}\right) + \log\left(1 + \frac{r}{2d_2}\right) + 2\log\frac{r}{8}. \tag{2.15}$$

Inequalities (2.14) and (2.15) then imply (2.10) with $a = 8e^{3\pi}$ completing the proof for the quasihyperbolic case.

The proof for the hyperbolic case follows directly from the following result.

2.16. Lemma. *Suppose that D is a simply connected proper subdomain of R^2 and that γ is a hyperbolic geodesic joining x_1 and x_2 in D. For each $x_0 \in \gamma \setminus \{x_1, x_2\}$ there exists a crosscut α of D containing x_0 which separates the components of $\gamma \setminus \{x_0\}$ in D and satisfies*

$$l(\alpha) \leqq c\, d(x_0, \partial D) \tag{2.17}$$

where c is an absolute constant.

Proof of Lemma 2.16. Let f be a conformal mapping of the unit disk B onto D normalized so that $y_j = f^{-1}(x_j)$ are points of the real axis L and $y_0 = 0$. Next let C_1 and C_2 denote the components of $\partial B \setminus L$. By Corollary 10.3 in [P1] we can choose for $j = 1, 2$ an open segment β_j joining 0 to C_j such that

$$l(f(\beta_j)) \leqq \frac{c}{2} d(f(0), \partial D) = \frac{c}{2} d(x_0, \partial D),$$

where c is an absolute constant. Then $\alpha = f(\beta_1 \cup \{0\} \cup \beta_2)$ is a crosscut of D with the desired properties.

Proof of Lemma 2.9 for the hyperbolic case. Suppose now that γ is a hyperbolic geodesic in D, let C denote the component of $C(x_0, r) \setminus \{x_1, x_2\}$ which joins x_1 and x_2 in D and let α be the crosscut described in Lemma 2.16. Since α separates x_1 and x_2, α must join x_0 and C in D. Hence

$$r \leqq l(\alpha) \tag{2.18}$$

and we obtain (2.10) with $a = c$ from (2.17) and (2.18).

3. Quasihyperbolic geodesics as cone arcs.

Suppose that D is a proper subdomain of $\mathbf{R}^n$. We derive in this section two criteria for a quasihyperbolic geodesic γ in D to be a cone arc. We begin with the following preliminary result.

3.1. LEMMA. *Suppose that γ is a rectifiable arc which joins x_1 to x_0 in D and that $c \geqq 1$. If*

$$k_D(\gamma(y_1, y_2)) \leqq c \log\left(1 + \frac{|y_1 - y_2|}{d(y_1, \partial D)}\right) \tag{3.2}$$

for all y_1, y_2 in γ with y_1 between x_1 and y_2, then γ is a b-cone arc where b depends only on c and a,

$$a = \sup_{y \in \gamma} \frac{d(y, \partial D)}{d(x_0, \partial D)} < \infty. \tag{3.3}$$

PROOF. We define inductively a sequence of points $y_1, \ldots, y_{m+1}$ in γ as follows. Set $y_1 = x_1$, suppose that y_j has been defined for some $j \geqq 1$ and set $d_j = d(y_j, \partial D)$. If

$$d(x_0, \partial D) \geqq 2d_j,$$

let y_{j+1} denote the first point of $\gamma(y_j, x_0)$ for which

$$d_{j+1} = d(y_{j+1}, \partial D) = 2d_j \tag{3.4}$$

as we traverse γ from y_j towards x_0; otherwise set $y_{j+1} = x_0$ and $m = j$. Next let $\gamma_j = \gamma(y_j, y_{j+1})$ and $l_j = l(\gamma_j)$. If $x \in \gamma_j$, then

$$d(x, \partial D) \leqq 2d_j$$

if $j = 1, \ldots, m - 1$ and

$$d(x, \partial D) \leqq a\, d(x_0, \partial D) \leqq 2ad_m$$

if $j = m$; hence

$$\frac{l_j}{d_j} \leqq 2a \int_{\gamma_j} d(x, \partial D)^{-1}\, ds = 2a\, k_D(\gamma_j) \tag{3.5}$$

for $j = 1, \ldots, m$. Next (3.2) implies that

$$k_D(\gamma_j) \leqq c \log\left(1 + \frac{l_j}{d_j}\right) \leqq c\left(\frac{l_j}{d_j}\right)^{1/2} \tag{3.6}$$

and we conclude that

$$l_j \leqq (2ac)^2 d_j \tag{3.7}$$

for all j.

Now fix $x \in \gamma$. Then $x \in \gamma_j$ for some $j \leqq m$ and

$$\log \frac{d_j}{d(x, \partial D)} \leqq k_D(y_j, x) \leqq k_D(\gamma_j) \leqq 2ac^2 \tag{3.8}$$

by Lemma 2.6 or Lemma 2.1 of [GP], (3.6) and (3.7). Hence by (3.7), (3.4) and (3.8),

$$\begin{aligned} l(\gamma(x_1, x)) &\leqq \sum_1^j l_i \leqq (2ac)^2 \sum_1^j d_i \leqq (2ac)^2 \sum_1^j 2^{i-j} d_j \\ &\leqq 8(ac)^2 d_j \leqq b\, d(x, \partial D) \end{aligned}$$

where $b = 8(ac)^2 e^{2ac^2}$. This is the desired inequality (1.3).

Condition (3.2) allows us to characterize the quasihyperbolic geodesics which are cone arcs.

3.9 THEOREM. *Suppose that γ is a quasihyperbolic geodesic joining x_1 to x_0 in D. If γ satisfies* (3.2), *then γ is a b-cone arc where b depends only on c in* (3.2) *and a in* (3.3). *Conversely, if γ is a b-cone arc, then γ satisfies* (3.2) *where c depends only on b.*

PROOF. The sufficiency is an immediate consequence of Lemma 3.1. For the necessity, since γ is a quasihyperbolic geodesic, it suffices to show there exists a constant c such that

$$k_D(y_1, y_2) \leqq c \log\left(1 + \frac{|y_1 - y_2|}{d(y_1, \partial D)}\right) \tag{3.10}$$

for all $y_1, y_2 \in \gamma$ with $y_1 \in \gamma(x_1, y_2)$.

Fix $y_1, y_2 \in \gamma$ and let $d = d(y_1, \partial D), t = |y_1 - y_2|, l = l(\gamma(y_1, y_2))$. If $t \leqq \frac{d}{2}$, then $d(y_2, \partial D) \geqq t$ and

$$k_D(y_1, y_2) \leqq \log\left(1 + \frac{2t}{d}\right) \leqq 2 \log\left(1 + \frac{t}{d}\right) \tag{3.11}$$

by Lemma 2.1; this is the required inequality (3.10) with $c = 2$. If $t > \frac{d}{2}$, choose $y \in \gamma$ so that $l(\gamma(y_1, y)) = \frac{d}{2}$. Then $|y_1 - y| \leqq \frac{d}{2}$ and

$$k_D(y_1, y) \leqq \log 2 \tag{3.12}$$

by (3.11). Next if γ is parametrized by arclength s with $\gamma(0) = y_1$, then for each $x \in \gamma(y_1, y_2)$

$$s \leqq l(\gamma(x_1, x)) \leqq b\, d(x, \partial D)$$

whence

$$k_D(y, y_2) = \int_{\gamma(y, y_2)} d(x, \partial D)^{-1}\, ds \leqq b \int_{d/2}^{l} s^{-1}\, ds = b \log \frac{2l}{d} \tag{3.13}$$

by (1.3). Finally

$$l \leqq l(\gamma(x_1, y_2)) \leqq b\, d(y_2, \partial D) \leqq b(d(y_1, \partial D) + |y_1 - y_2|) = b(t + d)$$

by (1.3), and since $b > 1$,

$$\begin{aligned} k_D(y_1, y_2) &\leqq \log 2 + b \log(2b) + b \log\left(1 + \frac{t}{d}\right) \\ &\leqq 2b \log(2b) + b \log\left(1 + \frac{t}{d}\right) \\ &\leqq \left(\frac{2b \log(2b)}{\log(3/2)} + b\right) \log\left(1 + \frac{t}{d}\right) \end{aligned}$$

by (3.12) and (3.13). Thus again we obtain inequality (3.10) with $c = c(b)$ and the proof for Theorem 3.9 is complete.

We derive next a second criterion for a quasihyperbolic geodesic γ joining x_1 to x_0 in D to be a cone arc. In this case, inequality (3.2) is replaced by an engulfing condition, namely that for some constant $c \geqq 1$,

$$\gamma(x_1, x) \subset \bar{B}(x,\ c\, d(x, \partial D)) \tag{3.14}$$

for all $x \in \gamma$.

3.15. Remark. It follows from [MS, pp. 385–386] that D is a John domain with center x_0 if and only if for each $x_1 \in D$ there exists an arc γ from x_1 to x_0 which satisfies (3.14) for some constant $c = c(D)$. Thus condition (3.14) characterizes John domains. However, an arbitrary arc γ which satisfies (3.14) need not be a b-cone arc with $b = b(c)$.

3.16 Theorem. *Suppose that γ is a quasihyperbolic geodesic joining x_1 to x_0 in D. If γ satisfies* (3.14), *then γ is a b-cone arc where b depends only on c and n. Conversely, if γ is a b-cone arc, then γ satisfies* (3.14) *where $c = b$.*

Proof. The necessity is an immediate consequence of inequality (1.3). For the sufficiency we again define inductively a sequence of points $y_1, \ldots, y_{m+1}$ in γ. Set

$y_1 = x_1$, suppose that y_j has been defined for some $j \geqq 1$ and set $d_j = d(y_j, \partial D)$. If

$$|x_0 - y_j| \geqq \tfrac{1}{2}d_j,$$

let y_{j+1} denote the last point of $\gamma(y_j, x_0)$ for which

$$|y_{j+1} - y_j| = \tfrac{1}{2}d_j$$

as we traverse γ from y_j towards x_0; otherwise let $y_{j+1} = x_0$ and $m = j$.

Now set $\gamma_j = \gamma(y_j, y_{j+1})$ and $l_j = l(\gamma_j)$. If B is any ball with $\bar{B} \subset D$, then $\bar{B} \cap \gamma$ is connected by Theorem 2.2 in [M] because γ is a quasihyperbolic. Hence it follows that

$$\gamma_j \subset \bar{B}(y_j, \tfrac{1}{2}d_j) \tag{3.17}$$

for $j = 1, \dots, m$ and that

$$|y_k - y_j| \geqq \tfrac{1}{2}d_j \tag{3.18}$$

for $1 \leqq j < k \leqq m$.

Since $|y_j - y_{j+1}| \leqq \tfrac{1}{2}d_j$,

$$t_D(y_i, y_{i+1}) \leq \log 2 \tag{3.19}$$

by Lemma 2.1 while

$$\log\left(1 + \frac{l_j}{d_j}\right) \leqq k_D(\gamma_j) \tag{3.20}$$

by Lemma 2.6. Because γ_j is a quasihyperbolic geodesic, these inequalities imply that $l_j \leqq d_j$, and with (3.14) we conclude that

$$l_j \leqq d_j \leqq (c+1)d_k \tag{3.21}$$

for $1 \leqq j \leqq k \leqq m$.

Choose an integer $p = p(c, n)$ so that $8^{-n}p > (c+1)^n$. Observe that if $m > p$, then for each $j \in (p, m]$ there exists an integer $\tilde{j}$ such that

$$1 \leqq j - \tilde{j} \leqq p, \quad d_{\tilde{j}} \leqq \tfrac{1}{2}d_j. \tag{3.22}$$

For if this were not the case we would have

$$d_k > \tfrac{1}{2}d_j \tag{3.23}$$

for $j - p \leqq k < j$. Then the balls $B_k = B(y_k, \tfrac{1}{8}d_j)$ would be disjoint by (3.18) and (3.23), they would lie in $B = B(y_j, (c+1)d_j)$ by (3.14), and we would obtain

$$p\Omega_n(\tfrac{1}{8}d_j)^n = \sum m(B_k) \leqq m(B) = \Omega_n((c+1)d_j)^n$$

contradicting our choice of the integer p.

Now fix $x \in \gamma$. Then $x \in \gamma_j$ for some integer $j \leqq m$. Next we can use inequality

(3.22) to define inductively a decreasing sequence of integers $j_1, \dots, j_{q+1}$ with $j_1 = j$ and $j_{q+1} = 0$ such that

$$1 \leqq j_k - j_{k+1} \leqq p, \quad d_{j_k} \leqq 2^{1-k} d_j \tag{3.24}$$

for $k = 1, \dots, q$. Then

$$\begin{aligned} l(\gamma(x_1, x)) &\leqq \sum_1^q (l_{j_k} + \dots + l_{j_{k+1}+1}) \\ &\leqq \sum_1^q (j_k - j_{k+1})(c+1) d_{j_k} \qquad (3.25) \\ &\leqq 2p(c+1) d_j \end{aligned}$$

by (3.21) and (3.24). Finally $x \in \bar{B}(y_j, \frac{1}{2} d_j)$ by (3.17). Hence

$$d(x, \partial D) \geqq \tfrac{1}{2} d_j \tag{3.26}$$

and we obtain (1.3) with $b = 4p(c+1)$ from (3.25) and (3.26). This completes the proof of Theorem 3.16.

We require the following hyperbolic analogue of Theorem 3.16 in what follows.

3.27. THEOREM. *Suppose that D is a simply connected domain in R^2 and that γ is a hyperbolic geodesic joining x_1 to x_0 in D. If γ satisfies* (3.14), *then γ is a b-cone arc where b depends only on c. Conversely, if γ is a b-cone arc, then γ satisfies* (3.14) *where $c = b$.*

PROOF. The necessity is clear. For the sufficiency we define the points $y_1, \dots, y_{m+1}$ in γ as in the proof for Theorem 3.16. If B is any disk with $\bar{B} \subset D$, then $\bar{B} \cap \gamma$ is connected by Theorem 2 in [J]; hence (3.17) and (3.18) hold as above. Next since D is simply connected, the Schwarz lemma and Koebe distortion theorem imply that

$$\tfrac{1}{4} d(x, \partial D)^{-1} \leqq \rho_D(x) \leqq d(x, \partial D)^{-1} \tag{3.28}$$

where ρ_D is the hyperbolic density in D. Thus for $1 \leqq j \leqq m$,

$$h_D(y_j, y_{j+1}) \leqq k_D(y_j, y_{j+1}) \leqq \log 2$$

and

$$\frac{1}{4} \log\left(1 + \frac{l_j}{d_j}\right) \leqq \frac{1}{4} k_D(\gamma_j) \leqq h_D(\gamma_j)$$

by (3.19), (3.20) and (3.28). Hence $l_j \leqq 15 d_j$,

$$l_j \leqq 15 d_j \leqq 15(c+1) d_k \tag{3.29}$$

for $1 \leqq j \leqq k \leqq m$ and the proof concludes as above with (3.29) in place of (3.21).

4. Simply connected John domains in $\mathbb{R}^2$.

We show next that quasihyperbolic and hyperbolic geodesics in a simply connected John domain D in $\mathbb{R}^2$ satisfy the cone condition (1.3).

4.1. THEOREM. *Suppose that D is a simply connected c-John domain in $\mathbb{R}^2$ with center x_0 and that x_1 is a point in D. If γ is either a quasihyperbolic or hyperbolic geodesic from x_1 to x_0 in D, then γ is a b-cone arc where b depends only on c.*

PROOF. Let a denote the absolute constant in Lemma 2.9. By Theorems 3.16 and 3.27, it is sufficient to show that γ satisfies the engulfing condition

$$\gamma(x_1,x) \subset \bar{B}(x,(a+2)(2c+1)d(x,\partial D)) \tag{4.2}$$

for all $x \in \gamma$.

Suppose that (4.2) does not hold for some $x \in \gamma$ and let $d = d(x,\partial D)$ and $r = (a+1)d$. Then there exists a point $z_1 \in \gamma(x_1,x)$ such that

$$(a+2)(2c+1)d < |z_1 - x| \leqq \operatorname{dia}(D), \tag{4.3}$$

and since D is a c-John domain with center x_0, we see that

$$|x_0 - x| \geqq d(x_0,\partial D) - d(x,\partial D) \geqq \frac{\operatorname{dia}(D)}{2c} - d > (a+1)d = r.$$

Thus x_0 and x are separated by $C(x,r)$. Then since $d < r$ and since D is simply connected, $C(x,r)\setminus D \neq \emptyset$ and there exists an open subarc C of $C(x,r) \cap D$ which separates x_0 and x in D. (See, for example, Theorem VI.7.1 in [N]). In particular, there exists a point $y_0 \in \gamma(x_0,x) \cap C$.

Suppose next that $\gamma(x_1,x) \cap C = \emptyset$ and let z_1 be as in (4.3). By hypothesis there exists a c-cone arc β joining z_1 to x_0 in D which must intersect C at some point z. With (4.3) we obtain

$$\operatorname{dia}(C) \geqq d(z,\partial D) \geqq \frac{1}{c} l(\beta(z_1,z)) \geqq \frac{1}{c}|z_1 - z| \geqq \frac{1}{c}(|z_1 - x| - |z - x|) > 2r,$$

contradicting the fact that C is a subarc of $C(x,r)$. We conclude that there exists a point $y_1 \in \gamma(x_1,x) \cap C$.

Now y_0, x, y_1 is an ordered triple of points on γ, $|y_0 - x| = |y_1 - x| = r$ and $C(x,r)$ contains a subarc which joins y_0 and y_1 in D. Hence Lemma 2.9 implies that

$$(a+1)d = r \leqq a\,d(x,\partial D) = ad$$

and we have a contradiction. Thus (4.2) holds for each $x \in \gamma$ and the proof for Theorem 4.1 is complete.

There are many ways to describe the class of simply connected John domains in R^2. The following characterization, reminiscent of Ahlfors' beautiful criterion for quasicircles, follows from results in Sections 2 and 3. It arose in the course of a coversation with *C*. Pommerenke; see [P2].

4.4. THEOREM. *Suppose that D is a simply connected bounded domain in* R^2. *Then D is a John domain if and only if there exists a constant a such that for each crosscut* α *of D*,

$$\min(\operatorname{dia}(D_1), \operatorname{dia}(D_2)) \leqq a \operatorname{dia}(\alpha) \tag{4.5}$$

where D_1 *and* D_2 *are the components of* $D \setminus \alpha$.

PROOF. Suppose that D is a John domain with center x_0, let α be a crosscut of D and let D_1 be a component of $D \setminus \alpha$ which does not contain x_0. If $x_1, x_2 \in D_1$, then for $j = 1, 2$ there exists a b-cone arc γ_j which joins x_j to x_0 and meets α in a point y_j; obviously

$$|y_1 - y_2| \leqq \operatorname{dia}(\alpha).$$

Then (1.3) and the fact that α joins y_j to ∂D imply that

$$|x_j - y_j| \leqq l(\gamma_j(x_j, y_j)) \leqq b\, d(y_j, \partial D) \leqq b \operatorname{dia}(\alpha)$$

for $j = 1, 2$. Thus

$$|x_1 - x_2| \leqq |x_1 - y_1| + |y_1 - y_2| + |x_2 - y_2| \leqq (2b + 1) \operatorname{dia}(\alpha)$$

and we obtain (4.5) with $a = 2b + 1$.

Suppose next that D satisfies condition (4.5) for some constant a. We show first there exists a point $x_0 \in D$ such that

$$\operatorname{dia}(D) \leqq 4ac\, d(x_0, \partial D), \tag{4.6}$$

where c is the absolute constant in Lemma 2.16. For this choose $y_1, y_2 \in D$ so that

$$\operatorname{dia}(D) \leqq 2|y_1 - y_2|,$$

let γ be the hyperbolic geodesic joining y_1 and y_2 in D and choose $x_0 \in \gamma$ so that $|y_1 - x_0| = |y_2 - x_0|$. Then by Lemma 2.16 there exists a crosscut α of D containing x_0 which separates y_1 and y_2 and satisfies

$$l(\alpha) \leqq c\, d(x_0, \partial D). \tag{4.7}$$

If D_1, D_2 denote the components of $D \setminus \alpha$, then (4.5) implies that

$$\begin{cases} \operatorname{dia}(D) \leqq 2|y_1 - y_2| \leqq 4|y_j - x_0| \\ \qquad \leqq 4 \min(\operatorname{dia}(D_1), \operatorname{dia}(D_2)) \leqq 4a\, l(\alpha) \end{cases} \tag{4.8}$$

and we obtain (4.6) from (4.7) and (4.8).

Now fix $x_1 \in D$, let γ be the hyperbolic geodesic which joins x_1 to x_0 *in D* and choose $x \in \gamma \setminus \{x_0, x_1\}$. Again by Lemma 2.16 there exists a crosscut α of D containing x which separates the components of $\gamma \setminus \{x\}$ and satisfies

$$l(\alpha) \leqq c\, d(x, \partial D). \tag{4.9}$$

Let D_0 and D_1 denote the components of $D \setminus \alpha$ which contain x_0 and x_1, respectively, and set $r = ac\, d(x, \partial D)$. If $d(x_0, \partial D) \leqq 3r$, then

$$\operatorname{dia}(D_1) \leqq \operatorname{dia}(D) \leqq 12acr \tag{4.10}$$

by (4.6). Otherwise since $a \geqq 1$ and $c \geqq 1$,

$$|x - x_0| \geq d(x_0, \partial D) - d(x, \partial D) > 2n \tag{4.11}$$

and with (4.9) and (4.11) we obtain

$$B(x_0, r) \subset D \setminus \alpha, \quad \operatorname{dia}(D_0) > 2r.$$

Then (4.5) and (4.9) imply that

$$\min(\operatorname{dia}(D_0), \operatorname{dia}(D_1)) \leqq r$$

and hence that

$$\operatorname{dia}(D_1) \leqq r. \tag{4.12}$$

Since $\gamma(x_1, x) \subset D_1 \cup \{x\}$, we conclude from (4.10) and (4.12) that

$$\gamma(x_1, x) \subset \bar{B}(x, 12(ac)^2 d(x, \partial D))$$

and thus by Theorem 3.27 that γ is a b-cone *arc* where $b = b(a)$. This completes the proof of Theorem 4.4.

5. Examples.

We conclude this paper with examples which show that a quasihyperbolic geodesic in a c-John domain need not be a b-cone arc with $b = b(c)$ unless $n = 2$ and D is simply connected. Thus these hypotheses on D in Theorem 4.1 are necessary.

5.1. Example. *For each $b \geqq 1$ there exists a doubly connected 10-John domain D_1 in R^2 with center x_0 and a point x_1 in D_1 such that any b-cone arc from x_1 to x_0 is not a quasihyperbolic geodesic.*

5.2. Example. *There exists an infinitely connected 10-John domain D_2 in R^2 with center x_0 and, for each $b \geqq 1$, a point x_1 in D_2 such that any b-cone arc from x_1 to x_0 is not a quasihyperbolic geodesic.*

5.3. Basic construction. For each $\sigma \in (0, \frac{1}{4}]$ and $\tau \in [0, \frac{1}{4}]$ set

$$\begin{cases} S_1 = \{z = u + iv\colon \sigma^4 \leqq u \leqq \sigma, v = \tau + u \tan\theta\}, \\ S_2 = \{z = u + iv\colon \sigma^4 \leqq u \leqq \sigma, v = \tau - u \tan\theta\} \end{cases} \tag{5.4}$$

where $\theta = \arcsin(1/10)$, and let

$$D_0 = B(0,2)\setminus(S_1 \cup S_2), \quad x_0 = -1, \quad x_1 = \sigma^3 + i\tau. \tag{5.5}$$

5.6. Lemma. *D_0 is a 10-John domain with center x_0.*

Proof. Fix $x = u + iv \in D_0$ and let

$$y = \begin{cases} \dfrac{x}{|x|} \text{ if } |x| \geqq 1, \\ (1 - v^2)^{1/2} + iv \text{ if } |x| < 1 \text{ and } |v - \tau| \leqq u\tan\theta, \\ -(1 - v^2)^{1/2} + iv \text{ if } |x| < 1 \text{ and } |v - \tau| > u\tan\theta. \end{cases} \tag{5.7}$$

Then it is easy to check that $\alpha = [x, y]$ is a 10-cone arc joining x to y in D_0. Next the unit circle contains an arc β joining y to x_0 with $l(\beta) \leqq \pi$ and $d(z, \partial D) \geqq \frac{5}{8}$ for $z \in \beta$. Hence $\gamma = \alpha \cup \beta$ is a 10-cone arc from x to x_0 in D_0.

5.8. Lemma. *If $b < \dfrac{6}{\sigma}$ and if γ is a b-cone arc from x_1 to x_0 in D_0, then γ is not a quasihyperbolic geodesic.*

Proof. Fix $b < \dfrac{6}{\sigma}$, suppose that γ is a b-cone arc joining x_1 to x_0 in D_0 and set

$$T_1 = \{z = \sigma^4 + i(\tau + t)\colon |t| \leqq \sigma^4 \tan\theta\}, \quad T_2 = \{z = \sigma + i(\tau + t)\colon |t| \leqq \sigma\tan\theta\}.$$

Then $\gamma \cap T_2 \neq \emptyset$ since otherwise we could find a point $w \in \gamma \cap T_1$ such that

$$\tfrac{3}{4}\sigma^3 \leqq \sigma^3 - \sigma^4 \leqq l(\gamma(x_1, w)) \leqq bd(w, \partial D_0) \leqq b\sigma^4 \tan\theta < \frac{b\sigma^4}{9}$$

contradicting our coice of b.

Next set $y_1 = \sigma^4 + i\tau$, $z_1 = -\frac{1}{2} + i\tau$ and let w_1 be the first point in $\gamma \cap T_2$ as we traverse γ from x_1 towards x_0. If $x \in \gamma(x_1, w_1)$, then

$$d(x, \partial D_0) \leqq \operatorname{Re}(x)\tan\theta < \frac{\operatorname{Re}(x)}{9}$$

and we obtain

$$k_{D_0}(\gamma) = \int_\gamma d(x, \partial D_0)^{-1}\, ds > 9\log\left(\frac{\operatorname{Re}(w_1)}{\operatorname{Re}(x_1)}\right) = 18\log\frac{1}{\sigma}. \tag{5.9}$$

Similarly if $x \in \alpha = [x_1, y_1]$, then

$$d(x, \partial D_0) \geqq \operatorname{Re}(x) \sin\theta = \frac{\operatorname{Re}(x)}{10}$$

and hence

$$k_{D_0}(x_1, y_1) \leqq \int_\alpha d(w, \partial D_0)^{-1}\, ds \leqq 10 \log\left(\frac{\operatorname{Re}(x_1)}{\operatorname{Re}(y_1)}\right) = 10 \log \frac{1}{\sigma}. \tag{5.10}$$

Next

$$d(y_1, \partial D_0) = \sigma^4 \tan\theta, \quad d(z_1, \partial D_0) \geqq \tfrac{1}{2} + \sigma^4, \quad |y_1 - z_1| = \tfrac{1}{2} + \sigma^4$$

and thus

$$k_{D_0}(y_1, z_1) \leqq \log(1 + (2 + \sigma^{-4}) \cot\theta) < 6 \log \frac{1}{\sigma} \tag{5.11}$$

by Lemma 2.1. Finally $d(x, \partial D_0) \geqq \frac{1}{2}$ for $x \in \beta = [z_1, x_0]$ and hence

$$k_{D_0}(z_1, x_0) \leqq 2l(\beta) < 2 \log \frac{1}{\sigma}. \tag{5.12}$$

Then (5.9), (5.10), (5.11) and (5.12) imply that

$$k_{D_0}(x_1, x_0) < 18 \log \frac{1}{\sigma} < k_{D_0}(\gamma) \tag{5.13}$$

and hence that γ is not a quasihyperbolic geodesic in D_0.

5.14 Proof for Example 5.1. Fix $b \geqq 1$, *let* $\theta = \arcsin(1/10)$ and choose $\sigma \in (0, \frac{1}{4})$ so that $b < \dfrac{6}{\sigma}$. Next set

$$S_1 = \{z = u + iv \colon \sigma^4 \leqq u \leqq \sigma,\ v = \tau + u \tan\theta\},$$

$$\tilde{S}_2 = \{z = u + iv \colon \sigma^4 \leqq u \leqq 2,\ v = \tau - u \tan\theta\}$$

and let

$$D_1 = B(0, 2) \setminus (S_1 \cup \tilde{S}_2).$$

Suppose that $x = u + iv \in D_1$. If $|x| < 1$, let y and α be as in the proof of Lemma 5.6. Then again there exists a subarc β of the unit circle such that $\gamma = \alpha \cup \beta$ is a 10-cone arc from x to y in D_1. If $|x| \geqq 1$, choose $\phi \in [-\pi, \pi]$ so that $x = |x| e^{i\phi}$ and let γ denote the arc defined by

$$x(t) = \begin{cases} |x|^{1-t} e^{i((1-t)\phi + t\pi)} & \text{if } \phi > -\theta, \\ |x|^{1-t} e^{i((1-t)\phi - t\pi)} & \text{if } \phi < -\theta, \end{cases} \qquad t \in [0, 1].$$

Then an elementary calculation shows that γ is again a 10-cone arc from x to x_0 in D_1. Thus D_1 is a 10-John domain.

Next suppose that γ is a b-cone arc from $x_1 = \sigma^3$ to $x_0 = -1$ in D_1. Then the proof of Lemma 5.8 with $\tau = 0$ implies that (5.9), (5.10), (5.11), (5.12) and (5.13) hold with D_1 in place of D_0. Hence γ is not a quasihyperbolic geodesic in D_1.

5.15. Proof for Example 5.2. Let $\theta = \arcsin(1/10)$, let

$$S_{1,j} = \{z = u + iv\colon\ \sigma_j^4 \leqq u \leqq \sigma_j,\ v = \tau_j + u\tan\theta\},$$

$$S_{2,j} = \{z = u + iv\colon\ \sigma_j^4 \leqq u \leqq \sigma_j,\ v = \tau_j - u\tan\theta\}$$

for $j = 1, 2, \ldots$, where $\sigma_j = \tau_j = 4^{-j}$, and set

$$D_2 = B(0,2)\setminus \bigcup_1^\infty (S_{1,j} \cup S_{2,j}).$$

Next fix $x = u + iv \in D_2$, let

$$(5.16)\qquad y = \begin{cases} \dfrac{x}{|x|} & \text{if } |x| \geqq 1, \\ (1 - v^2)^{1/2} + iv & \text{if } |x| < 1 \text{ and } |v - \tau_j| \leqq u\tan\theta \text{ for some } j, \\ -(1 - v^2)^{1/2} + iv & \text{if } |x| < 1 \text{ and } |v - \tau_j| > u\tan\theta \text{ for all } j, \end{cases}$$

and set

$$C_k = \{z = u + iv\colon\ 0 \leqq u < \infty,\ |v - \tau_k| \leqq u\tan\theta\}$$

for $k = 1, 2, \ldots$. Then

$$(S_{1,k} \cup S_{2,k}) \subset \partial C_k, \quad (S_{1,j} \cup S_{2,j}) \cap C_k = \emptyset \quad \text{for } j \neq k,$$

and again it is easy to show that $\alpha = [x, y]$ is a 10-cone arc from x to y. Hence D_2 is a 10-John domain as in the proof of Lemma 5.6.

Finally fix $b \geqq 1$, choose j so that $b\sigma_j < 6$ and let γ be a b-cone curve which joins $x_1 = \sigma_j^3 + i\tau_j$ to $x_0 = -1$ in D_2. Then again the proof of Lemma 5.8 with $\sigma = \tau = \sigma_j = \tau_j$ shows that γ is not a quasihyperbolic geodesic in D_2.

5.17. Remark. Similar examples exist in $\mathbf{R}^n$ for each $n \geqq 2$. For example, in the n-dimensional analogue of the domain D_2 we replace each set $S_{1,j} \cup S_{2,j}$ by the lateral surface $\sum_j$ of a frustum of an n-cone with vertex angle θ. Then when $n > 2$, the frustums $\sum_j$ can be joined by segments so that the resulting domain has a connected boundary.

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