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INTERSECTION PROPERTIES OF BALLS IN TENSOR PRODUCTS OF SOME BANACH SPACES

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1. Introduction.

There have been several papers in the literature dealing with tensor products of partially ordered linear spaces ([12]) and tensor products of compact convex sets ([23]), and in particular of Choquet simplexes ([3], [17]). In the last two references, it is proved for instance that Choquet simplexes are closed under certain tensor products. Since a Choquet simplex is characterized by the fact that $A(K)^*$ is an L^1-space, (see [1]), where $A(K)$ denotes the Banach space of continuous affine functions on K with the supremum norm, it seems natural to ask whether the general class of L^1-preduals have such permanence properties under (injective) tensor products. The answer to this question comes out as a special case of a more general result (Theorem 3.1) which characterizes L^1-preduals A, among Banach spaces, by intersection properties of balls in the *injective* tensor products of A with other Banach spaces. The proof, which is given in Section 3 of this paper, is non-trivial inasmuch as several preliminary resuls of independent interest have to be established en route to the theorem. We use this theorem in Section 4 to provide a characterization of L^1-spaces X by looking at a certain decomposition property (known to be dual to intersection property of balls – see Section 2 for more details) of *projective* tensor products of X with other Banach spaces.

Our results have been formulated and proved in the context of *complex* Banach spaces.

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2. Notations and resume of known results.

Throughout our discussion in this section and the next, A will denote a *complex* Banach space and A^* its dual space. For a Banach space X, its closed unit ball is written as X_1 and $B(a, r)$ denotes the *closed* ball in X with centre a and radius r. The set of extreme points of a convex set K will be denoted by ∂K.

Let n, k be integers with $n > k \geqq 3$. We say that A has the *almost n.k.* I.P. (to be read as the almost *n.k.* intersection property) if, for every family $\{B(a_j, r_j)\}_{j=1}^n$ of n balls in A such that any k of them intersect, we have $\bigcap_{j=1}^n B(a_j, r_j + \varepsilon) \neq \emptyset \; \forall \varepsilon > 0$. (If we can take $\varepsilon = 0$, we say that A has the *n.k.*I.P.). It was proved in [20] that A is an L^1-predual, i.e. A^* is isometric to an L^1-space, if and only if A has the 4.3.I.P.

Introducing the space $H^n(A^*) = \{(x_1 \ldots, x_n) \in (A^*)^n : \sum_{k=1}^n x_k = 0\}$ with the norm $\|(x_1, \ldots, x_n)\| = \sum_{k=1}^n \|x_k\|$, the following are known to be equivalent (see [19] & [20]):

(1) A has the almost *n.k.*I.P.

(2) Each extreme point in $H^n(A^*)_1$ has at most k non-zero components

(3) A^* has the $R_{n,k}$ property, i.e. if $(x_1, \ldots, x_n) \in H^n(A^*)$, there exist $(z_{i1}, \ldots, z_{in}) \in H^n(A^*)$, $\left(1 \leqq i \leqq \binom{n}{k}\right)$, such that

(i) $$(x_1, \ldots, x_n) = \sum_i (z_{i1}, \ldots, z_{in})$$

(ii) $$\|x_j\| = \sum_i \|z_{ij}\|, (\forall j),$$

and $(z_{i1}, \ldots, z_{in})$ has at most k non-zero components for all i.

In view of the equivalence of (1) and (3), we see that the $R_{n,k}$ property is "dual" to the almost *n.k.*I.P., as remarked in the introduction.

We shall make repeated use of the following result:

Let $(x_1, \ldots, x_n) \in H^n(A^*)_1$ with $x_i \neq 0$ and $x_i/\|x_i\| \in \partial A_1^* \; \forall i$. Then $(x_1, \ldots x_n) \in \partial H^n(A^*)_1$ if and only if $\{\|x_k\|, x_k\}_{k=1}^n \subseteq \mathsf{R} \times A^*$ are linearly independent over R. (The proof is the same as that given for Proposition 2.1 in [22]. It may be noted that the condition $x_i/\|x_i\| \in \partial A_1^* \; \forall i$ eliminates the need for assuming that A has the $(n + 1).n$.I.P. as in that proposition).

For unexplained notations and results concerning tensor products of Banach spaces, see [4].

3. Lemmata and Main Result.

Let Q be a compact Hausdorff space and E a Banach space. We denote by $C(Q, E)$ the space of continuous E-valued functions on Q with the supremum norm $\|x(\cdot)\|_\infty = \sup\{\|x(q)\| : q \in Q\}$. It is a standard fact (see [4]) that $C(Q, E)$ can be identified isometrically with the injective tensor product $C(Q) \check{\otimes} E$. We first prove a result which generalises Proposition 3.3 in [22] where E was assumed to be a finite-dimensional space with any norm.

LEMMA 1. *$C(Q, E)$ has the almost n.k.*I.P. *if E has the almost n.k.*I.P.

PROOF. We know, from [26] & [27], that there is a 1-1 correspondence $\Phi \leftrightarrow f_e$ between elements of $C(Q, E)^*$ and the space of all set functions f_e, defined on all Borel sets $e \subset Q$, with values in E^*, countably additive, regular and of bounded variation, endowed with the usual vector space operations and given the toal variation norm, with

$$\Phi[x(\cdot)] = \int \langle x(q), df_q \rangle,$$

$x(\cdot) \in C(Q, E)$. (This is the so-called *Gowurin integral* – see [27] for unexplained notations and resuls used here).

We will prove that if $A = C(Q, E)$, then

$$\partial H^n(A^*)_1 \subseteqq \left\{ (\partial_{q, f_i})_{i=1}^n : q \in Q, (f_1, \ldots, f_n) \in H^n(E^*), \sum_{i=1}^n \|f_i\| = 1 \right\}$$

where

$$\partial_{q,f}(e) = \begin{cases} f & \text{if } q \in e \\ 0 & \text{if } q \notin e, \end{cases}$$

(e denoting a Borel subset of Q). From this, it is easily seen that

$$\partial H^n(A^*)_1 = \{(\partial_{q, f_i})_{i=1}^n : q \in Q, (f_1, \ldots, f_n) \in \partial H^n(E^*)_1\}$$

and the lemma follows.

Suppose, to get a contradiction, that

$$(\Phi_1, \ldots, \Phi_n) \in H^n(A^*)_1 \setminus \overline{\text{conv}} \left\{ (\partial_{q, f_i})_{i=1}^n : q \in Q, (f_i)_{i=1}^n \in H^n(E^*), \sum_{i=1}^n \|f_i\| = 1 \right\}.$$

By the separation theorem, $\exists\, x_i(\cdot) \in C(Q, E)$, $1 \leqq i \leqq n$, such that

$$\sup \left\{ \left| \sum_{i=1}^n f_i(x_i(q)) \right| : q \in Q, \sum_{i=1}^n f_i = 0, \sum_{i=1}^n \|f_i\| = 1 \right\}$$
$$< \left| \sum_{i=1}^n \Phi_i(x_i(\cdot)) \right| = \left| \sum_{i=1}^n \int \langle x_i(q), df_q^i \rangle \right|,$$

where f^i is the vector measure associated with Φ_i, $(1 \leqq i \leqq n)$. On uniformly approximating each $x_i(\cdot)$ by simple functions $y_i(\cdot) = \sum_{j=1}^{m_i} x_{ij} C_{e_{ij}}$, where C_e denotes the characteristic function of the set e, we see that the strict inequality above is still preserved with $x_i(\cdot)$ replaced by a suitable $y_i(\cdot)$, $(1 \leqq i \leqq n)$.

Noting that $\sum_{i=1}^{n} f^i = 0$, the right hand side of the inequality is thus

$$\left| \sum_{i=1}^{n} \sum_{j=1}^{m_i} f^i_{e_{ij}}(x_{ij}) \right|.$$

The Borel sets associated with the simple functions y_i, $(1 \leqq i \leqq n)$, are displayed in their respective order as follows:

$$\begin{array}{l} e_{11}, e_{12}, \ldots, e_{1m_1} \\ e_{21}, e_{22}, \ldots, e_{2m_2} \\ \cdots\cdots \\ e_{n1}, e_{n2}, \ldots, e_{nm_n}. \end{array}$$

There are thus $m_1 m_2 \ldots m_n$ disjoint Borel sets of the form $e_{1j_1} \cap e_{2j_2} \cap \cap \ldots e_{nj_n}$, (where $1 \leqq j \leqq m_1, \ldots 1 \leqq j_n \leqq m_n$), which partition Q. It is not hard to see that the left hand side of the above inequality is

$$\sup \left\{ \left| \sum_{i=1}^{n} f_i(x_{ij_i}) \right| : 1 \leqq j_1 \leqq m_1, \ldots, 1 \leqq j_n \leqq m_n, \sum_{i=1}^{n} f_i = 0, \sum_{i=1}^{n} \|f_i\| = 1 \right\}.$$

Denoting this sypremum by α, our inequality takes the form

$$\alpha < \sum_{i=1}^{n} \sum_{j=1}^{m_i} f^i_{e_{ij}}(x_{ij}).$$

Now, we write

$$\sum_{j=1}^{m_1} f^1_{e_{1j}}(x_{ij}) = \sum_{\substack{1 \leqq i_2 \leqq m_2 \\ \vdots \\ 1 \leqq i_n \leqq m_n}} f^1_{e_{11} \cap e_{2i_2} \cap \ldots \cap e_{ni_n}}(x_{11}) + \ldots$$

$$+ \sum_{\substack{1 \leqq i_2 \leqq m_2 \\ \vdots \\ 1 \leqq i_n \leqq m_n}} f^1_{e_{1m_1} \cap e_{2i_2} \cap \ldots \cap e_{ni_n}}(x_{1m_1})$$

$$\vdots$$

$$\sum_{j=1}^{m_n} f^n_{e_{nj}}(x_{nj}) = \sum_{\substack{1 \leqq i_1 \leqq m_1 \\ \vdots \\ 1 \leqq i_{n-1} \leqq m_{n-1}}} f^n_{e_{n1} \cap e_{1i_1} \cap \ldots \cap e_{(n-1)i_{n-1}}}(x_{n1}) + \ldots$$

$$+ \sum f^1_{e_{nm_n} \cap e_{1i_1} \cap \ldots \cap e_{(n-1)i_{n-1}}}(x_{nm_n}).$$

Consider for example the term $f^1_{e_{11} \cap e_{21} \ldots \cap e_{n1}}(x_{11})$ which occurs in the first row of the above array and comes from the set $e_{11} \cap e_{21} \cap \ldots \cap e_{n1}$. This set occurs once, and only once, in each subsequent row. We group these terms:

$$f^1_{e_{11} \cap e_{21} \cap \ldots \cap e_{n1}}(x_{11}) + f^2_{e_{21} \cap e_{11} \cap \ldots \cap e_{n1}}(x_{21}) + \ldots + f^n_{e_{m1} \cap e_{11} \cap \ldots \cap e_{(n-1)1}}(x_{n1})$$

and note that by assumption

$$\begin{aligned} |f^1_{e_{11} \cap e_{21} \cap \ldots \cap e_{n1}}(x_{11}) &+ f^2_{e_{21} \cap e_{11} \cap \ldots \cap e_{n1}}(x_{21}) + \ldots + f^n_{e_{n1} \cap e_{11} \cap \ldots \cap e_{(n-1)1}}(x_{n1})| \\ &\leqq \alpha [\|f^1_{e_{11} \cap e_{21} \cap \ldots \cap e_{n1}}\| + \|f^2_{e_{21} \cap e_{21} \cap \ldots \cap e_{m1}}\| + \ldots \\ &\quad + \|f^n_{e_{n1} \cap e_{11} \cap \ldots \cap e_{(n-1)1}}\|]. \end{aligned}$$

It follows therefore by grouping terms in this manner that

$$\begin{aligned} \alpha^{-1} \left| \sum_{i=1}^{n} \sum_{j=1}^{m_i} f^i_{e_{ij}}(x_{ij}) \right| &\leqq \sum_{\substack{1 \leqq i_2 \leqq m_2 \\ \vdots \\ 1 \leqq i_n \leqq m_n}} \|f^1_{e_{11} \cap e_{2i_2} \cap \ldots \cap e_{ni_n}}\| \\ &+ \ldots + \sum_{\substack{1 \leqq i_2 \leqq m_2 \\ \vdots \\ 1 \leqq i_n \leqq m_n}} \|f^1_{e_{1m_1} \cap e_{2i_2} \cap \ldots \cap e_{ni_n}}\| \\ &+ \cdots\cdots\cdots\cdots \\ &+ \sum_{\substack{1 \leqq i_1 \leqq m_1 \\ \vdots \\ 1 \leqq i_{n-1} \leqq m_{n-1}}} \|f^n_{e_{n1} \cap e_{1i_1} \cap \ldots \cap e_{(n-1)i_{n-1}}}\| \\ &+ \cdots + \|f^n_{e_{nm_n} \cap e_{1i_1} \cap \ldots \cap e_{(n-1)i_{n-1}}}\| \\ &\leqq \|f^1_e\| + \ldots \|f^n_e\| \leqq 1 \end{aligned}$$

and we therefore have the contradiction $\alpha < \alpha$. This completes the proof of Lemma 1.

REMARK. It is possible to get a shorter proof of this lemma by using the approach employed in [22] to prove a special case, and this in turn was based on some ideas in Lindenstrauss' 1964 memoir "Extensions of Compact Operators". But we have preferred to retain the present proof as it is based, like the rest of the paper, on vector-measures.

We next state a result which may be considered a vector-valued analogue of a complex version of Choquet's theorem proved in [14]. (See also [9] and [25]). In the discussion that follows, use will be made of results from Choquet theory, as expounded in [1] and [24], for instance. We will also make free use of some of the notations and results from [6], [9] and [25]. Henceforth T denotes the unit circle in the complex plane and A denotes a closed subspace of $C(Q)$, the latter having the supnorm. We assume that A separates the points of Q but that it may not necessarily contain the constants. $q \to \Phi(q) \in A_1^*$ is the usual evaluation map with respect to A. We regard $A \check{\otimes} E$ as a closed subspace of $C(Q, E)$ in the usual way. (See Proposition 7 in [4]). Here E is any Banach space.

LEMMA 2. *Let $\Phi \in (A \check{\otimes} E)^*$. Then Φ can be represented by a vector measure $f: \mathscr{B}(Q) \to E^*$, countably additive, regular and of bounded variation, $\|f\| = \|\Phi\|$,*

$$\Phi[x(\cdot)] = \int \langle x(q), df_q \rangle, \quad (x(\cdot) \in A \check{\otimes} E),$$

with the property that all x-sections $(x \in E)$ of f are boundary measures with respect to $A \subsetneqq C(Q)$ in the sense defined in [25].

(Here $\mathscr{B}(Q)$ is the Borel σ-algebra on Q. If $\tilde{\Phi}$ denotes the functional in $C(Q, E)^*$ defined by f, the *x-section* μ_x of f is the complex measure representing $\tilde{\Phi} \mid C_x$ where $C_x = \{x\alpha(\cdot) : \alpha \in C(Q)\}$; note that $\|\mu_x\| = \|x\| \, \|\tilde{\Phi} \mid C_x\|$-see page 193 of [27].)

The proof of this result may be found in the paper "Integral representation by boundary vector measure", Canadian Math. Bulletin Vol. 25 [2], 1982, by P. Saab.

For complex measures λ, μ on Q, recall from [25] that $\lambda \approx \mu$ if $\lambda(g) = \mu(g)$ for all $g \in C(Q)$ such that $sg(q_1) = tg(q_2)$ whenever $s, t \in T$ and $q_1, q_2 \in Q$ and $sh(q_1) = th(q_2)$ for all $h \in A$. We will need two more results in the proof of our main Theorem 3.1.

LEMMA 4. *Let X and Y be Banach spces. Suppose that $\{\Phi_1, \ldots, \Phi_m\}$ (resp. $(\psi_n, \ldots, \psi_b)$ are m (resp. n) complex linearly independent functionals in X^* (resp. Y^*) with $\phi_i/\|\phi_1\| \in \partial X_1^*$, $1 \leqq i \leqq m$ (resp. $\psi_j/\|\psi_j\| \in \partial Y_1^*$, $1 \leqq j \leqq n$). If $\phi_0 = \sum_{i=1}^{m} z_i\phi_i \in X^*$ and $\psi_0 = \sum_{j=1}^{n} \omega_j\psi_j \in Y^*$, with none of the z_i's or ω_j's zero, then*

$$\frac{(-\phi_0 \otimes \psi_0, z_i\omega_j\phi_i \otimes \psi_j)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}}{\gamma} \in \partial H^{mn+1}(X \check{\otimes} Y)_1^*$$

where $\gamma = \|\phi_0\| \, \|\psi_0\| + \sum_{i,j} |z_i| \, |\omega_j| \, \|\phi_i\| \, \|\psi_j\|$.

PROOF. It is easy to see that $\{\phi_i \otimes \psi_j\}_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ are linearly independent functionals in $(X \check{\otimes} Y)^*$. Suppose that

$$0 = c_0(-\phi_0 \otimes \psi_0, \|\phi_0\| \, \|\psi_0\|) + \sum_{i,j} c_{ij}(z_i\omega_j\Phi_i \otimes \psi_j, |z_i| \, |\omega_j| \, \|\phi_i\| \, \|\psi_j\|)$$

for *real* scalars c_0 and $c_{ij} (1 \leqq i \leqq m, 1 \leqq j \leqq n)$.

First suppose that $c_0 = 0$. By the linear independence of the $\{\phi_i \otimes \psi_j\}_{i,j}$, it follows that $c_{ij} = 0 \, \forall i,j$.

If $c_0 \neq 0$ then

$$\phi_0 \otimes \psi_0 = \sum_{i,j} (c_{ij} c_0^{-1}) z_i \omega_j \phi_i \otimes \psi_j$$

and again by the linear independence of $\{\phi_i \otimes \psi_j\}_{i,j}$, it follows that $c_{ij} = c_0 \, \forall i,j$. Since

$$c_0 \|\phi_0\| \, \|\psi_0\| + \sum_{i,j} c_{ij} |z_i| \, |\omega_j| \, \|\phi_i\| \, \|\psi_j\| = 0$$

we get $c_0 = 0$, a contradiction. Thus, $c_0 = c_{ij} = 0 \, \forall i,j$ and we conclude the proof by first observing that

$$\frac{z_i \phi_i \otimes \omega_j \psi_j}{|z_i| \, \|\phi_i\| \, |\omega_j| \, \|\psi_j\|} = \left(\frac{z_i}{|z_i|} \frac{\phi_i}{\|\phi_i\|}\right) \otimes \left(\frac{\omega_j}{|\omega_j|} \cdot \frac{\psi_j}{\|\psi_j\|}\right) \in \partial(X \check{\otimes} Y)_1^*$$

(according to [21]) and then appealing to the result quoted at the end of section 2.

LEMMA 5. *Let X be a Banach space with the* 2k.2k − 1.I.P *and let* $(x_1, \ldots, x_j) \in \partial H^j(X^*)_1$ *with* $x_i \|x_i\|^{-1} \in \partial X_1^* (1 \leqq i \leqq j)$ *and* $j \leqq 2k - 1$. *Then*

$$\dim_{\mathbb{C}} \mathrm{sp}_{\mathbb{C}} \{x_1, \ldots, x_j\} \leqq k - 1.$$

PROOF. This was proved in [22, Theorem 2.6] for $j = 2k - 1$. Assume $j = (2k - 1) - 1 = 2(k - 1)$. So, $(x_1, \ldots, x_{2k-2}) \in \partial H^{2k-2}(X^*)_1$. Consider

$$\frac{((1 - i)x_1, (1 + i)x_1, 2x_2, \ldots, 2x_{2k-2})}{\alpha} \in H^{2k-1}(X^*)_1$$

(where $\alpha = 2\sqrt{2} \|x_1\| + 2 \|x_2\| + \ldots 2\|x_{2k-2}\|$). If this is extreme in $H^{2k-1}(X^*)_1$, we have the desired conclusion by the opening remark of the present proof. If this is not extreme, $\exists \, c_1, c_2 \ldots, c_{2k-1} \in \mathbb{R}$, not all of which are zero, such that

$$c_1 \langle (1 - i)x_1, \sqrt{2} \|x_1\| \rangle + c_2 \langle (1 + i)x_1, \sqrt{2} \|x_1\| \rangle + c_3 \langle 2x_2, 2 \|x_2\| \rangle + \ldots$$
$$+ c_{2k-1} \langle 2x_{2k-2}, 2 \|x_{2k-2}\| \rangle = 0.$$

First not that it is not possible to have $c_1 = c_2 = 0$ and some of $c_3, c_4, \ldots, c_{2k-1}$ not zero as that would contradict $\dim_{\mathbb{R}} \mathrm{sp}_{\mathbb{R}} \{x_1, \ldots, x_{2k-2}\} = 2k - 3$, a consequence of the extremality of $(x_1, \ldots, x_{2k-2})$ in $H^{2k-2}(X^*)_1$. (See Proposition 2.2 in [22]). Hence either $c_1 \neq 0$ or $c_2 \neq 0$ or both. If $c_1 \neq 0$ and $c_2 = 0$, we have a relation of the form

(1) $$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1\,2k-2} x_{2k-2} = 0,$$

where $a_{11} \in \mathbf{C} \setminus \mathbf{R}$ and $a_{1j} \in \mathbf{R}, (j \geqq 2)$. Similarly, if $c_1 = 0$ and $c_2 \neq 0$. If both c_1 and c_2 are non-zero, we have the relations $[(c_1 + c_2) + i(c_1 - c_2)]x_1 + 2c_3 x_2 + \ldots + 2c_{2k-1} x_{2k-2} = 0$,

$$\sqrt{2}\, c_1 \|x_1\| + \sqrt{2}\, c_2 \|x_2\| + 2c_3 \|x_2\| + \ldots + 2c_{2k-1} \|x_{2k-2}\| = 0.$$

If $c_1 = c_2$, the first relation gives, by the extremality of $(x_1, \ldots, x_{2k-2})$ in $H^{2k-2}(X^*)_1$, that

$$2c_1 - 2c_{2k-1} = 2c_3 - 2c_{2k-1} = \ldots = 0$$

whence $c_1 = c_2 = c_3 = \ldots = c_{2k-1}$ and we conclude from the second relation that $c_1 = c_2 = c_3 = \ldots = c_{2k-1} = 0$, a contradiction. Thus $c_1 \neq c_2$ and in this case we obtain a relation of the form (1).

Arguing now with $(x_2, x_1, x_3, \ldots, x_{2k-2})$ etc., we get relations similar to (1):

$$\sum_{j=1}^{2k-2} a_{lj} x_j = 0, (1 \leqq l \leqq 2j - 2, a_{jj} \in \mathbf{C} \setminus \mathbf{R}, a_{ij} \in \mathbf{R}, i \neq j),$$

and we conclude, as in the proof of Theorem 2.6 in [22] that

$$\dim_{\mathbf{C}} \mathrm{sp}_{\mathbf{C}} \{x_1, \ldots, x_{2k-2}\} \leqq \frac{2k-3}{2} = k - \frac{3}{2} \leqq k - 1.$$

If $(x_1, \ldots, x_{2k-3}) \in \partial H^{2k-3}(X^*)_1$, apply the above argument to $p = ((1 + i)x_1, (1 - i)x_1, 2x_2, \ldots, 2x_{2k-3}) \in H^{2k-2}(X^*)$. If $p \|p\|^{-1} \in \partial H^{2k-2}(X^*)_1$, we are back to the situation discussed at the beginning of this proof and the desired conclusion follows. If $p \|p\|^{-1} \notin \partial H^{2k-2}(X^*)_1$, we argue as earlier in the present proof by first appealing to the result quoted at the end of Section 2, and so on. Thix completes the proof. /

We can now state and prove the main result of this paper. We denote by $M(Q, X^*)$ the Banach space, under the total variation norm, of countably additive vector measures of bounded variation defined on the Borel σ-algebra $\mathscr{B}(Q)$ of a compact Hausdorff space Q and assuming values in a conjugate Banach space X^*. If $f \in M(Q, X^*)$ and $x \in X$, then f_x is the x-section of f defined earlier.

THEOREM 3.1 *Let A be a closed subspace of $C(Q)$ separating the points of Q. The following statements are equivalent:*

(i) *A is an L^1-predual*
(ii) *If $n > k \geqq 3$ and E is any Banach space with the almost n.k.I.P. then $A \check{\otimes} E$ also has the almost n.k.I.P.*
(iii) *If k is a fixed integer, $(k \geqq 2)$, and E is a finite-dimensional Banach space with 2k.2k–1.I.P. then $A \check{\otimes} E$ has the almost 2k.2k–1.I.P.*

PROOF: (i) $\Rightarrow$ (ii) Let $(\Phi_i)_{i=1}^n \in H^n(A \check{\otimes} E)_1^*$. By Lemma 2, we can find $f^i \in M(Q, E^*)$, $f^i \mid A \check{\otimes} E = \Phi_i$, $\|f^i\| = \|\Phi_i\|$ and $(f^i)_x = \mu_x^i$, a boundary measure with respect to A, $(1 \leqq i \leqq n)$. As $\sum_{i=1}^n \Phi_i = 0$, $\sum_{i=1}^n \Phi_i[xa(\cdot)] = 0 \, \forall a \in A$ and hence $\left(\sum_{i=1}^n \mu_x^i\right)\Big|_A = 0$. As A is an L^1-predual, by [6: Theorem 4.1] we have in the notation of [25] $\sum_{i=1}^n \mu_x^i \approx 0$ and therefore by [25, Proposition 3.5], $\sum_{i=1}^n \hom(\mu_x^i \circ \Phi^{-1}) = 0$. (Recall that $\Phi: Q \to A_1^*$ is the usual *evaluation map* $q \to \Phi(q)$). Now, $f^i \circ \Phi^{-1} \in M(\Phi(Q), E^*)$ and if $\alpha \in C(\Phi Q)$, $x \in E$, we have

$$\begin{aligned}\int \alpha d(f^i \circ \Phi^{-1})_x &= (f^i \circ \Phi^{-1})(x\alpha(\cdot)) \\ &= \int \alpha \circ \Phi d\mu_x^i \\ &= \int \alpha d(\mu_x^i \circ \Phi^{-1})\end{aligned}$$

and it follows that

$$(f^i \circ \Phi^{-1})_x = (f^i)_x \circ \Phi^{-1} = \mu_x^i \circ \Phi^{-1} \, \forall x \in E.$$

Let $F_i = \hom(f^i \circ \Phi^{-1}) \in M(A_1^*, E^*)$ be defined, for $1 \leqq i \leqq n$, by

$$F_i[x(\cdot)] = \int \langle \hom x(q), d(f^i \circ \Phi^{-1})_q \rangle$$

where $x(\cdot) \in C(A_1^*, E)$ and $\hom x(p) = \int_T \bar{t} x(tp) dt$ for $p \in A_1^*$ is defined in the usual way, (see e.g. [25]). (The integral defining $\hom x$ exists for functions of the form $\sum_{i=1}^n x_i \alpha_i(\cdot)$, $x_i \in E$, $\alpha_i \in C(A_1^*)$, $(1 \leqq i \leqq n)$, and exists in general for any $x(\cdot) \in C(A_1^*, E)$ as such a function can be uniformly approximated by functions of the former kind). Now, if $\alpha \in C(A_1^*)$,

$$\begin{aligned}(F_i)_x(\alpha) = F_i(x\alpha(\cdot)) &= [\hom(f^i \circ \Phi^{-1})](x\alpha(\cdot)) \\ &= (f^i \circ \Phi^{-1})[\hom(x\alpha)] \\ &= (f^i \circ \Phi^{-1})(x \hom \alpha) \\ &= (f^i \circ \Phi^{-1})_x(\hom \alpha) \\ &= \hom[(f^i \circ \Phi^{-1})_x](\alpha),\end{aligned}$$

whence $[\hom(f^i \circ \Phi^{-1})]_x = \hom[(f^i \circ \Phi^{-1})_x] = \hom[(f^i)_x \circ \Phi^{-1}] = \hom[\mu^i_x \circ \Phi^{-1}]$ and we have

$$\sum_{i=1}^{n} [\hom(f^i \circ \Phi^{-1})]_x = 0 \, \forall x \in E.$$

It follows therefore that $\sum_{i=1}^{n} F_i = 0$. If $F \in M(A_1^*, E^*)$ and support $(F) \subseteqq T\Phi(Q)$, (here support of F is the smallest closed set outside of which each F_x vanishes, $x \in E$), define $SF \in C(Q, E)^* = M(Q, E^*)$ by

$$(\mathrm{SF})[x(\cdot)] = \int_{T\Phi(Q)} \langle t_p x(q_p), dF_p \rangle, x(\cdot) \in C(Q, E),$$

where $p = t_p \Phi(q_p) \in T\Phi(Q)$. (See the proof of Lemma 2 and [25]). Note first that the map from $T\Phi(Q)$ into E defined by $p \to t_p x(q_p)$ is a bounded measurable function with values in E. This follows easily by looking at functions $x = \sum_{i=1}^{n} x_1 \alpha_i(\cdot)$, $(x_i \in E, \alpha_i \in C(Q))$ and then using the density of such functions in $C(Q, E)$. Thus, the above integral is well-defined.

We check that $SF^i|_{A \otimes E} = \Phi_i, (1 \leqq i \leqq n)$. If $x \in E$, $a(\cdot) \in A$,

$$\begin{aligned} SF^i(xa(\cdot)) &= \int_{A_1^*} \langle t_p x\, a(q_p), d[\hom(f^i \circ \Phi^{-1})]_p, (p \in A_1^*) \\ &= \int_{A_1^*} \langle x\, a(t_p \Phi(q_p)), d[\hom(f^i \circ \Phi^{-1})]_p \rangle \\ &= [\hom(f^i \circ \Phi^{-1})]_x(\tilde{a}) \\ &= \hom[(f^i)_x \circ \Phi^{-1}](\tilde{a}), \text{ by what we observed before,} \\ &= (f^i_x) \circ \Phi^{-1}(h \circ m\, \tilde{a}) \\ &= f^i_x(\tilde{a} \circ \Phi) = f^i_x(a) = f^i(xa) = \Phi_i(xa). \end{aligned}$$

Once again, by the density of functions of the form $\sum_{i=1}^{n} x_i a_i(\cdot)$ in $A \check{\otimes} E$ and the continuity of each Φ_i, we see that $SF^i|_{A \otimes E} = \Phi_i, (1 \leqq i \leqq n)$. Thus, $\|SF^i\| \geqq \|\Phi_i\|$ but, on the other hand, it is easily seen that

$$\|SF^i\| \leqq \|F^i\| = \|\hom(f^i \circ \Phi^{-1})\| \leqq \|f^i\| = \|\Phi_i\|$$

and we therefore have $\|SF^i\| = \|\Phi_i\|, (1 \leqq i \leqq n)$. Since $\sum_{i=1}^{n} SF^i = 0$, by Lemma 1 we can write

$$(SF^1, \ldots, SF^n) = \sum_{i=1}^{\binom{n}{k}} (g_{i1}, \ldots, g_{in}), \tag{2}$$

where $(g_{i1}, \ldots, g_{in}) \in H^n\, C(Q, E)^*$ for each i, at most k components of $(g_{i1}, \ldots, g_{in})$ are non-zero for each i. Restricting these functionals to $A \check{\otimes} E$, we get

$$(\Phi_1, \ldots, \Phi_n) = \sum_i (g_{i1}\,|_{A \check{\otimes} E}, \ldots, g_{in}\,|_{A \check{\otimes} E})$$

and one sees easily that conditions (i) and (ii) in the statement (3) of the $R_{n,k}$ property stated in Section 2 are fulfilled (as these are filfilled in (2) above). This completes the proof.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (1). Since C has the $2k.2k$-1.I.P., $A = A \check{\otimes} C$ has the almost $2k.2k$-1.I.P. Suppose if possible that $(x_1, \ldots, x_{2k-1}) \in \partial H^{2k-1}(A^*)_1$ with all $x_i \neq 0$, $x_i \|x_i\|^{-1} \in \partial A_1^*$, $(1 \leqq i \leqq 2k-1)$. (This means that A does not have the almost $2k-1$. $2k-2$.I.P.) From [22, Theorem 2.6], we may assume that $x_1, x_2, \ldots, x_{k-1}$ are linearly independent (over C) and $x_k, x_{k+1}, \ldots, x_{2k-1}$ are expressible uniquely as linear combinations of $x_1, \ldots, x_{k-1}$. This means that for at least one of $x_k, x_{k+1}, \ldots, x_{2k-1}$, say x_k, we can write

$$x_k = z_1 x_1 + z_2 x_2 + \ldots + z_m x_m, (2 \leqq m \leqq k-1)$$

and all z_i's *not zero*. We choose a positive integer n such that $k \leqq mn \leqq 2(k-1)$. For such an n, necessarily $\leqq k-1$ as $m \geqq 2$, look at the following vectors in $\mathbf{C}^n$:

$$\left[\begin{pmatrix} \omega \\ \omega^2 \\ \omega^3 \\ \vdots \\ \omega^n \end{pmatrix}, \begin{pmatrix} \omega^2 \\ (\omega^2)^2 \\ (\omega^2)^3 \\ \vdots \\ (\omega^2)^n \end{pmatrix}, \begin{pmatrix} \omega^3 \\ (\omega^3)^2 \\ (\omega^3)^3 \\ \vdots \\ (\omega^3)^n \end{pmatrix}, \begin{matrix} \ldots, \\ \ldots, \\ \ldots, \\ \ldots, \\ \ldots, \end{matrix} \begin{pmatrix} \omega^{n+1} \\ (\omega^{n+1})^2 \\ (\omega^{n+1})^3 \\ \vdots \\ (\omega^{n+1})^n \end{pmatrix}\right] \in \partial H^{n+1}(l_\infty(n))_1$$

where $l_\infty(n)$ refers to $\mathbf{C}^n$ with the l_∞-norm and ω is a primitive $(n+1)^{\text{th}}$ root of unity, $\omega^{n+1} = 1$. (The fact that each row in the above array adds to zero is easily verified by taking the sum of a geometric prograssion, and that the vectors are extreme is checked by considering the so-called Vandermonde $(n+1) \times (n+1)$ determinant

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ \omega & \omega^2 & \ldots & \omega^{n+1} \\ \omega^2 & (\omega^3)^2 & \ldots & (\omega^{n+1})^2 \\ \vdots & \vdots & & \vdots \\ \omega^n & (\omega^2)^n & \ldots & (\omega^{n+1})^n \end{vmatrix}$$

which is non-zero as $\omega^1, \omega^2, \ldots, \omega^{n+1}$ are all distinct (see [18]) and applying the result quoted in Section 2.) If we denote the columns of the preceding matrix by $\psi_0, \psi_1, \ldots, \psi_n$ (note that $\psi_1, \ldots, \psi_n$ are linearly independent over C again by the Vandermonde determinant) we have

$$-\psi_0 = \psi_1 + \ldots + \psi_n$$

and hence

$$-(x_k \otimes \psi_0) = \sum_{\substack{1 \leqq j \leqq m \\ 1 \leqq i \leqq n}} (z_j x_j) \otimes \psi_i.$$

By Lemma 4,

$$\frac{(x_k \otimes \psi_0, z_j x_j \otimes \psi_i)_{\substack{1 \leqq j \leqq m \\ 1 \leqq i \leqq n}}}{\gamma} \in \partial H^{mn+1}(A \check{\otimes} l_1(n))_1^*$$

where $\gamma = \|x_k\| \, \|\psi_0\| + \sum_{i,j} |z_j| \, \|x_j\| \, \|\psi_i\|$ and $l_1(n)$ denotes C^n with the l_1-norm. Since $n \leqq k-1$, $l_1(n)$ has the $2k.\ 2k-1$. I.P. by Helly's Theorem and therefore by hypothesis, $A \check{\otimes} l_1(n)$ has the almost $2k.\ 2k-1$. I.P. Noting that $mn+1 \leqq 2k-1$ and that by proof of Lemma 4, the number of (complex) linearly independent functionals in

$$\frac{1}{\gamma}(x_k \otimes \psi_0, z_j x_j \otimes \psi_i)_{\substack{1 \leqq j \leqq m \\ 1 \leqq i \leqq n}}$$

is mn and that by our choice of n, $mn \geqq k$, we get a contradiction to the assertion of Lemma 5. This contradiction shows that A must have the almost $2k-1$. $2k-2$. I.P. Arguing in a similar manner, we proceed 'downwards" till we get that the extreme points of $H^4(A^*)$ are of the form $(z_1 g, z_2 g, z_3 g, 0)$, $(g \in \partial A_1^*$, $(z_1, z_2, z_3) \in \partial H^3(C)_1$,) which means that A has the 4.3.I.P., i.e. A is an L^1-predual. This completes the proof of Theorem 3.1.

We have immediately the following

COROLLARY 3.2. *L^1-preduals are closed under injective tensor products.*

4. A characterization of L^1-spaces by the $R_{n,k}$ property.

We will prove in this section a result "dual" to Theorem 3.1 and which will characterize L^1-spaces by means of the $R_{n,k}$ property. (See in this connection Theorem 5.1 in [20].

THEOREM 4.1. *The following statemens are equivalent for a Banach space X.*
(i) *The Banach space X is isometric to some L^1-space.*

(ii) *If* $n > k \geqq 3$ *and* E^* *is a conjugate space with the Radon-Nikodym property* (R.N.P) *and the* $R_{n,k}$ *proporty, then* $X \hat{\otimes} E^*$ *has the* $R_{n,k}$ *property.*

(iii) *If* $k\,(\geqq 2)$ *is a fixed integer and* E *is a finite dimessional Banach space with the* $R_{2k,2k-1}$ *property, then* $X \hat{\otimes} E$ *has the* $R_{2k,2k-1}$ *property.*

PROOF. (i) $\Rightarrow$ (ii) If $X \cong L^1(\mu)$ for some measure space $(\Omega, \mathcal{Q}, \mu)$, then $X \hat{\otimes} E^* \cong L^1(\mu, E^*)$. (See [4: page 228]). To show that $L^1(\mu, E^*)$ has the $R_{n,k}$ property, notice that one only need to decompose finitely many functions in $L^1(\mu, E^*)$ at a time. Since L^1-functions have σ-finite support, it is easy to see that there is no loss of generality in assuming that μ it self is a σ-finite measure. Again by considering the measure υ defined on $\mathcal{Q}$ by $\upsilon(A) = \sum \frac{1}{2^n} \frac{\mu(A \cap A_n)}{\mu(A_n)}$ where $0 < \mu(A_n) < \infty$, $A_n \in \mathcal{Q}$, A_n's are disjoint and $\Omega = \cup A_n$, it is easy to verify that $L^1(\mu, E^*)$ is isometric to $L^1(\upsilon, E^*)$. Since the $R_{n,k}$ property is invariant under isometries, for the remainder of the proof we are going to assume that $(\Omega, \mathcal{Q}, \mu)$ is a finite measure space.

Step 1: Suppose Q is a compact Hausdorff space, $\mathcal{Q}$ the Borel σ-field on Q and μ a finite regular Borel measure on $\mathcal{Q}$.

Since E^* has the $R_{n,k}$ property it follows from the result of Lima, quoted in the introduction that E has the almost *n.k.*I.P. Therefore by Lemma 1 of the previous section, the space $C(Q, E)$ has the almost *n.k.*I.P. Applying Lima's result a second time gives us that $C(Q, E)^*$ has the $R_{n,k}$ property.

We shall complete the proof in this case by showing that $L^1(\mu, E^*)$ is isometric to a constrained subspace (i.e. a subspace, complemented by a projection of norm one) of $C(Q, E)^*$ and appealing to the easily verifiable fact that the $R_{n,k}$ property is hereditary for constrained subspaces.

As before we identify $C(Q, E)^*$ as the space of E^*-valued, countably additive, regular, vector measures of bounded variation with the total variation norm. Clearly the map which associates for each $f \in L^1(\mu, E)$, the corresponding E^*-valued measure $\int_A f\,d\mu$ is an isometric embedding of $L^1(\mu, E^*)$ into $C(Q, E)^*$. It is well-known that for any $F \in C(Q, E)^*$ we have the decomposition $F = F_a + F_s$ where $F_a, F_s \in C(Q, E)^*$ and $F_a \ll \mu$ and F_s is "singular" *w.r.t.* μ (see [4] page 31). Since X^* has the R.N.P, the derivative $\frac{dF_a}{d\mu}$ exists in $L^1(\mu, X^*)$ and the map $F \to \frac{dF_a}{d\mu}$ is a projection of norm one (as in the classical, scalar-valued case) from $C(Q, E)^*$ onto $L^1(\mu, E^*)$.

Step 2: To deduce the theorem for a general finite measure space $(\Omega, \mathcal{Q}, \mu)$ from Step 1, we make an appeal to the proof of Lemma 1 in [29]. As in that proof, if

Q denotes the Maximal ideal space of $L^\infty(\mu)$ (the usual scaler-valued function space) and $\hat{\mu}$ a regular Borel measure whose value on a clopen set $\hat{A}(A \in \mathcal{Q}, {}^\wedge$ denotes the usual Gelfand-transform) is determined by $\hat{\mu}(\hat{A}) = \mu(A)$, it is clear that $L^1(\mu, E^*)$ is isometric to $L^1(\hat{\mu}, E^*)$ and the conclusion now follows from Step 1.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i) We first remark that if F is a Banach space with the $R_{n,k}$ property, then F^* has the almost $n.k.$I.P. To see this observe that $H^n(F)^{**} \cong H^n(F^{**})$ and hence $H^n(F)$ is isometric to a w^*-dense subspace of $H^n(F^{**})$ in the natural embedding. Now the $R_{n,k}$ property for F^{**} follows from the w^*-lower-semi continuity of the dual norm and the fact that F has the $R_{n,k}$ property. Apply the theorem of Lima, quoted in Section 2, to conclude that F^* has the almost $n.k.$I.P.

Next observe that sine C has the $R_{2k,2k-1}$ property, $X \hat{\otimes} \mathbf{C} = X$ also has the $R_{2k,2k-1}$ property. Now, if $E = l_\infty(k-1) = [l_1(k-1)]^*$, then by assumption $X \hat{\otimes} E$ has the $R_{2k,2k-1}$ property and so $(X \hat{\otimes} E)^*$ has the $2k.\,2k-1$. I.P. by the remark in the preceding paragraph. From [4], we know that

$$(X \hat{\otimes} E^* = \{ +: E \to X^*;\ + \text{ a bounded operator, with oprator norm}\} \\ = (X^*)^{k-1},$$

the latter space having the norm

$$\|(x_1^*, \ldots, x_{k-1}^*)\| = \sup\left\{\left|\sum_{i=1}^{k-1} x_i^*(x)\right| : x \in X_1\right\}.$$

But by [4: example 4, page 223],

$$l_1(k-1) \check{\otimes} X^* = \{G;\ G \text{ vector measure from } \{1,2,\ldots,k-1\} \text{ to } X^* \text{ with semivariation norm}\} \\ = (X^*)^{k-1}, \text{ by an easy application of Goldstine's theorem.}$$

We have thus shown that $(X \hat{\otimes} l_\infty(k-1))^* = X^* \check{\otimes} l_1(k-1)$ has the $2k.\,2k-1$.I.P. The proof of (iii) $\Rightarrow$ (i) in Theorem 3.1 now shows that X^* is an L^1-predual and so by [20: Theorem 4.1], X^* is an $E(n)$-space $\forall n \geqq 3$. By the w^*-compactness of balls in X^* and a result of Hustad [15], X^* is a P_1-space and by [10] (see [11] and [28] for the complex version of this result) it follows that X is an L^1 – space.

Remark 1. If E is a Banach space such that E^* has the Radon-Nikodym property then the proof of (i) $\Rightarrow$ (ii) in Theorem 3.1 follows comparatively painlessly by combining Theorem 5.3 [16] and the (somewhat easier) result (i) $\Rightarrow$ (ii) of the above theorem.

Remark 2. It is possible to prove that the space of Bochner integrable functions has the $R_{n,k}$ property, under a more general hypothesis on E than the one considered in (ii) of Theorem 4.1 (the Corollary below is one such instance).

However since that result is in a spirit different from that of this paper it shall be presented elsewhere.

COROLLARY 4.2. *If* E^* *has the* $R_{n,k}$ and the R.N.P. *properties and* $F \subseteq E^*$ *is a closed hereditary subspace,* (i.e. $f \in F$, $f = g + h$ *with* g, $h \in E^*$ *and* $\|f\| = \|g\| + \|h\| \Rightarrow g \in F$, *see* [19]) *then* $L^1(\mu, F)$ *has the* $R_{n,k}$ *property for any measure space* $(\Omega, \mathscr{Q}, \mu)$.

PROOF. As was remarked during the proof of Theorem 4.1, there is no loss of generality in assuming that the measure μ is finite. It is easy to check that in the natural embedding $L^1(\mu, F)$ is a hereditary subspace of $L^1(\mu, E^*)$. Now since $L^1(\mu, E^*)$ has the $R_{n,k}$ property by Theorem 4.1, it follows from the definition of the $R_{n,k}$ property that the hereditary subspace $L^1(\mu, F)$, inherits the same.

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