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THE SIMPLICITY OF THE QUOTIENT ALGEBRA $M(A)/A$ OF A SIMPLE C^*-ALGEBRA

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Abstract.

It is shown that the C^*-algebra $M(A)/A$, where A is a σ-unital stably semi-finite C^*-algebra and $M(A)$ is the multiplier algebra of A, is simple if and only if either A has a continuous dimension scale or A is elementary.

Let A be a C^*-algebra, and denote by A^{**} the enveloping von Neumann algebra of A. The multiplier algebra $M(A)$ is the idealiser of A in A^{**}. We denote by $\mathscr{K}$ the C^*-algebra of all compact operators on an infinite dimensional separable Hilbert space, and by $\mathscr{B}(H)$ the C^*-algebra of all bounded operators on H. It is well known that $M(\mathscr{K}) = \mathscr{B}(H)$ and $M(\mathscr{K})/\mathscr{K}$ is simple. The ideal structure of the C^*-algebra $M(A)/A$ for A a simple AF C^*-algebra has been studied in [5], [7] and [6], and for A a factorial simple C^*-algebra has been studied in [8]. In the present note we shall show that in the case of a σ-unital, stably semi-finite C^*-algebra, $M(A)/A$ is simply if and only if either A has a continuous dimension scale or A is elementary. We shall also show that for every σ-unital purely infinite C^*-algebra A, $M(A)/A$ is simple.

1. Preliminaries.

1.1. Let B be a dense hereditary*-subalgebra of a C^*-algebra A, and a, b elements of B. Following Cuntz, we write $a \precsim b$ if there are x, y in A such that $a = xby$. We write $a \lesssim b$ if there is a sequence $\{a_n\}$ in B such that $a_n \precsim b$ and $a_n \to a$. This relation is transitive and reflexive. We write $a \approx b$ if $a \lesssim b$ and $b \lesssim a$. We say that a is orthogonal to b ($a \perp b$) if $ab = ba = a^*b = ba^* = 0$.

Let A be a simple C^*-algebra, $K(A)$ its Pedersen ideal, $\mathscr{F}$ the algebra of operators of finite rank on an infinite-dimensional separable Hilbert space H and $\mathscr{K}$ the C^*-algebra of compact operators on H. We denote by $\mathscr{F} \otimes K(A)$ the algebraic tensor product of $\mathscr{F}$ and $K(A)$. We call an element x in $\mathscr{F} \otimes K(A)$ infinite if $y \lesssim x$ for every y in $\mathscr{F} \otimes K(A)$.

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There are three possibilities for a simple C^*-algebra A.

(i) $\mathscr{F} \otimes K(A)$ contains only finite elements. In this case we shall call A stably semi-finite.

(ii) $\mathscr{F} \otimes K(A)$ contains non-zero finite and infinite elements.

(iii) All non-zero elements in $\mathscr{F} \otimes K(A)$ are infinite.

It is not known if case (ii) can appear. If A has a lower semi-continuous semi-finite trace, then A is stably semi-finite.

We call a function d: $\mathscr{F} \otimes K(A) \to \mathrm{R}_+$ a dimension function (on $K(A)$) if

(a) $d(x) = 0$ if and only if $x = 0$

(b) $x \lesssim y$ implies $d(x) \leqq d(y)$

(c) $d(x \dotplus y) = d(x) + d(y)$ for all orthogonal x, y in $\mathscr{F} \otimes K(A)$.
A dimension function also satisfies.

(d) $d(x + y) \leqq d(x)) + d(y)$ for all $x, y \in \mathscr{F} \otimes K(A)$.

Given $x \in \mathscr{F} \otimes K(A)$, we denote by $\langle x \rangle$ the $\approx$-equivalence class of x in $\mathscr{F} \otimes K(A)$. Let F be the free abelian group generated by $\{\langle x \rangle \mid x \in \mathscr{F} \otimes K(A)\}$ and let R be the subgroup of F generated by all elements of the form $\langle x \rangle + \langle y \rangle - \langle x_1 + y_1 \rangle$ ($\mathrm{x}_1 \in \langle \mathrm{x} \rangle$, $\mathrm{y}_1 \in \langle \mathrm{y} \rangle$, $\mathrm{x}_1 \perp y_1$). We denote by $\Delta(\mathrm{A})$ the quotient F/R. $\Delta(A)$ is an ordered group with the order induced by "$\lesssim$". We shall use "$\leqq$" for the order. $\Delta(\mathrm{A}) \neq \{0\}$ if and only if A is stably semi-finite. Moreover, there is a bijective correspondence between non-zero positive homomorphisms h: $\Delta(A) \to \mathrm{R}$ and dimension functions d on $K(A)$ given by $d(x) = h(\langle x \rangle)$, and if $\Delta(\mathrm{A}) \neq \{0\}$, $K(A)$ admits a dimension function. For the details of the relations $\lessapprox$ and "$\lesssim$", dimension functions and the ordered group, readers are referred to [1], [2], [3] and [7].

1.2. Given $\varepsilon > 0$, let f_ε be the continuous function on R defined by

$$f_{\varepsilon(t)} = \begin{cases} 0 & \text{if } t \in (-\infty, \varepsilon/2] \\ \text{linear} & \text{if } t \in [\varepsilon/2, \varepsilon]. \\ 1 & \text{if } t \in [\varepsilon, \infty) \end{cases}$$

If $a \in A$, set

$$A_a = A_{|a|} = \bigcup_{\varepsilon > 0} f_\varepsilon(|a|) A f_\varepsilon(|a|).$$

1.3. We now identify $p \otimes K(A)$ with $K(A)$ and $p \otimes A$ with A for a fixed one dimensional projection p in $\mathscr{F}$. Suppose that a and b are in $K(A)$, and $\langle a \rangle \leqq \langle b \rangle$. So $a \lesssim b$ in $\mathscr{F} \otimes K(A)$ and $a^*a \lesssim b^*b$. Since $a \approx a^*a$, $b^*b \approx b$ both in $K(A)$ and $\mathscr{F} \otimes K(A)$, we may assume that $0 \leqq a$ and $0 \leqq b$. There are $x_n \in \mathscr{F} \otimes K(A)$ such that $x_n \lessapprox b$, $x_n \to a$. We can find a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ such that

$$f_{\varepsilon_n}(a) x_n f_{\varepsilon_n}(\mathrm{a}) \to \mathrm{a}$$

since $f_{\varepsilon_n}(a) x_n f_{\varepsilon_n}(a) \lessapprox b$, $a \lesssim b$ in $K(A)$.

1.4. If A is a stably semi-finite simple C^*-algebra and u is a non-zero positive element in $K(A)$, then $\langle u\rangle$ is an order unit (see [2, 4.2]). A positive homomorphism h: $\Delta(A) \rightarrow \mathrm{R}$ is called a state (with respect to $\langle u\rangle$) if $h(\langle u\rangle) = 1$. The collection $S = S_u(\Delta(A))$ of all states on $\Delta(A)$ is a convex compact subset of the locally convex space $\mathrm{R}^{\Delta(A)}$ of all functions f: $\Delta((A) \rightarrow \mathrm{R}$ with the product topology. S is the set of all the dimension functions d on $K(A)$ such that $d(u) = 1$. We define a positive homomorphism θ: $D(A) \rightarrow \operatorname{Aff} S$, $g \rightarrow \hat{g}$ by setting $\hat{g}(h) = h(g)$, where $\operatorname{Aff} S$ is the set of all continuous real affine functions on S.

Let us say that $g \in \Delta(A)$ is infinitesimal if $-\varepsilon u \leqq g \leqq \varepsilon u$ for every positive rational number ε. (If $\varepsilon = p/q$, $p, q \in \mathrm{N}$, then $g \leqq \varepsilon u$ means that $qg \leqq pu$).

The notation "$\hat{g} \gg 0$" for $g \in \Delta(A)$ means $\hat{g}(d) > 0$ for all $d \in S$.

1.5. PROPOSITION (*Corollary of* [4, 4.2]). *The homomorphism* θ: $\Delta(A) \rightarrow \operatorname{Aff} S$ *determines the order on* $\Delta(A)$ *in the sense that* $\Delta(A)^+ = \{g \in \Delta(A) \mid \hat{g} \gg 0\} \cup \{0\}$. *Hence we have* $g \in \ker \theta$ *if and only if* g *is infinitesemal.*

PROOF. To apply 4.2 of [4] one need note only that the ordered group Δ(A) is unperforated.

1.6. When a is a positive element in a C^*-algebra A, we shall denote by $[a]$ the range projection of a in the enveloping von Neumann algebra A^{**}. Suppose that A is σ-unital (and non-unital), and let e be a strictly positive element of A. By choosing a proper sequence of continuous functions h_n, we can construct an approximate identity $\{e_n = h_n(e)\}$ for A satisfying

(i) $g_n = e_n - e_{n-1} \neq 0$ ($\mathrm{e}_0 = 0$), and $g_m g_n = 0$ if $|m - n| \geqq 2$,

(ii) There are $a_n \in A_+$, $a_n \neq 0$ such that $0 \leqq a_n \leqq [a_n] \leqq g_n$, $a_n g_n = g_n a_n = a_n$ and $a_n g_m = g_m a_n = 0$, if $n \neq m$.

Any subsequence $\{e_{n_k}\}$ of $\{\mathrm{e}_n\}$ is also an approximate identity satisfying (i) and (ii).

2. The results.

2.1. DEFINITION. Let A be a simple C^*-algebra. Call A purely inifite if every two non-zero elements are $\approx$-equivalent. (This definition is weaker than [8, 2.3]).

It is clear that every simple C^*-algebra in the case (1.1) (iii) is purely infinite.

2.2. LEMMA. *Let f be a continuous function on* $[-1, 1]$. *For every* $\varepsilon > 0$ *there is a constant* $M > 0$ *such that for any two self-adjoint elements a and b in the unit ball of a* C^*-*algebra* A,

$$\|f(a) - f(b)\| \leqq M\, \|a - b\| + \varepsilon$$

PROOF. For each integer k,

$$\begin{aligned}&\|a^{k+1} - b^{k+1}\|\\ &= \|a(a^k - b^k) + (a - b)b^k\|\\ &\leqq \|a^k - b^k\| + \|a - b\|.\end{aligned}$$

Thus we have

$$\|a^k - b^k\| \leqq k\,\|a - b\|$$

for all k. Therefore for every polynomial $p(t)$),

$$\|p(a) - p(b)\| \leqq M(p)\,\|a - b\|$$

where $M(p)$ is a constant depending only on p. By the Weierstrass approximation theorem, there is a polynomial p such that

$$\sup\{|f(t) - p(t)| \mid t \in [-1, 1]\} < \varepsilon/2.$$

Thus

$$\begin{aligned}&\|f(a) - f(b)\|\\ &\leqq \|f(a) - p(a)\| + \|p(a) - p(b)\| + \|p(b) - f(b)\|\\ &< \varepsilon/2 + M(p)\,\|a - b\| + \varepsilon/2\\ &= M(p)\,\|a - b\| + \varepsilon.\end{aligned}$$

2.3. THEOREM. *Let A be a σ-unital simple C^*-algebra. If A is purely infinite then $M(A)/A$ is simple.*

PROOF. Suppose that J is an ideal of $M(A)$ properly containing A. Choose a positive element x in $J\backslash A$. Let $\{e_n\}$ and $\{g_n\}$ be as in 1.6. Passing to a subsequence if necessary, we may assume that

$$\begin{aligned}&\|(1 - e_{n+1})xe_n\| < 1/2^n \text{ and}\\ &\|e_n x(1 - e_{n+1})\| < 1/2^n\end{aligned}$$

for all n. Then the elements

$$\sum_{n=1}^{\infty} (1 - e_{n-1})xg_n,\ \sum_{n=1}^{\infty} g_n x(1 - e_{n+1}),\ \sum_{n=3}^{\infty} g_{n-2}xg_n \text{ and } \sum_{n=3}^{\infty} e_{n-2}xg_n$$

are in A. Therefore the element

$$y = x - \sum_{n=1}^{\infty} (1 - e_{n+1})xg_n$$

is in $J\backslash A$. Since

$$y - \sum_{n=3}^{\infty} e_{n-2}xg_n = e_2xg_1 + e_3xg_2 + \sum_{n=3}^{\infty} g_{n+1}xg_n + \sum_{n=3}^{\infty} g_nxg_n + \sum_{n=3}^{\infty} g_{n-1}xg_n,$$

one of the last three elements must be in $J\backslash A$. Suppose that $\sum_{n=3}^{\infty} g_{n+1}xg_n$ is in $J\backslash A$. Since

$$\left[\sum_{n=3}^{\infty} g_{n+1}xg_n\right]^* \left[\sum_{n=3}^{\infty} g_{n+1}xg_n\right] = \sum_{n=3}^{\infty} g_nxg_{n+1}^2xg_n + \sum_{n=3}^{\infty} g_nxg_{n+1}g_{n+2}xg_{n+1} + \sum_{n=4}^{\infty} g_nxg_{n+1}g_nxg_{n-1},$$

and

$$\|g_nxg_{n+1}g_{n+2}xg_n\| = \|g_nxg_{n+2}g_{n+1}xg_{n+1}\| \leqq 1/2^n,$$

$$\|g_nxg_{n+1}g_nxg_{n-1}\| = \|g_nxg_nxg_{n+1}xg_{n-1}\| \leqq 1/2^{n-1},$$

$$\sum_{n=3}^{\infty} g_nxg_{n+1}g_{n+2}xg_{n+1} + \sum_{n=4}^{\infty} g_nxg_{n+1}g_nxg_{n-1} \text{ is in } A.$$

Thus $\sum_{n=3}^{\infty} g_nxg_{n+1}^2xg_n$ is in $J\backslash A$. Similarly, if $\sum_{n=3}^{\infty} g_{n-1}xg_n$ is in $J\backslash A$, $\sum_{n=3}^{\infty} g_nxg_{n-1}^2xg_n$ is in $J\backslash A$. In either case, it follows that $\sum_{n=1}^{\infty} g_nx^2g_n$ is in $J\backslash A$. By changing notation, we may therefore assume that $\sum_{n=1}^{\infty} g_nxg_n$ is in $J\backslash A$.

So $\sum_{k=1}^{\infty} g_{2k}xg_{2k}$ and $\sum_{k=1}^{\infty} g_{2k+1}xg_{2k+1}$ are in J and one of them is in $J\backslash A$. We may assume that $y = \sum_{k=1}^{\infty} g_{2k}xg_{2k}$ is in $J\backslash A$. Hence for a sufficiently small $\delta > 0$, $f_\delta(y) \in J\backslash A$. Since $g_{2k}xg_{2k} \perp g_{2j}xg_{2j}$ if $k \neq j$,

$$f_\delta(y) = \sum_{k=1}^{\infty} f_\delta(g_{2k}xg_{2k}).$$

Without loss of generality, we may assume that

$$f_\delta(g_{2k}xg_{2k}) \neq 0 \text{ for each } k.$$

Then $f_\delta(g_{2k}xg_{2k}) \approx g_k$ for each k. Let M_k be the constant in Lemma 2.2 such that

$$\|a^{1/2} - b^{1/2}\| \leqq M_k \|a - b\| + 1/2^k$$

for all $a, b \in A_s$, $\|a\| \leqq 1$, $\|b\| \leqq 1$, $k = 1, 2, \ldots$. For every $\varepsilon > 0$ and k, there is $x_k \in K(A)$ such that

$$x_k \lessapprox f_\delta(g_{2k}xg_{2k}) \text{ and}$$

$$\|x_k - g_k\| < \frac{\varepsilon}{2^k(M_k + 1)}.$$

We may assume that $0 \leqq x_k \leqq 1$ for each k. By [1, 1.7] there is $z_k \in A$ such that $z_k z_k^* = x_k$ and $z_k^* z_k \leqq [f_\delta(g_{2k} \times g_{2k})] \leqq f_{\delta/2}(g_{2k} \times g_{2k})$. Hence $z_k z_j^* = 0$, if $k \neq j$. So $\left[\sum_{k=1}^n z_k\right]\left[\sum_{k=1}^n z_k\right]^* = \sum_{k=1}^n z_k z_k^*$ and $\left\|\sum_{k=1}^n z_k z_k^*\right\|$ is bounded. Thus $\left\{\sum_{k=1}^n z_k\right\}$ is bounded. It is then easy to see that $\sum_{k=1}^n z_k$ converges in the right strict topology to an element $z\left[=\sum_{k=1}^\infty z_k\right]$ in the right multipliers $\mathrm{RM}(A)$. To show that $\sum_{k=1}^n z_k$ converges strictly to z, it is enough to show that for each n, $g_n \sum_{k=N}^\infty z_k$ converges (in norm) to zero as $N \to \infty$. Write $z_k = (z_k z_k^*)^{1/2} u_k$. Then

$$\left\|\sum_{k=N}^\infty z_k - \sum_{k=N}^\infty g_k u_k\right\| \leqq \sum_{k=N}^\infty \|(z_k z_k^*)^{1/2} - g_k\| < \sum_{k=N}^\infty \varepsilon/2^k + \sum_{k=N}^\infty 1/2^k \to 0,$$

as $N \to \infty$. Since $g_n \sum_{k=N}^\infty g_k u_k = 0$, for $N > n + 1$, we conclude that $\left\|g_n \sum_{k=N}^\infty z_k\right\| \to 0$ as $N \to \infty$. So $z \in M(A)$. From

$$z f_{\delta/2}(y) = \left[\sum_{k=1}^\infty z_k\right]\left[\sum_{k=1}^\infty f_{\delta/2}[g_{2k}xg_{2k}]\right] = \sum_{k=1}^\infty z_k,$$

we conclude that $z \in J$. On the other hand,

$$\|zz^* - 1\| = \left\|\sum_{k=1}^\infty z_k z_k^* - \sum_{k=1}^\infty g_k\right\| < \varepsilon,$$

so $1 \in J$.

2.4. Definition. Let A be a σ-unital, stably semi-finite, simple C^*-algebra. If A is not unital, fix a strictly positive element e and choose $\{e_n\}$ as in 1.6. We define

$$\hat{1}(d) = \lim_{n \to \infty} d(e_n) \text{ for every } d \in S_u(\Delta(A))$$

for some fixed $u \in K(A)^+ \backslash \{0\}$. We shall say that A has a continuous dimension scale if $\hat{1}$ is a continuous function on $S = S_u(\Delta(A))$ for some strictly positive element e. For convenience, we shall also say that every unital simple C^*-algebra has a continuous dimension scale. It is clear that the definition does not depend on the choice of u. Later we shall see that $\lim_{n\to\infty} d(e_n)$ is continuous for every approximate identity $\{e_n\}$ as described in 1.6 if it is continuous for one of them.

We now fix $u \in K(A)^+\backslash\{0\}$, and a strictly positive element e and an approximate identity $\{e_n\}$ as in 1.6.

2.5. For every $a \in M(A)_+$, we define

$$\tilde{d}(a) = \sup\{d(b) \mid b \leqq a,\ b \in A_e\},$$

$d \in S$. Then $\hat{1}(d) = \tilde{d}(1)$ for every $d \in S$. If $a \in AA_eA$, then $\langle a^*a \rangle = \langle a \rangle$ and a^*a has the form b^*x^*xb with $b \in A$ and $x \in A_e$. So there is $c \in (A_e)^+$ such that $\langle c \rangle = \langle a \rangle$. Hence, if $a \in AA_eA$,

$$\tilde{d}(a) = d(a) \text{ for each } d \in S.$$

2.6. Set

$$I_0 = \{a \in M(A) \mid \exists a_n \in AA_eA \text{ such that}$$
$$\tilde{d}((a - a_n)^*(a - a_n)) \to 0 \text{ uniformly on } S\}.$$

Clearly I_0 is a *-invariant subset of $M(A)$. Suppose that $a, b \in I_0$, and $\tilde{d}((a - a_n)^*(a - a_n)) \to 0$, $\tilde{d}((b - b_n)^*(b - b_n)) \to 0$ uniformly on S, where $a_n, b_n \in AA_eA$. Since for each k,

$$e_k(a + b - a_n - b_n)^*(a + b - a_n - b_n)e_k$$
$$= e_k(a - a_n)^*(a - a_n)e_k + e_k(b - b_n)^*(b - b_n)e_k$$
$$+ e_k(b - b_n)^*(a - a_n)e_k + e_k(a - a_n)^*(b - b^n)e_k$$

and

$$d[e_k(b - b_n)^*(a - a_n)e_k]$$
$$\leqq d[e_k(b - b_n)^*(b - b_n)e_k],$$

we conclude that

$$\tilde{d}((a + b - a_n - b_n)^*(a + b - a_n - b_n))$$
$$\leqq 2[(\tilde{d}((a - a_n)^*(a - a_n)) + \tilde{d}((b - b_n)^*(b - b_n))] \to 0$$

uniformly on S. Therefore I_0 is a *-invariant linear space. Suppose that $b \in M(A)$, $a \in I_0$ and $a_n \in AA_eA$ are such that

$$\tilde{d}((a - a_n)^*(a - a_n)) \to 0 \text{ uniformly on } S.$$

Then $ba_n \in AA_eA$ and

$$\tilde{d}((ba - ba_n)^*(ba - ba_n))$$
$$= \tilde{d}((a - a_n)^*b^*b(a - a_n)) \leqq \tilde{d}((a - a_n)^*(a - a_n)) \to 0$$

uniformly on S.

So I_0 is an ideal of $M(A)$. We denote by I the closure of I_0. Clearly, I is a closed ideal of $M(A)$ containing A.

2.7. LEMMA. *Let A be a non-elementary, σ-unital, non-unital, stably semi-finite simple C^*-algebra, and let I be as defined in 2.6.Then I contains A properly.*

PROOF. Clearly, I contains A. Let $\{a_n\}$ be as in 1.6, and fix n. For each $k > 0$, as shown in [8, 2.7] there are $\varepsilon > 0$ and $h_1, \ldots, h_k \in K(A)_+$ such that $h_i \perp h_j$ for $i \neq j$,

$$h_1 \gtrsim h_2 \gtrsim \ldots \gtrsim h_k \text{ and}$$

$$f_{\frac{1}{2}\varepsilon}(a_n) \geqq [f_\varepsilon(a_n)] \geqq \sum_{i=1}^{k} h_i.$$

Thus

$$\langle f_{\frac{1}{2}\varepsilon}(a_n)\rangle \geqq \langle h_1\rangle + \ldots + \langle h_k\rangle \geqq k\langle h_k\rangle,$$

whence $\tilde{d}(h_k) = d(h_k) \leqq k^{-1}d[f_{\frac{1}{2}\varepsilon}(a_n)] = k^{-1}\tilde{d}[f_{\frac{1}{2}\varepsilon}(a_n)]$ for $d \in S$. We conclude that for each n, there is $x_n \in (A_e)^+$ such that $\|x_n\| = 1, x_n \leqq g_n$ and $\tilde{d}(x_n) = d(x_n) \leqq 1/2^n$ for all $d \in S$. It is clear that $x = \sum_{n=1}^{\infty} x_n \in I \backslash A$.

2.8. THEOREM. *Let A be a σ-unital stably semi-finite simple C^*-algebra. Then $M(A)/A$ is simple if and only if either A has a continuous dimension scale or A is elementary.*

PROOF. Suppose that $M(A)/A$ is simple and A is neither unital nor elementary. By 2.7, $1 \in I$. Thus there is $a \in I^+$ such that

$$\|1 - a\| < 1/4.$$

This implies $\tilde{d}(1) = \tilde{d}(a)$ so $\hat{1}(d) = \tilde{d}(1)$ is continuous on S and A has a continuous dimension scale.

If A is elementary, it is well known that $M(A)/A$ is simple. Now suppose that $\hat{1}(d)$ is continuous on S. By Dini's theorem,

$$\tilde{d}(1 - e_n) \to 0$$

uniformly on S. Passing to a subsequence if necessary, we may assume that

$$\tilde{d}(1 - e_n) < \frac{1}{2^n}$$

uniformly on S. Therefore both $\sum_{n=1}^{\infty} d(g_{2n})$ and $\sum_{n=1}^{\infty} d(g_{2n+1})$ converge uniformly on S.

Suppose that J is a closed ideal properly containing A. Choose $w \in J^+ \backslash A$. As in 2.3, we may assume that $y = \sum_{k=1}^{\infty} g_{2k} w g_{2k} \in J^+ \backslash A$. Therefore for a sufficiently small $\delta > 0$, $f_\delta(y) \in J^+ \backslash A$. Since $f_\delta(y) = \sum_{k=1}^{\infty} f_\delta(g_{2k} w g_{2k})$, we may assume that $f_\delta(g_{2k} w g_{2k}) = \not= 0$ for each k. Since S is compact, then $\inf\{d(f_\sigma(g_{2k} w g_{2k})) \mid d \in S\} > 0$ for each k. Choose an integer n_0 such that

$$\sum_{k > n_0} d(g_{2k+1}) < \inf\{d(f_\delta(g_2 w g_2)) \mid d \in S\}$$

for all $d \in S$. Since $\inf\{d(f_\delta(g_{2k} w g_{2k})) \mid d \in S\} > 0$ for each k, we can find a partition of the set $\{n_0 + 1, n_0 + 2, \ldots\}$ into finite subsets $N_1, N_2 \ldots$ (of consecutive integers) such that for each $n = 1, 2, \ldots$,

$$\sum_{k \in N_n} d(g_{2k+1}) < d(f_\varepsilon(g_{2n} w g_{2n}))$$

for all $d \in S$. Then by 1.5 and 1.3,

$$\sum_{k \in N_n} g_{2k+1} \leqq f_\delta(g_{2n} w g_{2n}) \text{ in } K(A).$$

For any $\varepsilon > 0$, there are $x_n \in K(A)$ such that

$$x_n \lessapprox f_\delta(g_{2n} w g_{2n}) \text{ and}$$

$$\left\| x_n - \sum_{n \in N_n} g_{2k+1} \right\| < \varepsilon/2^n.$$

We may assume that $0 \leqq x_n \leqq 1$. It follows from [1, 1.7] that there are $z_n \in A$ such that

$$z_n z_n^* = x_n \text{ and}$$

$$z_n^* z_n \leqq [f_\delta(g_{2n} w g_{2n})] \leqq f_{\frac{1}{2}\delta}(g_{2n} w g_{2n}).$$

As in 2.3, this implies that $z = \sum_{n=1}^{\infty} z_n$ is in J and

$$zz^* = \sum_{n=1}^{\infty} z_n z_n^*.$$

Hence

$$\left\| zz^* - \sum_{k > n_0} g_{2k+1} \right\| < \varepsilon.$$

Therefore $\sum_{k>n_0} g_{2k+1}$ is in J, and hence so is $\sum_{k=0}^{\infty} g_{2k+1}$. Similarly $\sum_{k=1}^{\infty} g_{2k}$ is in J. Hence $1 \in J$.

2.9. From 2.8, together with its proof, we see that if A has a continuous dimension scale then for any e and $\{e_n\}$ in 1.6, $\hat{1}(d) = \lim_{n\to\infty} d(e_n)$ is continuous.

Choose a separable, algebraically simple AF C^*-algebra A such that $M(A)/A$ is not simple, or equivalently A has no continuous scale (see [7]). By 2.8, $\hat{1}(d)$ is not continuous. Since $e \in K(A)$, $d(e)$ is continuous on S. So

$$\sup\{d(a) \mid a \in A_e\} \neq d(e) \text{ for some } d \in S.$$

If d is lower semi-continuous, then it is easy to check that

$$\sup\{d(a)) \mid a \in A_e)\} = d(e)$$

Thus we conclude the following:

2.10. COROLLARY. *If A is a separable algebraically simple* AF *C^*-algebra without continuous scale then there is at least one dimension function d on A which is not lower semi-continuous. Consequently, d is not determined by a trace.*

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REFERENCES

1. J. Cuntz, *The structure of multiplication and addition in simple C^*-algebras*, Math. Scand. 40 (1977), 215–233.
2. J. Cuntz, *Dimension functions on simple C^*-algebras*, Math. Ann. 233 (1978), 145–153.
3. J. Cuntz, *The internal structure of simple C^*-algebras*, in *Operator Algebras and Applications* (ed. R. V. Kadison), Proc. Sympos. Pure Math. 38, Amer. Math. Soc., Providence, 1981, Part 1, 85–115.
4. E. Effros, *Dimensions and C^*-algebras*, CBMS Regional Conf. Ser. in Math. No. 46, Amer. Math. Soc., Providence, 1981.
5. G. A. Elliott, *Derivations of matroid C^*-algebras, II*, Ann. of Math. 100 (1974), 407–422.
6. G. A. Elliott, *The ideal structure of the multiplier algebra of an AF algebra*, C.R. Math. Rep. Acad. Sci. Canada 9 (1987), 225–230.
7. H. Lin, *On ideals of multiplier algebras of simple AF C^*-algebras*, Proc. Amer. Math. Soc. 104 (1988), 239–244.
8. H. Lin, *The multipliers and quasi-multipliers of simple C^*-algebras*, preprint.

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