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PROPER HOLOMORPHIC MAPPINGS FROM STRONGLY PSEUDOCONVEX DOMAINS IN C^2 TO THE UNIT POLYDISC IN C^3

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Introduction.

It has been known for a long time that if M is an n dimensional Stein manifold, then there exists a proper holomorphic mapping from M to C^{n+1} [4]. In this paper we shall be studying proper holomorphic maps between bounded domains. In particular people have been interested in finding out whether there exist proper holomorphic maps from the unit ball $B^n, n \geqq 2$, in C^n to the unit polydisc Δ^m in C^m for some m.

It has been known for a while that if $n \geqq m$ then this is not possible (see [9]) and it is also known that it is not possible to find a proper holomorphic map from Δ^m into B^n for any $m \geqq 2$ and any n (see [9]).

For a while one thought that finding a proper holomorphic map from B^n to $\Delta^{m(n)}$ for some large $m(n)$ would be a step in the direction of solving the inner function problem.

Then Hakim and Sibony [5] developed a method so that for every strictly positive continuous function φ on the boundary ∂B^n of the unit ball B^n in C^n and for every $\varepsilon > 0$ you can find a function f such that:

1. f is holomorphic in B^n.
2. $|f(p)| \leqq \max\{\varphi(z) : z \in \partial B^n\}$ for all p in B^n.
3. $\lim_{r \to 1} |f(rz)| = \varphi(z)$ for all $z \in \partial B^n - A$ where A has surface measure $< \varepsilon$.
4. $f(0) = 0$, hence f is not constant in B^n.

By repeateadly using this method and making sure that what was obtained was converging absolutely on all compact subsets of B^n, E. Løw [8] was able to

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produce a series $F := \sum f_j$ such that $F(0) = 0$ and the radial limits of norm F is one almost everywhere on the boundary of $\mathbf{B}^n$. In other words F is an inner function.

With some more modifications of the Hakim, Sibony method. Løw [7] was able to produce functions $f_1, f_2, \ldots, f_{m(n)} : \mathbf{B}^n \to \mathbf{C}^n$ such that

a. $f_1, f_2, \ldots, f_{m(n)}$ are holomorphic in $\mathbf{B}^n$.

b. $|f_i(p)| \leqq 1$ for all $p \in \mathbf{B}^n, i = 1, 2, \ldots, m(n)$.

c. If $\psi(p) := \sum \{\log(1 - |f_i(p)|)| \quad i = 1, 2, \ldots, m(n)\}$, then

$\lim_{j \to \infty} \psi(p_j) = -\infty$ whenever $\{p_j\}$ is a sequence in $\mathbf{B}^n$ converging to some boundary point p_0 of $\mathbf{B}^n$.

In other words $F := (f_1, f_2, \ldots, f_{m(n)})$ is a proper holomorphis map from $\mathbf{B}^n$ to $\Delta^{m(n)}$.

Also A. B. Aleksandrov [1] has given a proof of the existence of a proper holomorphic map from $\mathbf{B}^n$ to $\Delta^{m(n)}$ for some $m(n)$.

In both of these papers $m(n)$ was much larger than $n + 1$, so the remanining question was whether there exists a proper holomorphic mapping from $\mathbf{B}^n$ to Δ^{n+1}.

In this paper we shall prove that such a mapping does indeed exist when $n = 2$. Since the proof is only based on the fact that $\mathbf{B}^2$ is strongly pseudoconvex we get the following theorem:

THEOREM. *Let D be a strongly pseudoconvex domain in $\mathbf{C}^2$ with a C^∞-boundary. Then there exists a proper holomorphic mapping fom D to Δ^3.*

The proof of this theorem is based on ideas which can be found in Løw's paper [7].

1.

For a general n we are looking for a mapping $F = (f_1, \ldots, f_{n+1})$ and numbers $\{d_j\}_j$ and $\{\delta_j\}_j$ such that $\delta_j, d_j \to 0$ when $j \to \infty$ and each f_i is holomorphic and nonconstant on D and:

(i) $|f_i(p)| < 1$ when $p \in D$ for each $i = 1, \ldots, n + 1$

(ii) $\max\{|f_i(p)| \quad i = 1, \ldots, n + 1\} > 1 - \delta_j$ when $p \in D$ and $\operatorname{dist}(p, \partial D) < d_j$.

Instead of looking for property (ii) we shall look for (ii′) saying:

(ii′) $\sum \{\log(1 - |f_i(p)|)| \quad i = 1, \ldots, n + 1\} < (n + 1)\log \delta_j$ when $p \in D$ and $\operatorname{dist}(p, \partial D) < d_j$.

In this is the case, then of course there is at least one i such that $\log(1 - |f_i(p)|) < \log \delta_j$, hence $|f_i(p)| > 1 - \delta_j$ and (ii) follows.

Now to the case $n = 2$. The construction is done by hand and it is very technical.

First we shall find $W_1, \ldots, W_N \subset \partial B^2$ such that $W_1 \cup W_2 \cup \ldots \cup W_N = \partial B^2$ and such that whenever (f_1, f_2, f_3) are holomorphic in D and continous on $\bar{D}$ then the following inductive procedure is possible:
Choose $\{\varepsilon_j\}$, $i = 1, 2, \ldots$, such that $\sum \varepsilon_j^2$ diverges but $\sum \varepsilon_j^3$ converges and

1. If U is any subset of W_j, $j = 1, \ldots, N$, we can find $r_1 > 0$ and h_1^1, h_2^1, h_3^1 such that, h_1^1, h_2^1, h_3^1 are smooth on $\bar{D}$ and holomorphic in D:

a. $|h_1^1(p)|, |h_2^1(p)|, |h_3^1(p)| < \varepsilon_1^3$ when $p \in \bar{D}$ and $\operatorname{dist}(p, U) < 6r_1^{\frac{1}{2}}$.

b. $|f_1 + h_1^1|, |f_2 + h_2^1|, |f_3 + h_3^1| < 1$ on $\bar{D}$.

c. $\log(1 - |f_1(q) + h_1^1(q)|) + \log(1 - |f_2(q) + h_2^1(q)|)$
$+ \log(1 - |f_3(q) + h_3^1(q)|) < \log(1 - |f_1(q)|) + \log(1 - |f_2(q)|) +$
$\log(1 - |f_3(q)|) - R_1(q)$
where $-\varepsilon_1^3 < R_1(q)$ for all $q \in \bar{D}$ and $R_1(q) > (1/20)\varepsilon_1^2$ when $\operatorname{dist}(q, U) < r_1$ and $\max\{1 - |f_j(p)| \quad j = 1, 2, 3\} \geqq \varepsilon_1^{0.1}$.

Of course r_1 will depend on f_1, f_2 and f_3 and it will depend on W_j.

Let $f_1^1 := f_1 + h_1^1, f_2^1 := f_2 + h_2^1$ and $f_3^1 = f_3 + h_3^1$, then each of these functions is holomorphic in D and continuous on $\bar{D}$.

In the inductive step the functions $f_1^{i-1}, f_2^{i-1}, f_3^{i-1}$ are defined such that they are all holomorphic in D, continuous on $\bar{D}$ and all of them have norm less than 1 on $\bar{D}$.

1. We can find $0 < r_i < r_{i-1}$ and functions h_1^i, h_2^i and h_3^i which are holomorphic in D and smooth on D and:

a. $|h_1^i(p)|, |h_2^i(p)|, |h_3^i(p)| < \varepsilon_i^3$ when $p \in \bar{D}$ and $\operatorname{dist}(\rho, U) > 6_i^{r\frac{1}{2}}$.

b. $|f_1^{i-1} + h_1^i|, |f_2^{i-1} + h_2^i|, |f_3^{i-1} + h_3^i| < 1$ on $\bar{D}$.

c. $\log(1 - |f_1^{i-1}(q) + h_1^i(q)|) + \log(1 - |f_2^{i-1}(q) + h_2^i(q)|) +$
$+ \log(1 - |f_3^{i-1}(q) + h_3^i(q)|) < \log(1 - |f_1^{i-1}(q)|) + \log(1 - |f_2^{i-1}(q)|) +$
$\log(1 - |f_3^{i-1}(q)|) - R_i(q)$ where $R_i(q) > -\varepsilon_i^3$ for all $q \in \bar{D}$ and $R_i(q) > (1/20)\varepsilon_i^2$ when $\operatorname{dist}(q, U) < r_i$ and $\max\{1 - |f_j(p)| \quad j = 1, 2, 3\} \geqq \varepsilon_i^{0.1}$.

Let $f_1^i := f_1^{i-1} + h_1^i, f_2^i := f_2^{i-1} + h_2^i$ and $f_3^i = f_3^{i-1} + h_3^i$, then each of these functions are holomorphic in D and continuous on $\bar{D}$.

It will be clear from the constructions of the functions h_1^i, h_2^i, h_3^i that the r_i's do depend on the the functions f_1, f_2, f_3 and the neighborhood W_j. Furthermore the r_i's will have to go to 0 rather rapidly when the ε_i's goes to zero.

Now we let $U_1 = W_1$, $f_1 = f_2 = f_3 = 3/4$ and we add the h_1^i's to the initial f_1, h_2^i's to f_2 and h_3^i's to f_3 and obtain:

d. $|f_1(p) - f_1^i(p)|, |f_2(p) - f_2^i(p)|, |f_3(p) - f_3^i(p)| < \varepsilon_1^3 + \ldots + \varepsilon_i^3$ when $\text{dist}(p, W_1) > 6r_1^{\frac{1}{4}}$ and $p \varepsilon \bar{D}$.

e. $\log(1 - |f_1^i(p)|) + \log(1 - |f_2^i(p)|) + \log(1 - |f_3^i(p)|) <$ $\log(1 - |f_1(p)|) + \log(1 - |f_2(p)|) + \log(1 - |f_3(p)|) +$ $\varepsilon_1^3 + \ldots + \varepsilon_i^3$ for all $p \in \bar{D}$.

f. $\log(1 - |f_1^i(p)|) + \log(1 - |f_2^i(p)|) + \log(1 - |f_3^i(p)|) < \log(1 - |f_1(p)|) +$ $+ \log(1 - |f_2(p)|) + \log(1 - |f_3(p)|) - (1/20)(\varepsilon_1^2 + \ldots + \varepsilon_i^2)$ when $p \in \bar{D}$ and $\text{dist}(p, W_1) < r_i$ and $\max\{1 - |f_j(p)| \quad j = 1,2,3\} \geqq \varepsilon_i^{0.1}$.

Since $\sum \varepsilon_i^3$ is converging and we can of course choose ε_1 as small as we want, d. and e. implies that the original functions are not changed much away from W_1. But $\sum \varepsilon_i^2$ is diverging, hence if i is large enough, say $i \geqq i_0$, then as long as $\text{dist}(p, W_1) < r_i$

$$* \quad \log(1 - |f_1^i(p)|) + \log(1 - (1 - |f_2^1(p)|) + \log(1 - |f_3^i(p)|) < 4\log(\delta_1).$$

Notice that since we have only added a finite number of smooth terms to f_1, f_2 and f_3, then $f_1^{i_0}, f_2^{i_0}$ and $f_3^{i_0}$ are continuous on $\bar{D}$ and holomorphic in D.

Next we let U_2 be $W_2 \setminus \{p \in \bar{D} | \quad \text{dist}(p, W_1) < r_{i_0}\}$ and now we let $f_1 = f_1^{i_0}$, $f_2 = f_2^{i_0}$ and $f_3 = f_3^{i_0}$ from above. We should have had some indices on these new initial functions, but in this way we would eventually drown in indices. So we just have to keep in mind that f_1, f_2, f_3 are not the same functions as we started out to adjust near W_1, but in fact are the adjusted ones.

We start over again and find a new i_0 and new $f_1^{i_0}, f_2^{i_0}, f_3^{i_0}, r_1$ and r_{i_0}. By taking a minimum we may assume that the new r_{i_0} is equal to the previous one, such that:

d. $|f_1(p) - f_1^{i_0}(p)|, |f_2(p) - f_2^{i_0}(p)|, |f_3(p) - f_3^{i_0}(p)| <$ $\varepsilon_1^3 + \ldots \varepsilon_{i_0}^3$ when $\text{dist}(p, U_2) > 6_1^{\frac{1}{4}}$.

e. $\log(1 - |f_1^{i_0}(p)|) + \log(1 - |f_2^{i_0}(p)|) + \log(1 - |f_3^{i_0}(p)|) <$ $\log(1 - |f_1(p)|) + \log(1 - |f_2(p)|) + \log(1 - |f_3(p)|) +$ $\varepsilon_1^3 + \ldots \varepsilon_{i_0}^3$ for all $p \in \bar{D}$.

f. $\log(1 - |f_1^{i_0}(p)|) + \log(1 - |f_2^{i_0}(p)|) + \log(1 - |f_3^{i_0}(p)|) <$ $\log(1 - |\mathrm{f}_1(p)|) + \log(1 - |f_2(p)| + \log(1 - |f_3(p)|) -$ $(1/20)(\varepsilon_1^2 + \varepsilon_2^2 + \ldots + \varepsilon_{i_0}^2)$ is large enough and when $p \in \bar{D}$ and $\text{dist}(p, U_2) < r_{i_0}$.

Notice that statement e. and f. together with the previous statement(*), gives:

f'. $\log(1 - |f_1^{i_0}(p)|) + \log(1 - |f_2^{i_0}(p)|) + \log(1 - |f_3^{i_0}(p)|) <$ $4\log(\delta_1) + \varepsilon_1^3 + \ldots + \varepsilon_{i_0}^3$ when $\text{dist}(p, U_2 \cup W_1) < r_{i_0}$.

Again that we have only added a finite number of smooth terms to the new f_1, f_2 and f_3 so the new $f_1^{i_0}$, $f_2^{i_0}$ and $f_3^{i_0}$ are continuous on $\bar{D}$ and holomorphic in D.

Now we let $U_3 = W_3 \setminus (U_2 \cup W_1), U_4 = W_4 \setminus (U_3 \cup U_2 \cup W_1), \ldots, U_N = W_N \setminus (U_{N-1} \cup \ldots \cup U_2 \cup W_1)$ and f_1^{io}, $f_2 = f_2^{io}$, $f_3 = f_3^{io}$ from the previous adjustment. Eventually we end up with functions $f_1^{io}, f_2^{io}, f_3^{io}$ continuous on $\bar{D}$ and holomorphic in D and positive numbers $r_1 > r_{i_0}$ where:

g. $|(3/4) - f_1^{io}(p)|, |(3/4) - f_2^{io}(p)|, |(3/4) - f_3^{io}(p)| < N(\varepsilon_1^3 + \ldots + \varepsilon_{i_0}^3)$ when $\operatorname{dist}(p, \partial D) > 6r_1^{\frac{1}{4}}$.

h. $\log(1 - |f_1^{io}(p)|) + \log(1 - |f_1^{io}(p)|) + \log(1 - |f_1^{io}(p)|) + \log(1 - |f_1^{io}(p)|) < 3\log(1/4) + N(\varepsilon_1^3 + \ldots + \varepsilon_{i_0}^3)$ for all $p \in D$.

i. Since $U_N \cup \ldots \cup U_2 \cup W_1 = \partial D$ we get $\log(1 - |f_1^{io}(p)|) + \log(1 - |f_2^{io}(p)|) + \log(1 - |f_3^{io}(p)|) < 3\log(1/4) - (1/20)(\varepsilon_1^2 + \ldots + \varepsilon_{i_0}^2) + N(\varepsilon_1^3 + \ldots + \varepsilon_{i_0}^3)$ when $p \in \bar{D}$ and $\operatorname{dist}(p, \partial D) < r_{i_0}$ is the smallest number needed in the above construction.

Now $\sum \varepsilon_i^2$ is diverging while $\sum \varepsilon_i^3$ is converging. Hence if i_0 is large enough, then $(1/20)(\varepsilon_1^2 + \ldots + \varepsilon_{i_0}^2) - N(\varepsilon_1^3 + \ldots + \varepsilon_{i_0}^3)$ is larger than $-(9/2)\log(\delta_1)$. Let $d_1 := r_{i_0}$.

Next we do the above procedure over again, except this time we shall start by adding $h_1^{i_0+1}, h_2^{i_0+1}, h_3^{i_0+1}$, to f_1^{io}, f_2^{io} and f_3^{io}. We also need to make sure that $3r_{i_0+1} < r_{i_0}$. We obtain functions $f_1^{i_1}, f_2^{i_1}, f_3^{i_1}$ such that:

j. $|f_1^{io}(p) - f_1^{i_1}(p)|, |f_2^{io}(p) - f_2^{i_1}(p)|, |f_3^{io}(p) - f_3^{i_1}(p)| < \varepsilon_{i_0+1}^3 + \ldots + \varepsilon_{i_1}^3$ when $\operatorname{dist}(p, \partial D) > 6(r_{i_0+1})^{\frac{1}{4}}$.

k. $\log(1 - |f_1^{i_1}(p)|) + \log(1 - |f_2^{i_1}(p)|) + \log(1 - |f_3^{i_1}(p)|) < \log(1 - |f_1^{io}(p)|) + \log(1 - |f_2^{io}(p)|) + \log(1 - |f_3^{io}(p)|) + \varepsilon_{i_0+1}^3 + \ldots + \varepsilon_{i_1}^3$ for all $p \in \bar{D}$.

l. $\log(1 - |f_1^{i_1}(p)|) + \log(1 - |f_2^{i_1}(p)|) + \log(1 - |f_3^{i_1}(p)|) < \log(1 - |f_1^{i_1}(p)| + \log(1 - |f_2^{i_1}(p)|) + \log(1 - |f_3^{i_1}(p)|) - (1/20)(\varepsilon_{i_0+1}^2 + \ldots + \varepsilon_{i_1}^2)$ when $p \in \bar{D}$ and $\operatorname{dist}(p, \partial D) < r_{i_1}$.

Again since $\sum \varepsilon_i^2$ is diverging it is clear that if i_1 is large enough, then:

$$\log(1 - |f_1^{i_1}(p)|) + \log(1 - |f_2^{i_1}(p)|) + \log(1 - |f_3^{i_1}(p)|) < (9/2)\log(\delta_2) \text{ when } p \in \bar{D} \text{ and } \operatorname{dist}(p, \partial D) < r_{i_1}.$$

Let $d_2 = r_{i_1}$.

Now when we keep going we end up with functions f_1, f_2 and f_3 where:

I. f_1, f_2 and f_3 are holomorphic and nonconstant in D.

II. $|f_1|, |f_2|, |f_3| \leqq 1$ in D.

III. There exist sequences $\{d_j\}$ and $\{\delta_j\}$ $d_j, \delta_j \to 0$, of positive real numbers such that $\log(1 - |f_1(p)|) + \log(1 - |f_2(p)|) + \log(1 - |f_3(p)|) < 3\log(\delta_j)$ when $p \in D$ and $(p, \partial D) < d_j$.

The point II. and III. follows from the above constructions and the fact that $\sum \varepsilon_i^3$ is converging, hence we can assume that $(9/2)\log(\delta_j) + N\sum \varepsilon_i^3 < 3\log(\delta_j)$ for all j.

As for I. we only have to observe that if K is a compact in D when $\operatorname{dist}(K, \partial D) > 6r_i^{\frac{1}{3}}$ when i is large enough. Hence the sum of the n's which are added is absolutely convergent on every compact in D.

2.

Let f_1, f_2 and f_3 be continuous on $\bar{D}$ and holomorphic in D such that $|f_1|, |f_2|$ and $|f_3|$ are all less than 1, then choose $\varepsilon > 0$ so that $(1 - |f_i(p)|) > 7\varepsilon^3$ for all $p \in \bar{D}$ and for all $i = 1, 2, 3 \ldots$

If D a smoothly bounded strongly pseudoconvex domain in $\mathbf{C}^2$ then there exist local coordinates (x, y, z) on ∂D (Darboux coordinates) such that $\partial/\partial x$ and $\partial/\partial y + x\,\partial/\partial z$ is generating the complex tangent plane of ∂D [2]. We can use these coordinates to construct functions $u_{a,b,I}$ where a and b are real numbers and I is an interval in R and:

i. $u_{a,b,I}$ is C^∞-smooth on $\bar{D}$ and holomorphic in D.
ii. $\operatorname{Re} u_{a,b,I}(p) > 0$ when $p \in \bar{D} \setminus N_{a,b,I}$ where $N_{a,b,I} :=$
$\{q \mid x(q) \in I, y(q) = a, z(q) = b$ and $w(q) = 0\}$ and $u_{a,b,I}$ is zero on $N_{a,b,I}$.
iii. $u_{a,b,I}(p) = \psi(x(p)) + i2[(z(p) - b) - x(p)(y(p) - a)]$
$+ 2(z(p) - b - x(p)(y(p) - a))^2 + (y(p) - b)^2 - w(p)$
$+ O((y(p) - a)^3, (z(p) - b)(y(p) - a)^2, (z(p) - b)^2(y(p) - a), w^2(p))$
where $w \mid \partial D = 0$, $w < 0$ in D and $w(p) \sim \operatorname{dist}(p, \partial D)$ in D.

We can choose ψ to be a nonnegative real valued funtion on R such that $\psi \mid I = 0$ and $\psi(x) = \exp\{-1/\operatorname{dist}(x, I)\}$ when x is not contained in I. A proof of this can be found in [3].

Choose $r > 0$ and $\varepsilon > 0$ such that $r \ll \varepsilon$ and let $\varphi_{a,b,I}(p) = \exp\{\log \varepsilon/r) u_{a,b,I}(p)$. Then we have the following lemma.

LEMMA 1. *Let* $V(I, a, b; t) = \{p \in \bar{D} \mid (y(p) - a)^2 + 2(z(p) - b - x(p)(y(p) - a))^2 - w(\mathrm{p}) + \exp\{-1/\operatorname{dist}(x(p), I)\} < t\}$.

If x, y and z are chosen right, then:

i. $|\varphi_{a,b,I}(p)| \leqq 1$ *for all p in* $\bar{D}$.
ii. $|\varphi_{a,b,I}(p)| > (1/2)\varepsilon$ *when* $p \in V(I, a, b; r)$.
iii. $|\varphi_{a,b,I}(p)| < \varepsilon^{(k+1)/2}$ *when* $p \in \bar{D} \setminus V(I, a, b; kr)$ *and* $|y(p) - a|, |z(p) - b|$ *and* $|w(p)|$ *are less than some fixed constant, say c and* $k \geqq 2$.

Note that c does not depend on r or ε.

PROOF. If $p \in V(I, a, b;; f)$, then $0 \leqq \operatorname{Re} u_{a,b,I}(p) < r + r^{\frac{3}{2}}$, hence $|\varphi_{a,b,I}(p)| = \exp\{(\log \varepsilon/r) \operatorname{Re} u_{a,b,I}(p)\} > \exp\{\log \varepsilon/r)(r + r^{\frac{3}{2}})$ if $\varepsilon < 1$.

But this implies that $|\varphi(p)| > \exp\{\log\varepsilon + r^{\frac{1}{4}}\log\varepsilon\} = \varepsilon\exp\{r^{\frac{1}{4}}\log\varepsilon\} > (1/2)\varepsilon$ since $r \ll \varepsilon$. This gives point i. in the lemma.

For point ii. we can observe that $\operatorname{Re} u(p) > kr - |O((kr)^{\alpha+1})|$, $\alpha > 0$ is independant of ε, k and r, when $p \in \bar{D} \setminus V(l, a, b, kr)$ so if kr is not to large then $kr - |O((kr)^{1+\alpha})| > ((k+1)r)/(2)$. Hence

$$|\varphi(p)| < \exp\{(\log\varepsilon/r)((k+1)r/2)\} = \varepsilon^{(k+1)/2}.$$

Before we proceed we should make sure that we have chosen $r > 0$ so small that $|f_i(p) - f_i(q)| < \varepsilon^4$ whenever $i = 1, 2$ or 3 and $p, q \in V(l, a, b; l\,2r^{\frac{1}{4}})$ for some a, b and l where length I is less than or equal to $-4(1/\log r)$. Here we can see how r will depend on ε and f_1, f_2 and f_3.

Now each $W_1, \ldots, W_N \subset \partial D$ should be chosen to be contained in some coordinate neighborhood as above we should in fact choose each W to be the inverse image under the mapping (x, y, z) of $[x_0 - c/4, x_0 + c/4] \times [y_0 - c/4, y_0 + c/4] \times [z_0 - c/4, z_0 + c/4]$ for some $(x_0, y_0, z_0) \in \mathsf{R}$. Here c is the constant mentioned in lemma l. For simlicity let us assume that $x_0 = y_0 = z_0 = 0$. It is clear from the Darboux theorem that we can choose $W_1, \ldots, W_N$ this way and still make zure that $W_1 \cup \ldots \cup W_N = \partial D$.

Next we make the following inductive choice of points in R:

1. $x_{j_1} = (2 + 4j_1)(\pm 1/\log r)$ and $I_{j_1} = [x_{j_1} - 2(\pm 1/\log r), x_{j_1} + 2(\pm 1/\log t)\}$ $j_1 = 1, 2, \ldots, J_1$ where J_1 is chosen so that the union of all the I_{j_1} covers all of $[-c/4, c/4]$.
2. $a_{j_2} = j_2(1.1)r^{\frac{1}{4}}$ where j_2 is an integer and $0 < |j_2| < c/(4.4r^{\frac{1}{4}}) := J_2$.
3. Choose v such that $1 < v(1.2)/(1.1)$ and such that $v(1.1)r^{\frac{1}{4}}\log\varepsilon/r$ is some integer time 2π. Then we let $b_{j_3}v(1.1)r^{\frac{1}{4}}$ where j_3 is an integer and $0 < |j_3| \leqq c/((4.4)vr^{\frac{1}{4}}) := J_3$.

Now we let $u_{j_1, j_2, j_3} = u_{a,b,l}$ where $a = a_{j_2}$, $b = b_{j_3}$ and $l = l_{j_1}$ and then we let $\varphi_{j_1, j_2, j_3} = \exp\{(\log\varepsilon/r)u_{j_1, j_2, j_3}\}$. Finally let p_{j_1, j_2, j_3} in ∂D be the point such that $x(p_{j_1, j_2, j_3}) = x_{j_1}$, $y(p_{j_1, j_2, j_3}) = a_{j_2}$ and $z(p_{j_1, j_2, j_3}) = b_{j_3}$.

Now we want to define $\beta_1, \beta_2, \beta_3$ in the following way:
$\beta_1(\mathrm{q}) = (1/4)(1 - |f_1(q)|)e^{i\arg(f_1(q))}$, $\beta_2(q) = (1/4)(1 - |f_2(q)|)e^{i\arg(f_2(q))}$ and $\beta_3(\mathrm{q}) = (1/4)(1 - |f_3(q)|)e^{i\arg(f_3(q))}$. Note that $|f_i(q) + 4\beta_i(q)| = |f_i(q)| + (1 - |f_i(q)|) = 1$ for all $i = 1, 2, 3$ and all q's.

Make sure that $r > 0$ is chosen so small that $|1 - (\beta_i(q)/\beta_i(p)| < \varepsilon^4$ whenever p and q are contained in $V(I_{j_1}, a_{j_2}, b_{j_3})$.

Now we are ready to define the h's. They will need to be adjusted later on so for now we will call them $\tilde{h}_1, \tilde{h}_2$ and $\tilde{h}_3$.

$$\text{I.} \quad \tilde{h}_1 := \sum\{\beta_1(p_{j_1, j_2, j_3})\varphi_{j_1, j_2, j_3}(p) \mid |j_1| = 1, \ldots, J_1, |j_2| = 1, \ldots, J_2, |j_3| = 1, \ldots, J_3\}$$

II. $\tilde{h}_2 := \sum\{-\beta_2(p_{j_1,j_2,j_3})\varphi_{j_1,j_2,j_3}(p) \mid \; |j_1| = 1,\dots,J_1, |j_2| = 2,4,6,\dots$
$|j_3| = 1,\dots J_3\}$

Note that $|j_2|$ is only running through the even integers between 1 and J_2.

III. $\tilde{h}_3 := \sum\{-\beta_3(p_{j_1,j_2,j_3})\varphi_{j_1,j_2,j_3}(p) \mid \; |j_1| \mp 1,\dots,J_1, |j_2| = 1,3,5,\dots$
$|j_3| = 1,\dots,J_3\}$

Note that $|j_2|$ is only running through the odd integers between 1 and J_2.

Now let us show that $\tilde{h}_1$, $\tilde{h}_2$ and $\tilde{h}_3$ at least locally does have the properties needed.

The first thing we should notice is that $V(I_{j_1}, a_{j_2}, b_{j_3}; r) := V_{j_1,j_2,j_3}$ then $\cup\{V_{j_1,j_2,j_3} \mid \; |j_1| = 1,\dots,J_1, |j_2| = 1,\dots,J_2$ and $|j_3| = 1,\dots,J_3\}$ does cover the set $\{p \in \bar{D} \mid \; \operatorname{dist}(p, W) < (1/2)r\}$.

Next we want to prove:

LEMMA 2.

a. $|f_i(p) - \tilde{h}_1(q)| < 1$ *for all* q *and* $i = 1,2,3$.

b. $|\tilde{h}_1(p)|, |\tilde{h}_2(p)|, |\tilde{h}_3(p)|) \leqq \varepsilon^3$ *when* $\operatorname{dist}(p, W) > 6r^{\frac{1}{4}}$.

c. $\log(1 - |f_1(p) + \tilde{h}_1(p)|) + \log(1 - |f_2(p) + \tilde{h}_2(p)|)$
$+ \log(1 - |f_3(p) + \tilde{h}_3(p)|) \leqq \log(1 - |f_1(p)|) + \log(1 - |f_2(p)|)$
$+ \log(1 - |f_3(p)|) - (1/20)\varepsilon^2$ *when* $p \in V_{j_1,j_2,j_3}$ *for any* j_1, j_2 *and* j_3.

d. $\log(1 - |f_1(p) + \tilde{h}_1(p)|) + \log(1 - |f_2(p) + \tilde{h}_2(p)|) +$
$\log(1 - |f_3(p) + \tilde{h}_3(p)|) < \log(1 - |f_1(p)|) + \log(1 - |f_2(p)|) +$
$\log(1 - |f_3(p)|) + \varepsilon^3$ *for all* $p \in \bar{D}$.

PROOF. The first thing we should notice is that $|f_i(p) + \tilde{h}_i(p)| \leqq |f_i(p)| + \sum\{|\beta_i(p)|\,|\varphi_{j_1,j_2,j_3}(p)|$. Now the norms of the φ's are decreasing when the distance to W is increasing, in other words the last term will attain it maximum near W. Hence if $|f_i(p) + \tilde{h}_i(p)| > 1$ for some p, then this happens near W. In fact it will suffice to study the norm when $p \in V_{j_1,j_2,j_3}$ for some j_1, j_2, j_3.

Next we observe that if we freeze j_1, j_2 and j_3, say we call them j_1^0, j_2^0 and j_3^0, then there are no more than k^2 of the V_{j_1,j_2,j_3}'s in the set $V(I_{j_1^0}, a_{j_2^0}, b_{j_3^0}; (k + 1r)) \setminus V(I_{j_1^0}, a_{j_2^0}, b_{j_3^0}; kr)$. Also for such a choice of j_1, j_2, j_3 we know that $|\varphi_{j_1,j_2,j_3}(p)| < \varepsilon^{(k+1)/2}$. In addition we see that there are no more than 6^3 of the V_{j_1,j_2,j_3}'s in the set $V(I_{j_1^0}, a_{j_2^0}, b_{j_3^0}, 6r)$.

Let us take a close look at the situation:

i. $|\varphi_{j_1,j_2,j_3}(p) < \varepsilon$ if (j_1, j_2, j_3) if $|j_1 - j_1^0| > 1$ or $|j_2 - j_2^0| > 1$
or $|j_3 - j_3^0| > 1$.

ii. $|\varphi_{j_1,j_2,j_3}(p)| < \varepsilon^{0.01}$ if $j_1 \neq j_1^0$.

iii. $|\beta_i(p_{j_1,j_2,j_3}) - (1/4)(1 - |f_i(p)|)| < \varepsilon^4$ if
$V_{j_1,j_2,j_3} \cap V(I_{j_1^0}, a_{j_2^0}, b_{j_3^0})$

From all of this we get that $|f_i(p) - \tilde{h}_i(p)|$
$\leqq |f_i(p)| + (1 - |f_j(p)|)(6^3\varepsilon + 24\varepsilon^{0.01} + 3/4) + \sum\{k^2\varepsilon^{(k+1)/2} | \quad k = 6, 7, \dots\}$
$+ 6^3\varepsilon^4 \leqq |f_i(p)| + (1 - |f_i(p)|)(5/6) + \varepsilon^3$ if $\varepsilon > 0$ is small enough.

But $(1 - \max|f_1(p)|) > 7\varepsilon^3$, hence $|f_i(p) + \tilde{h}_i(p)| < 1 - (1/6)7\varepsilon^3 + \varepsilon^3 < 1$. So a. follows.

Now let us do b. if $\operatorname{dist}(p, W) > 6r^{\frac{1}{4}}$ then p can not be in $V(I_{j_1}, a_{j_2} b_{j_3}; 6r)$ for any choice of j_1, j_2 and j_3, hence $|\varphi_{j_1,j_2,j_3}(p)| < \varepsilon^3$. In fact if we use the above argument we get:

$|\tilde{h}_i(p)| < \sum\{k^2\varepsilon^{(k+1)/2} | \quad k = 6, 7, \dots\} < \varepsilon^3$ for each $i = 1, 2, 3$.
Note that this estimate holds whenever p is not contained in some $V(I_{j_1}, a_{j_2}, b_{j_3}; 6r)$.

The final and crucial part of this proof is c. and d. We shall prove these simultaneously.

The statement in d. follows from a. and b. if $\operatorname{dist}(p, W) \geqq 6r^{\frac{1}{4}}$. So we have to look at the situation where this inequality does not hold.

If $\operatorname{dist}(p, W) < 6r^{\frac{1}{4}}$, then $p \in V(I_{j_1}, a_{j_2} b_{j_3}; 6r^{\frac{1}{4}})$ for some j_1, j_2, j_3.
The proceed we need a little lemma:

LEMMA 3. *If* $0 \leqq x < 1$ *then* $\log(1 - x) < -x - (1/4)x^2$.

PROOF. When we expand $\log(1 - x)$ in Taylor series we obtain that $\log(1 - x) = -x - (1/2(1 + c^2))x^2$ where $0 \leqq c \leqq x < 1$. Hence $\log(1 - x) < -x - (1/4)x^2$.

If $p \in V(I_{j_1^0}, a_{j_2^0}, b_{j_3^0}; 6r^{\frac{1}{4}})$ for some j_1^0, j_2^0, j_3^0, then either $|\beta_i(p_{j_1,j_2,j_3}) - \beta_i(p)| < \varepsilon^4$ or φ_{j_1,j_2,j_3} belongs to the group of no more than k^2 functions where $|\varphi_{j_1,j_2,j_3}(p)| < \varepsilon^{(k+1)/2}$ and $k \geqq 6$. Hence

$\log(1 - |f_1(p) + \tilde{h}_1(p)|) + \log(1 - |f_2(p) + \tilde{h}_2(p)|) + \log(1 - |f_3(p) + \tilde{h}_3(p)|)$
$< \log(1 - |f_1(p) + \sum\{\beta_1(p)\varphi_{j_1,j_2,j_3}(p)| \quad |j_1| = 1, 2, \dots, |j_2| = 1, 2, \dots, |j_3| = 1, 2, \dots\}|)$
$+ \log(1 - |f_2(p) + \sum\{\beta_2(p)\varphi_{j_1,j_2,j_3}(p)| \quad |j_1| = 1, 2, \dots, |j_2| = 1, 2, \dots, |j_3| = 1, 2, \dots\}|)$
$+ \log(1 - |f_3(p) + \sum\{\beta_3(p)\varphi_{j_1,j_2,j_3}(p)| \quad |j_1| = 1, 2, \dots, |j_2| = 1, 2, \dots, |j_3| = 1, 2, \dots\}|)$
$+ (1/2)\varepsilon^3$
$= \log(1 - ||f_1(p)| + (1/4)(1 - |f_1(p)|)\sum\{\varphi_{j_1,j_2,j_3}(p)| \quad |j_1|, |j_2|, |j_3| = 1, 2, \dots\}|)$
$+ \log(1 - ||f_2(p)| + (1/4)(1 - |f_2(p)|)\sum\{\varphi_{j_1,j_2,j_3}(p)| \quad |j_1|, |j_3| = 1, 2, \dots, |j_2| = 2, 4, \dots\}|)$
$+ \log(1 - ||f_3(p)| + (1/4)(1 - |f_3(p)|)\sum\{\varphi_{j_1,j_2,j_3}(p)| \quad |j_1|, |j_3| = 1, 2, \dots, |j_2| = 1, 3, \dots\}|)$
$+ \varepsilon^3 := \Phi(p) + \varepsilon^3$.

Note that for simplicity we will call the main expression Φ. Before we go on we need to observe that if a and b are real numbers, $|a|, |b| < 1$ and $a > 0$ then $|a + ib| \geqq a + (1/4)b^2$. Now $|f_i|, |\tilde{h}_i| < 1$ and $|Re\, \tilde{h}_i| < |f_i|$. Hence:
$\Phi(p) < \log(1 - |f_1(p)| - (1/4)(1 - |f_1(p)|)\sum\{\mathrm{Re}\,\varphi_{j_1,j_2,j_3}(p)| \quad |j_1|, |j_2|, |j_3| = 1, 2, \dots\}$
$-(1/4)^3(1 - |f_1(p)|)^2(\sum\{\mathrm{Im}\,\varphi_{j_1,j_2,j_3}(p)| \quad |j_1|, |j_2|, |j_3| = 1, 2, \dots\})^2)$
$+\log(1 - |f_2(p)| - (1/4)(1 - |f_2(p)|)\sum\{\mathrm{Re}\,\varphi_{j_1,j_2,j_3}(p)| \quad |j_1|, |j_3| = 1, 2, \dots$ and $|j_2| = 2, 4, \dots\}$

$-(1/4)^3(1-|f_1(p)|)^2(\sum\{\operatorname{Im}\varphi_{j_1,j_2,j_3}(p)|\quad |j_1|,|j_3|=1,2,\ldots \text{ and } |j_2|=2,4,\ldots\})^2)$
$+\log(1-|f_3(p)|-(1/4)(1-|f_3(p)|)\sum\{\operatorname{Re}\varphi_{j_1,j_2,j_3}(p)|\quad |j_1|,|j_3|=1,2,\ldots$ and $|j_2|=1,3,\ldots\}$
$-(1/4)^3(1-|f_3(p)|)^2(\sum\{\operatorname{Im}\varphi_{j_1,j_2,j_3}|\quad |j_1|,|j_3|=1,2,\ldots \text{ and } |j_2|=1,3,\ldots\})^2)$
$+\varepsilon^3.$

If we apply lemma 3 we get:

$\Phi(p)<\log(1-|f_1(p)|)+\log(1-|f_2(p)|)+\log(1-|f_3{}^3(p)|)$
$-(1/4)\sum\{\operatorname{Re}\varphi_{j_1,j_2,j_3}(p)|\quad |j_1|,|j_2|,|j_3|=1,2,\ldots\}-(1/4)^3(1-|f_1(p)|)(\sum\operatorname{Im}\varphi_{j_1,j_2,j_3}(p)|$
$|j_1|,|j_2|,|j_3|=1,2,\ldots\})^2-[(1/4)\sum\operatorname{Re}\varphi_{j_1,j_2,j_3}(p)|\quad |j_1|,|j_2|,|j_3|=1,2,\ldots\}+$
$(1/4)^3(1-|f_1(p)|)(\sum\operatorname{Im}\varphi_{j_1,j_2,j_3}(p)|\quad |j_1|,|j_2|,|j_3|=1,2,\ldots\})^2]^2$

$+(1/4)\sum\{\operatorname{Re}\varphi_{j_1,j_2,j_3}(\mathrm{p})|\quad |j_1|,|j_3|=1,2,\ldots \text{ and } |j_2|=2,4,\ldots\}$
$-(1/4)^3(1-|f_2(p)|)(\sum\{\operatorname{Im}\varphi_{j_1,j_2,j_3}(p)|\quad |j_1|,|j_3|=1,2,\ldots \text{ and } |j_2|=2,4,\ldots\})^2$
$-[(1/4)\sum\{\operatorname{Re}\varphi_{j_1,j_2,j_3}(p)|\quad |j_1|,|j_3|=1,2,\ldots \text{ and } |j_2|=2,4,\ldots\}$
$+(1/4)^3(1-|f_2(p)|)\sum\{\operatorname{Im}\varphi_{j_1,j_2,j_3}(p)|\quad |j_1|,|j_3|=1,2,\ldots \text{ and } |j_3|=2,4,\ldots\}]^2$

$+(1/4)\sum\{\operatorname{Re}\varphi_{j_1,j_2,j_3}(p)|\quad |j_1|,|j_3|=1,2,\ldots \text{ and } |j_2|=1,3,\ldots\}$
$-(1/4)^3(1-|f_3(p)|)(\sum\operatorname{Im}\varphi_{j_1,j_2,j_3}(p)|\quad |j_1|,|j_3|=1,2,\ldots \text{ and } |j_2|=1,3,\ldots\})^2$
$-[(1/4)\sum\{\operatorname{Re}\varphi_{j_1,j_2,j_3}(p)|\quad |j_1|,|j_3|=1,2,\ldots \text{ and } |j_2|=1,3,\ldots\}$
$+(1/4)^3(1-|f_3(p)|)(\sum\{\operatorname{Im}\varphi_{j_1,j_2,j_3}(p)|\quad |j_1|,|j_3|=1,2,\ldots \text{ and } |j_2|=1,3,\ldots\})^2]^2+\varepsilon^3$

Observe that the real parts are cancelling each other. Again we should do a rather trivial observation. Namely if $z=x+iy$ is a complex number and $|z|\geqq t$ and $d,e,f>0$ are constants, then $dx^2+(ex+fy)^2>gt^2$ where $g=(1/4)\min(d,e,f)$.

From this we get:

$\Phi(p)<\log(1-|f_1(p)|)+\log(1-|f_2(p)|)+\log(1-|f_3(p)|)$
$-(1/4)^4(1-|f_1(p)|)(|\sum\{\varphi_{j_1,j_2,j_3}(p)|\quad |j_1|,|j_2|,|j_3|=1,2,\ldots\}|)^2$
$-(1/4)^4(1-|f_2(p)|)(|\sum\{\varphi_{j_1,j_2,j_3}(p)|\quad |j_1|,|j_3|=1,2,\ldots \text{ and } |j_2|=2,4,\ldots\}|)^2$
$-(1/4)^4(1-|f_3(p)|)(|\sum\{\varphi_{j_1,j_2,j_3}(p)|\quad |j_1|,|j_3|=1,2,\ldots \text{ and } |j_2|=1,3,\ldots\}|)^2+\varepsilon^3.$

From this d. follows. Now it remains to prove that this last expression is less that or equal to $\log(1-|f_1(p)|)+\log(1-|f_2(p)|)+\log(1-|f_3(p)|)-(1/20)\varepsilon^2$ whenever $p\in\bar{D},\max\{1-|f_i(p)|:i=1,2,3\}\geqq\varepsilon^{0.1}\}$ and also $p\in V_{j_1^0},\mathrm{a}_{j_2^0},\mathrm{b}_{j_3^0}$ for some j_1^0,j_2^0,j_3^0.

We may assume that $a_{j_1^0}<y(p)<a_{j_1^0+1}$ and that j_2^0 is odd which means that j_2^0+1 is even. From the work done to prove a. we know that
$\sum\varphi_{j_1,j_2,j_3}(p)|\ V_{j_1,j_2,j_3}\cap V(I_{j_1^0},a_{j_2^0},b_{j_3^0};tr)=\emptyset\}|\quad<\varepsilon^3.$

Now let us take a close look at φ_{j_1,j_2,j_3}:

$\varphi_{j_1,j_2,j_3}(p)=exp\{\log\varepsilon/r)[\psi_{j_1}(x(p))+(y(p)-a_{j_2})^2+(z(p)-b_{j_3})^2-w(p)$
$+i2(z(p)-b_{j_3}-x(p)(y(p)-a_{j_2}))+O((y(p)-a_{j_2})^3,w^2(p))]\}=$
$\exp\{(\log\varepsilon/r)\alpha(p)\}\exp\{\theta(p)\}\exp\{O((p)-a_{j_2}(p))^3,w^2(p))\}$
where $\alpha_{j_1,j_2,j_3}(p):=\psi_{j_1}(x(p))+(y(p)-a_{j_2})^2+(z(p)-b_{j_3})^2-w(p),$
$\theta_{j_1,j_2,j_3}(p):=i(z(p)-b_{j_3}-x(p)(y(p)-a_{j_2})).$

Notice that the b_{j_3}'s are chosen that if j_2 is fixed, then $\theta_{j_1,j_2,j_3}(p) - \theta_{j_1}, a_{j_2^0}, b_{j_3}(p) = m2\pi$ where m is an integer. Hence $\sum\{\psi_{j_1,j_2,j_3}(p): j_1, j_3 = 1, 2, \ldots\} = (\cos(\theta_{j_1^0}, a_{j_2^0}, b_{j_3^0}(p)) + i\sin\theta_{j_1^0}, a_{j_2^0}, b_{j_3^0}(p))$
$\sum\{|\varphi_{j_1}, a_{j_2^0}, b_{j_3}(p)|| \quad |j_1|, |j_3| = 1, 2, \ldots\}$.

We know that $|a_{j_2^0} - a_{j_2^0+1}| = (1.1)r^{\frac{1}{4}}$ so we may assume that $0 \leqq y(p) - a_{j_2^0} \leqq (0.6)r^{\frac{1}{4}}$. Hence there is some j_1 and j_2 such that $|\varphi_{j_1}, a_{j_2^0}, b_{j_3}(p)| \geqq \varepsilon^{0.7}$. Now j_2^0 is odd so this implies that $|\sum\{\varphi_{j_1}, a_{j_2}, b_{j_3}(p)| \quad |j_1|, |j_3| = 1, 2, \ldots$ and $|j_2| = 1, 3, \ldots$, i.e j_2 is running through the odd integers between 1 and $J_2\} \geqq |\varepsilon^{0.7} - 6^3\varepsilon^{1.1} - \varepsilon^3| \geqq \varepsilon^{0.8}$.

From this we obtain that $\Phi(p) < \log(1 - |f_1(p)| + \log(1 - |f_2(p)|) + \log(1 - |f_3(p)|) - (1/4)^4(1 - |f_3(p)|)(\varepsilon^{0.8})^2 + \varepsilon^3 > \log(1 - |f_1(p)|) + \log(1 - |f_2(p)|) + \log(1 - |f_3(p)|) - \varepsilon^2$ when $(1 - |f_3(p)|) > \varepsilon^{0.1}$.
This completes the proof of part d.

Finally we need to remember that the estimates we have for $\tilde{h}_1, \tilde{h}_2, \tilde{h}_3$ are only local estimates. To obtain functions whith the same properties as these in a global sense we use a cut off function χ such that χ is zero whenever these estimates can not be obtained and $\chi = 1$ whenever $|\tilde{h}_i| > \varepsilon^4$, $i = 1, 2, 3$. Simultanously we should make sure that the first order derivatives of χ are no larger than $1/(\varepsilon^{\frac{1}{4}})$. Finally we let $h_i(p) := \chi\tilde{h}_i(p) - g_i(p)$, where $\bar{\partial} g_i(p) = \bar{\partial}(\chi\tilde{h}_i)(p)$ and $|g_i) < C\varepsilon^{\frac{7}{4}}$. This is possible since $|\bar{\partial}\chi\tilde{h}_i(p)| < \varepsilon^{\frac{7}{4}}$, hence by [5] such g_i's exist. It is easy to see that h_1, h_2, h_3 also satifies a., b., c., and d. in lemma 2.

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