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A CONSTRUCTION OF ANALOGS OF THE BLOCH-WIGNER FUNCTION

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0. Introduction.

D. Wigner has constructed a function

$$D(z) = (\log|z|)\arg(1-z) - \operatorname{Im}\int_0^z \log(1-t)\frac{dt}{t}$$

(see [1]). $D(z)$ is a single valued function from $\mathbf{C}$ to $\mathbf{R}$. Notice that $\int_0^z \log(1-t)\frac{dt}{t}$ as well as $\operatorname{Im}\int_0^z \log(1-t)\frac{dt}{t}$ are multivalued functions. Let us define $L_1(z) = \log(1-z)$ and $L_n(z) = \int_0^z L_{n-1}(t)\frac{dt}{t}$ for $n > 1$. D. Ramakrishnan has constructed some analogs of the Bloch-Wigner function for functions $L_n(z)$ (see [3]). In this note we construct also some analogs of the Bloch-Wigner function. We mention that $L_n(z) = -Li_n(z)$ in the notation used in [2] Chapter 7, which is the standard reference on the subject.

1. Analogs of the Bloch-Wigner function.

Let $\operatorname{Re}(z)$ (resp. $\operatorname{Im}(z)$) be the real (resp. imaginary) part of a complex number z. Our main result is the following:

THEOREM 1.
i)

$$\mathcal{L}_{2k+1}(z) = \sum_{s=0}^{2k}\frac{(-1)^s}{s!}(\operatorname{Re}L_{2k+1-s}(z))(\log|z|)^s - \frac{1}{(2k+1)!}(\log|1-z|)(\log|z|)^{2k}$$

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for $k > 0$ defines a single valued, real analytic function on $\mathbb{C}\backslash\{0,1\}$ *to* $\mathbb{R}$. *This function extends to a continuous functions from* $\mathbb{C}$ *to* $\mathbb{R}$ *if we set* $Ł_{2k+k}(0) = 0$ *and* $Ł_{2k+1}(1) = \operatorname{Re} L_{2k+1}(1)$.
ii)

$$Ł_{2k}(z) = \sum_{s=0}^{2k-1} \frac{(-1)^s}{s!}(\operatorname{Im} L_{2k-s}(z))(\log|z|)^s$$

for $k > 0$ defines a single valued, real analytic function from $\mathbb{C}\backslash\{0,1\}$ *to* $\mathbb{R}$. *This function extends to a continuous function from* $\mathbb{C} \cup \infty$ *to* $\mathbb{R}$ *if we set* $Ł_{2k}(0) = Ł_{2k}(1) = Ł_{2k}(\infty) = 0$.

2. Proof.

Let us set $[x] = n$ if x is a real number, $n \leqq x < n + 1$ and n is an integer. Let $()^S$ (resp. $()^T$) denote the function $()$ after the monodromy transformation around 0 (resp. 1). The monodromy transformation of $L_n(z)$ is given by $L_n(z))^S = L_n(z)$ and $(L_n(z))^T = L_n(z) + \frac{2\pi i}{(n-1)!}(\log z)^{n-1}$. This implies the following lemma.

LEMMA 1. *The monodromy of the functions* $\operatorname{Re}(L_n(z))$ *and* $\operatorname{Im}(L_n(z))$ *is given by the following formulas:*

$(\operatorname{Re} L_n(z))^S = \operatorname{Re} L_n(z), (\operatorname{Im} L_n(z))^S = \operatorname{Im} L_n(z)$,

$$(\operatorname{Re} L_n(z))^T = \operatorname{Re} L_n(z) + (-2\pi) \sum_{l=0}^{[\frac{n-2}{2}]} \frac{(-1)^l}{(2l+1)!\,(n-2l-2)!}(\log|z|)^{n-2l-2} (\arg z)^{2l+1}$$

and

$$(\operatorname{Im} L_n(z))^T = \operatorname{Im} L_n(z) + 2\pi \sum_{l=0}^{[\frac{n-1}{2}]} \frac{(-1)^l}{(2l)!\,(n-2l-1)!}(\log|z|)^{n-2l-1}(\arg z)^{2l}$$

It follows from Lemma 1 that after the monodromy transformation of $L_{2k+1}(z)$ (resp. $L_{2k}(z)$) around 1 the coefficient at $(\log|z|)^{2k-2i-1}(\arg z)^{2i+1}$ (resp. $(\log|z|)^{2k-2i-1}(\arg z)^{2i}$) is equal to

$$(-2\pi) \sum_{s=0}^{2k-2i-1} \frac{(-1)^s(-1)^i}{s!(2i+1)!(2k-2i-s-1)!} \left(\text{resp. } 2\pi \sum_{s=0}^{2k-2i-1} \frac{(-1)^s(-1)^i}{s!(2i)!(2k-2i-s-1)!}\right).$$

These sums are binomial expansions of $(-1+1)^{2k-2i-1}$ up to a multiplicative constant $\frac{(-2\pi)(-1)^i}{(2k-2i-1)!(2i+1)!}$ $\left(\text{resp. } \frac{2\pi(-1)^i}{(2k-2i-1)!(2i)!}\right)$. Hence they are equal to zero. The functions clearly have trivial monodromy around 0. It remains to show that $\lim_{z\to\infty} Ł_{2k}(z) = 0$. The arguments given below are due to the referee. For $z \in \mathbb{C}\backslash\mathbb{R}$

write $z = re^{i\theta}$ and note that $L_n(z) = \int_0^r L_{n-1}(te^{i\theta})\frac{dt}{t}$. Now differentiating the above expression for $Ł_{2k}(re^{i\theta})$ most of the terms cancel out and we are left with

$$\begin{aligned}\frac{\partial Ł_{2k}(re^{i\theta})}{\partial r} &= -\frac{1}{(2k-1)!}\left(\frac{\partial \operatorname{Im} L_1(re^{i\theta})}{\partial r}\right)(\log r)^{2k-1} \\ &= \frac{\sin\theta}{(2k-1)!(r^2+1-2r\cos\theta)}(\log r)^{2k-1}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{\partial Ł_{2k}(r^{-1}e^{i\theta})}{\partial r} &= \frac{-\sin\theta}{(2k-1)!(r^{-2}+1-2r^{-1}\cos\theta)r^2}(-\log r)^{2k-1} \\ &= \frac{\sin\theta}{(2k-1)!(r^2+1-2r\cos\theta)}(\log r)^{2k-1}.\end{aligned}$$

Thus $Ł_{2k}(r^{-1}e^{i\theta})$ and $Ł_{2k}(re^{i\theta})$ have the same derivative and hence are equal since they agree for $r = 1$. It follows that $\lim_{r\to\infty} Ł_{2k}(re^{i\theta}) = \lim_{r\to\infty} Ł_{2k}(r^{-1}e^{i\theta}) = 0$.

We have got the functions $Ł_n(z)$ investigating behaviour of $L_k(z)$'s at ∞. Using more advanced methods we can show that $\lim_{z\to\infty} Ł_{2k+1}(z)$ is finite.

REFERENCES

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