MATH. SCAND. 65 (1989), 143–151

ON POWER SERIES AND MAHLER'S U-NUMBERS

M. H. ORYAN

1. Introduction.

Let

$$f(x) = \sum_{n=0}^{\infty} c_n x^{e_n} \tag{1}$$

be a power series with non-zero rational coefficients $c_n = b_n/a_n$ (a_n, b_n integers and $a_n > 1$) and increasing integers e_n satisfying the following conditions

$$\liminf_{n\to\infty} \frac{\log a_{n+1}}{\log a_n} = \sigma > 1, \tag{2}$$

$$\limsup_{n\to\infty} \frac{\log |b_n|}{\log a_n} = \theta < 1, \tag{3}$$

$$\lim_{n\to\infty} \frac{\log a_n}{e_n} = +\infty. \tag{4}$$

It follows from (2) and (3) that the radius of convergence of (1) is infinity and from (2) that the number

$$u = \limsup_{n\to\infty} \frac{\log \{\operatorname{lcm}(a_0, a_1, \ldots, a_n)\}}{\log a_n}$$

is finite with $1 \leqq u \leqq \sigma/(\sigma - 1)$.

In this paper we prove at first by using a theorem of LeVeque [4; Theorem 4-15, p. 148] which is a generalization of the Thue-Siegel-Roth Theorem the following

THEOREM 1. *Let $f(x)$ be a power series as in* (1) *such that* (2), (3) *and* (4) *hold. Suppose that α in a non-zero algebraic number of degree m smaller than $\sigma(1-\theta)/2u$. Then the number $f(\alpha)$ is transcendental.*

Received October 19, 1988

For the case $e_n = n$ (4) follows from (2) and we obtain from the Theorem 1 the following

COROLLARY. *Let $f(x)$ be a power series as in* (1) *such that* (2), (3) *and $e_n = n$ hold. Suppose that α is a non-zero algebraic number of degree $m < \sigma(1-\theta)/(2u)$. Then the number $f(\alpha)$ is transcendental.*

Moreover we give some sufficient conditions for $f(\alpha)$ to be or not a U-number according to Mahler's classification for transcendental numbers. We prove the following

THEOREM 2. *Let $f(x)$ be a power series as in* (1) *such that* (2), (3) *and* (4) *hold. Suppose that α is a non-zero algebraic number of degree $m < \sigma(1-\theta)/(2u)$ with algebraic conjugates $\alpha^{(1)} = \alpha, \alpha^{(2)}, \ldots, \alpha^{(m)}$.*

1°) *If*

$$\limsup_{n\to\infty} \frac{\log a_{n+1}}{\log a_n} < +\infty, \tag{5}$$

$$\limsup_{n\to\infty} \frac{e_{n+1}}{e_n} < +\infty \tag{6}$$

and if no $\alpha^{(i)}/\alpha^{(j)}$ $(i \neq j)$ is a root of unity then $f(\alpha)$ is not a U-number, i.e. *it is either an S-number or a T-number.*

2°) *If*

$$\limsup_{n\to\infty} \frac{\log a_{n+1}}{\log a_n} = +\infty \tag{7}$$

then $f(\alpha)$ is a U-number of degree $\leqq m$.

For the case $e_n = n$ we give a necessary and sufficient condition for $f(\alpha)$ to be a U-number.

THEOREM 3. *Let $f(x)$ be a series as in* (1) *such that* (2), (3) *and $e_n = n$ hold. Suppose that α is a non-zero algebraic number of degree $m < \sigma(1-\theta)/(2u)$. Then the condition* (7) *is a necessary and sufficient condition for $f(\alpha)$ to be a U-number.*

For the proof of the theorem 2 and the theorem 3 we use essentially the following theorem of Baker [1; Theorem 1, p. 98].

THEOREM (*Baker*). *Suppose that ξ is a real or a complex number and $\kappa > 2$. Let $\alpha_1, \alpha_2, \ldots$ be a sequence of distinct numbers in an algebraic number field K with field heights $H_K(\alpha_1), H_K(\alpha_2), \ldots$ such that*

$$|\xi - \alpha_i| < (H_K(\alpha_i))^{-\kappa} \tag{8}$$

and

$$\limsup_{i\to\infty} \frac{\log H_K(\alpha_{i+1})}{\log H_K(\alpha_i)} < +\infty \tag{9}$$

hold. Then ξ is either an S-*number or a* T-*number.*

2. Lemmas.

The following lemmas are used in the proofs.

LEMMA 1. *Let α be an algebraic number of degree m and height H. Suppose d is a positive integer such that $d\alpha$ is an algebraic integer. Then*

$$H \leqq (2d \max(1, \overline{|\alpha|}))^m.$$

PROOF. See Cijsouw and Tijdeman [2; Lemma 1, p. 302].

LEMMA 2. *Suppose that K is an algebraic number field of degree N and ζ is an algebraic number in K with field height $H_K(\zeta)$. Let the field conjugates of ζ be $\zeta^{(1)} = \zeta, \zeta^{(2)}, \ldots, \zeta^{(N)}$ and the coefficients of x^N in the field equation of ζ, with relatively prime integer coefficients, be t. Then*

$$t \prod_{i=1}^{N} (1 + |\zeta^{(i)}|) < 6^N H_K(\zeta).$$

Further, if $j_1, \ldots, j_s$ are s integers with $1 \leqq j_1 < \ldots < j_s \leqq N$ then

$$t\,\zeta^{(j_1)} \ldots \zeta^{(j_s)}$$

is an algebraic integer.

PROOF. See LeVeque [4; Theorem 4-2, pp. 124–125, and Theorem 2-21, pp. 63–65].

LEMMA 3. *Let ζ_1 and ζ_2 be different conjugates of an algebraic number of degree m and of height H. Then*

$$|\zeta_1 - \zeta_2| \geqq (4m)^{-(m-2)/2}((m+1)H)^{-(2m-1)/2}.$$

PROOF. See Güting [3; Theorem 8, p. 158].

In the remainder of this paper the inequalities hold for all sufficiently large indices and the real numbers $\varepsilon_1, \varepsilon_2, \ldots$ are positive and sufficiently small such that they are not depending on the varying indices.

LEMMA 4. *Let $f(x)$ be a series as in* (1) *such that* (2), (3), (4), (6) *and* $1 < \sigma(1-\theta)$ *hold. Suppose that α is a non-zero algebraic number of degree m with algebraic* *conjugates* $\alpha^{(1)} = \alpha, \alpha^{(2)}, \ldots, \alpha^{(m)}$ *such that no* $\alpha^{(i)}/\alpha^{(j)}$ $(i \neq j)$ *is a root of unity. Let* $\beta_n = \sum_{v=0}^{n} c_v \alpha^{e_v} (n = 0, 1, 2, \ldots)$. *Then the length of any sequence of consecutive terms* β_n *of degree* $< m$ *is bounded.*

PROOF. Let $K = Q(\alpha)$ then $[K:Q] = m$, $\beta_n \in K$. It follows from (2) that the sequence $\{a_n\}$ is monotonically increasing for all sufficiently large n and it follows further from (3)

$$(10) \qquad |b_n| < a_n^{\theta+\varepsilon_1} < a_n \qquad (0 < \varepsilon_1 < 1 - \theta).$$

We assume that the assertion of the lemma is not true. Then there must exist a sequence $\{\Sigma_s\}$ such that

$$\Sigma_s = \{\beta_{n_s+1}, \ldots, \beta_{n_s+q_s}\} \qquad (n_s, q_s \to \infty \text{ as } s \to \infty)$$

with $\deg \beta_v < m$ for $n_s + 1 \leqq v \leqq n_s + q_s$, where $\deg \beta$ denotes the degree of the algebraic number β.

Let $\beta_n^{(1)} = \beta_n, \beta_n^{(2)}, \ldots, \beta_n^{(m)}$ be the field conjugates of β_n. For a pair $(i,j) (1 \leqq i < j \leqq m)$ the equations

$$\beta_v^{(i)} = \beta_v^{(j)} \qquad (v = n, n+1, n+2)$$

can not be satiesfied simultaneoulsy. For otherwise we would get from

$$\frac{\beta_{n+2}^{(i)} - \beta_{n+1}^{(i)}}{\beta_{n+1}^{(i)} - \beta_n^{(i)}} = \frac{\beta_{n+2}^{(j)} - \beta_{n+1}^{(j)}}{\beta_{n+1}^{(j)} - \beta_n^{(j)}}$$

that $(\alpha^{(i)}/\alpha^{(j)})^{e_{n+2} - e_{n+1}} = 1$ which is a contradiction. It follows from this, from $q_n \to \infty$ and the finiteness of the number of the pairs (i,j) that there exists an index pair (i,j) and a subsequence $\{\Sigma_s'\}$ of $\{\Sigma_s\}$ such that for every s it is possible to find terms $\beta_{n_t}, \beta_{n_{t+1}} \in \Sigma_s'$ with $n_{t+1} - n_t \geqq 2$, $\beta_{n_t}^{(i)} = \beta_{n_t}^{(j)}$, $\beta_{n_{t+1}}^{(i)} = \beta_{n_{t+1}}^{(j)}$ and $\beta_v^{(i)} \neq \beta_v^{(i)}$ $(n_t < v < n_{t+1})$. Because of $\beta_{n_t+1}^{(i)} \neq \beta_{n_t+1}^{(j)}$ it follows that $(\alpha^{(i)})^{e_{n_t+1}} \neq (\alpha^{(j)})^{e_{n_t+1}}$. Furthermore we have $\beta_{n_{t+1}}^{(i)} - \beta_{n_t}^{(i)} = \beta_{n_{t+1}}^{(j)} - \beta_{n_t}^{(j)}$ and hence

$$(11) \qquad \sum_{v=1}^{n_{t+1}-n_t} c_{n_t+v}((\alpha^{(i)})^{e_{n_t+v}} - (\alpha^{(j)})^{e_{n_t+v}}) = 0.$$

It follows from (2) and (10)

$$(12) \qquad |c_{n+1}/c_n| \leqq a_n^{-(1-\theta-\varepsilon_1)(\sigma-\varepsilon_2)+1} \qquad (1 < \sigma - \varepsilon_2).$$

From $1 < \sigma(1 - \theta)$ we get $(1 - \theta - \varepsilon_1)(\sigma - \varepsilon_2) - 1 > 0$. By (11) and (12) we obtain

$$(13) \qquad |(\alpha^{(i)})^{e_{n_t+1}} - (\alpha^{(j)})^{e_{n_t+1}}| < 2(n_{t+1} - n_t - 1) \max(1, \overline{|\alpha|})^{e_{n_{t+1}}} \left| \frac{c_{n_t+2}}{c_{n_t+1}} \right|.$$

We have $H(\alpha^{e_{n_t}+1}) < \gamma_1^{e_{n_t}} + 1$ and from Lemma 3 we obtain

$$|(\alpha^{(i)})^{e_{n_t}+1} - (\alpha^{(j)})^{e_{n_t}+1}| \geqq \gamma_2 \gamma_3^{-e_{n_t}+1}, \tag{14}$$

where the real contants $\gamma_1, \gamma_2, \gamma_3$ are positive and independent of n_t.

If $n_{t+1} - n_t$ is bounded for $t \to \infty$ then there exists a real constant $B > 0$ with $n_{t+1} - n_t - 1 \leqq B$. Hence it follows from (6), (12), (13) and (14) a contradiction because of (4).

Hence $n_{t+1} - n_t$ is not bounded for $t \to \infty$. Therefore there exists an index pair $(p, q) \neq (i,j)$ and a subsequence $\{\Sigma_s''\}$ of $\{\Sigma_s'\}$ such that for every s it is possible to find terms $\beta_{n_u}, \beta_{n_{u+1}} \in \Sigma_s''$ with $n_{u+1} - n_u \geqq 2$, $n_t < n_u < n_{u+1} < n_{t+1}$, $\beta_{n_u}^{(p)} = \beta_{n_u}^{(q)}$, $\beta_{n_{u+1}}^{(p)} = \beta_{n_{u+1}}^{(q)}$ and $\beta_v^{(p)} \neq \beta_v^{(q)} (n_u < v < n_{u+1})$. We can show similarly that $n_{u+1} - n_u$ is not bounded for $u \to \infty$. If we go on, we get such terms in Σ_s for sufficiently large s with all different field conjugates because the number of the distinct pairs (i,j) is finite. This contradicts the definition of Σ_s. Hence the lemma is proved.

LEMMA 5. *Let $f(x)$ be a series as in* (1) *such that* (2), (3), $e_n = n$ and $1 < \sigma(1 - \theta)$ *hold. Suppose that α is a non-zero algebraic number of degree m. Let $\beta_n = \sum_{v=0}^{n} c_v \alpha^v$* ($n = 0, 1, 2, \ldots$). *Then the length of any sequene of consecutive terms β_n of degree $< m$ is bounded.*

PROOF. From $e_n = n$ we obtain (4) and (6). In this case $e_{n+2} - e_{n+1} = 1$ therefore for a pair (i,j) $(1 \leqq i < j \leqq m)$ the equations

$$\beta_v^{(i)} = \beta_v^{(j)} \qquad (v = n, n + 1, n + 2)$$

can not be satiesfied simultaneously because of $\alpha^{(i)} \neq \alpha^{(j)}$. The remainder of the proof is similar to the proof of the lemma 4.

LEMMA 6. *Let $f(x)$, α and β_n be as in Lemma* 4 *(respectively as in Lemma* 5*). If $\{\beta_{n_k}\}$ is the subsequence of the terms of degree m in $\{\beta_n\}$ then there is an integer k_0 such that $\beta_{n_k} \neq \beta_{n_{k+1}}$ holds for all integers $k \geqq k_0$.*

PROOF. If the assertion of the lemma were not true then $\beta_{n_k} = \beta_{n_{k+1}}$ would hold for infinitely many k. Hence it would follow for inifinitely many k

$$1 + \sum_{v=2}^{n_{k+1}-n_k} \frac{c_{n_k+v}}{c_{n_k+1}} \alpha^{e_{n_k+v} - e_{n_k+1}} = 0. \tag{15}$$

By Lemma 4 (respectively by Lemma 5) the number of the terms in (15) is bounded and by (4), (6) and (12)

$$\lim_{k\to\infty} \frac{c_{n_k+v}}{c_{n_k+1}} = 0 \qquad (v = 2, 3, \ldots, n_{k+1} - n_k).$$

Therefore we would get a contradiction from (15) and this proves Lemma 6.

3. Proofs.

PROOF OF THEOREM 1. Let $A_n = \operatorname{lcm}(a_0, a_1, \dots, a_n)$. It follows from (2) that $a_n \leqq A_n \leqq a_n^{u+\varepsilon_3}$. We get from Lemma 1 and from (4) that

$$H(\beta_n) \leqq a_n^{um+\varepsilon_4}. \tag{16}$$

Let $\xi = f(\alpha)$. It follows from (2), (4), (10) and (16)

$$\begin{aligned} |\xi - \beta_n| &\leqq a_{n+1}^{-(1-\theta-\varepsilon_5)} \qquad (0 < \varepsilon_5 < 1 - \theta) \\ &\leqq a_n^{-(1-\theta-\varepsilon_5)(\sigma-\varepsilon_2)} \\ &\leqq H(\beta_n)^{-\kappa} \end{aligned} \tag{17}$$

where $\kappa = (1 - \theta - \varepsilon_5)(\sigma - \varepsilon_2)/(um + \varepsilon_4)$. Because of $m < \sigma(1 - \theta)/(2u)$ we obtain $\kappa > 2$. From the theorem of LeVeque [4; Theorem 4-15, p. 148] we get that ξ is transcendental.

PROOF OF THEOREM 2. It follows from $m < \sigma(1 - \theta)/(2u)$ and $1 \leqq u$ that $1 < \sigma(1 - \theta)$. We consider the sequence $\{\beta_{n_k}\}$ $(k \geqq k_0)$ in Lemma 6. We have for the terms of this sequence

$$H(\beta_{n_k}) = H_K(\beta_{n_k}). \tag{18}$$

Let t_{n_k} be the coefficient of x^m in the field equation of β_{n_k} with relatively prime integer coefficients. We put

$$\Lambda = t_{n_{k+1}} t_{n_k} \operatorname{Norm}(\beta_{n_{k+1}} - \beta_{n_k}) \tag{19}$$

where

$$\operatorname{Norm}(\beta_{n_{k+1}} - \beta_{n_k}) = \prod_{i=1}^{m} (\beta_{n_{k+1}}^{(i)} - \beta_{n_k}^{(i)}). \tag{20}$$

Since $\beta_{n_{k+1}} \neq \beta_{n_k}$ it follows from (19) that Λ is the sum of products of conjugates of $\beta_{n_{k+1}}$ and β_{n_k}, all multiplied by $t_{n_{k+1}} t_{n_k}$. It follows from Lemma 2 that Λ is a rational integer and hence we obtain

$$|\Lambda| \geqq 1. \tag{21}$$

We now find an upper bound for $|\Lambda|$. From

$$|\beta_{n_{k+1}}^{(i)} - \beta_{n_k}^{(i)}| \leqq (1 + |\beta_{n_{k+1}}^{(i)}|)(1 + |\beta_{n_k}^{(i)}|),$$

and from Lemma 2 and (18) we obtain

$$t_{n_k} \prod_{i=1}^{m} (1 + |\beta_{n_k}^{(i)}|) \leq 6^m H(\beta_{n_k}). \tag{22}$$

It follows from (19) and (20) that

$$|\Delta| = |\beta_{n_{k+1}} - \beta_{n_k}|\,|t_{n_{k+1}} - t_{n_k} \prod_{i=2}^{m} (\beta^{(i)}_{n_{k+1}} - \beta^{(i)}_{n_k})|$$
$$\leqq |\beta_{n_{k+1}} - \beta_{n_k}|\, 6^{2m} H(\beta_{n_{k+1}}) H(\beta_{n_k})$$

and

$$|\Delta| \leqq |\beta_{n_{k+1}} - \beta_{n_k}|\, 6^{2m} (\max\{H(\beta_{n_{k+1}}), H(\beta_{n_k})\})^2.$$

We obtain from (17) and $\sigma - \varepsilon_2 > 1$ that

$$(23) \qquad |\beta_{n_{k+1}} - \beta_{n_k}| \leqq |\xi - \beta_{n_{k+1}}| + |\xi - \beta_{n_k}| \leqq 2a_{n_k}^{-(1-\theta-\varepsilon_4)}$$

and hence from (21) and (23)

$$(24) \qquad a_{n_k}^{1-\theta-\varepsilon_4} \leqq 2\cdot 6^{2m} (\max\{H(\beta_{n_{k+1}}), H(\beta_{n_k})\})^2,$$

and therefore

$$\max\{H(\beta_{n_{k+1}}), H(\beta_{n_k})\} \to \infty \quad \text{as} \quad k \to \infty.$$

Thus from (24) on taking logarithms it follows that

$$(25) \qquad (1 - \theta - \varepsilon_4) \log a_{n_k} \leqq (2 + \varepsilon_5) \max\{\log H(\beta_{n_{k+1}}), \log H(\beta_{n_k})\}.$$

We define now inductively a sequence $\{k_i\}$. Let k_1 be the least positive integer for which (16), (17), (18) and (25) hold. Let i be a positive integer and we suppose that k_i has been defined and we take k_{i+1} as $k_i + 1$ or $k_i + 2$ according as $H(\beta_{n_{k_i+1}})$ is or is not greater than $H(\beta_{n_{k_i+2}})$. Then by definition

$$(26) \qquad \max\{\log H(\beta_{n_{k_{i-1}+2}}), \log H(\beta_{n_{k_{i-1}+1}})\} = \log H(\beta_{n_{k_i}}).$$

By (5) there is a constant $c > 1$ such that

$$(27) \qquad \log a_{n+1} < c \log a_n.$$

Hence from the definition of k_i it follows for all i

$$(28) \qquad c^{-A} \log a_{n_{k_i}} < \log a_{n_{k_{i-1}+1}},$$

where A is an upper bound for $n_{k+1} - n_k$ by Lemma 4. From (25), (26) and (28) we obtain

$$(29) \qquad (1 - \theta - \varepsilon_4) c^{-A} \log a_{n_{k_i}} < (2 + \varepsilon_5) \log H(\beta_{n_{k_i}})$$

for all i. Hence we obtain from (16), (27) and (29)

$$\text{(30)}\qquad \frac{\log H(\beta_{n_{k_{i+1}}})}{\log H(\beta_{n_{k_i}})} \leqq \frac{(um+\varepsilon_3)\log a_{n_{k_{i+1}}}}{\dfrac{1-\theta-\varepsilon_4}{c^A(2+\varepsilon_5)}\log a_{n_{k_i}}} \leqq \frac{(um+\varepsilon_3)c^A\log a_{n_{k_i+1}}}{\dfrac{1-\theta-\varepsilon_4}{c^A(2+\varepsilon_5)}\log a_{n_{k_i}}}.$$

We obtain from (5) and (30)

$$\text{(31)}\qquad \limsup_{i\to\infty} \frac{\log H(\beta_{n_{k_{i+1}}})}{\log H(\beta_{n_{k_i}})} < \infty .$$

Finally we define a subsequence $\{\beta_{t_j}\}$ of $\{\beta_{n_{k_i}}\}$ so that we take $t_1 = 1$ and for each integer $j \geqq 1$ we take t_{j+1} as the least integer in $\{n_{k_i}\}$ greater than t_j for which $H(\beta_{t_j})$ is less than $H(\beta_{t_{j+1}})$. It is possible to find such an index since the number of the algebraic numbers in K with bounded field height is finite and if in the sequence $\{\beta_n\}$ a term is repeated infinitely many times, then ξ must be in K because of the definition of β_n. Then we have

$$H(\beta_{t_1}) < H(\beta_{t_2}) < \dots$$

and

$$\frac{\log H(\beta_{t_{j+1}})}{\log H(\beta_{t_j})} \leqq \frac{\log H(\beta_{t_{j+1}})}{\log H(\beta_{t_{j+1}-1})}$$

hence

$$\text{(32)}\qquad \limsup_{j\to\infty} \frac{\log H(\beta_{t_{j+1}})}{\log H(\beta_{t_j})} < +\infty .$$

Moreover the terms of $\{\beta_{t_j}\}$ are all distinct because their heights are all distinct. We have further for the sequence $\{\beta_{t_j}\}$ from (17)

$$\text{(33)}\qquad |\xi - \beta_{t_j}| < (H(\beta_{t_j}))^{-\kappa}$$

with $\kappa > 2$. We obtain from (32) and (33) the conditions (8) and (9) of Baker's Theorem for the sequence $\{\beta_{t_j}\}$ and hence the first part of the theorem 2 is proved.

For the proof of the second part we consider the sequence $s_n := (\log a_{n+1})/(\log a_n)$. It follows from (7) that the sequence $\{s_n\}$ contains a subsequence $\{s_{n_j}\}$ with $\lim_{j\to\infty} s_{n_j} = +\infty$. We consider now the sequence $\{\beta_{n_j}\}$. No term in $\{\beta_{n_j}\}$ can be repeated infinitely many times because of the transcendence of ξ. Hence there is a subsequence $\{\beta_{n_{j_q}}\}$ of $\{\beta_{n_j}\}$ with pairwise different terms and monotonically increasing heights. For this subsequence we get from (16) and (17)

$$\text{(34)}\qquad |\xi - \beta_{n_{j_q}}| \leqq H(\beta_{n_{j_q}})^{-\frac{1-\theta-\varepsilon_4}{um+\varepsilon_3} s_{n_{j_q}}}.$$

Because of $\deg \beta_{n_{j_q}} \leqq m$ and $\lim_{q\to\infty} s_{n_{j_q}} = +\infty$ we get from (34) that ξ is a U-number of degree $\leqq m$. From the equivalence of the Mahler's and Koksma's classification of transcendental numbers it follows that ξ is a U-number of degree $\leqq m$.

PROOF OF THEOREM 3. The proof of this theorem is similar to that of Theorem 2 (use Lemma 5 instead of Lemma 4).

REFERENCES

1. A. Baker, *On Mahler's classification of transcendental numbers*, Acta Math. 111 (1964), 97–120.
2. P. L. Cijsouw, R. Tijdeman, *On the transcendence of certain power series of algebraic numbers*, Acta Arith. 23 (1973), 301–305.
3. R. Güting, *Approximation of algebraic numbers by algebraic numbers*, Michigan Math. J. 8 (1961), 149–159.
4. W. J. Leveque, *Topics in Number Theory*, *vol.* 2, Addision-Wesley, Massachusetts, 1961.

UNIVERSITY OF ISTANBUL
FACULTY OF SCIENCE
MATHEMATICS DEPARTMENT
VEZNECILER-ISTANBUL
TURKEY