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HAUSDORFF DIMENSION AND QUASISYMMETRIC MAPPINGS

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A. It is well known that quasisymmetric mappings, although sharing some of the properties of higher dimensional quasiconformal mappings, can be very singular in the measure theoretic sense. This behaviour is in stark contrast to that of quasiconformal mappings which, for instance, preserve null-sets. It is known from the time of Beurling-Ahlfors [BA] that this is not true for quasisymmetric mappings.

The purpose of this note is to show that the behaviour of quasisymmetric mappings can be worse than just not being absolutely continuous with respect to the linear measure. We will construct a quasisymmetric map of the unit interval I and a subset $Y \subset I$ such that the Hausdorff dimensions of both fY and $I \setminus Y$ are less than 1. Thus we can give an affirmative answer to the question of the existence of quasisymmetric mappings not preserving sets of Hausdorff dimension 1 which has been posed by Hayman and Hinkkanen in [HH, p. 64].

On the other hand, Hölder-continuity of quasisymmetric maps implies that there is a positive lower bound on the dimension of the image of a set of Hausdorff dimension $d > 0$ which depends only on d and the quasisymmetry constant of the map. This has been proved for quasisymmetric maps of general metric spaces in [TV, 3.18] and [HH, Theorem 11] gives some explicit estimates for the present situation.

We remark that n-dimensional quasiconformal maps, $n > 1$, preserve sets of Hausdorff dimension n, as was shown by Gehring and Väisälä [GV]. We may note, that by Hölder-continuity, quasisymmetric and quasiconformal maps preserve sets of Hausdorff dimension 0.

B. We work in the following setting. Let I be the interval $[0, 1]$. We consider increasing self-maps of I and such a map f is *quasisymmetric* if there is $k \geqq 1$ such

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that

$$\frac{1}{k} \leqq \frac{f(x+t)-f(x)}{f(x)-f(x-t)} \leqq k \tag{1}$$

for all distinct $x, x+t, x-t \in I$. If (1) is true we say that f is k-quasisymmetric and the smallest number $k \geqq 1$ such that (1) is true is the *quasisymmetry constant* of f.

The Hausdorff dimension of a set X is denoted by $\dim_H X$. We can now formulate our result more precisely as

THEOREM. *There are a quasisymmetric self-map f of I and a set $Y \subset I$ such that the Hausdorff dimensions of both $I \setminus Y$ and fY are less than* 1. *Furthermore, given $k > 1$ and $d > 0$, f and Y can be so chosen that either*

(a) *f is k-quasisymmetric, or*

(b) $\dim_H fY < d$ *and* $\dim_H I \setminus Y < d$.

It is slightly more convenient to consider maps of the interval I rather than of the real line. This is an inessential restriction since, if $f\colon I \to I$ is k-quasisymmetric, then we can extend f as in [LV, II.7.2] to a k'-quasisymmetric homeomorphism of R by

$$\begin{aligned} &f(x) = -f(-x) \ \text{ if } \ x \in -I, \ \text{ and} \\ &f(x+2n) = f(x) + 2n \ \text{ if } \ x \in I \cup (-I), \quad n \in \mathsf{Z}, \end{aligned} \tag{2}$$

where $k' = k'(k)$ and $k' \to 1$ as $k \to 1$ (see C below).

Thus our theorem is also valid for quasisymmetric maps of R although for definitiveness we have formulated it for maps of I.

C. We make here use of the fact that an increasing homeomorphism f of I is quasisymmetric as soon as the numbers

$$k_{nj} = \frac{f((j+1)2^{-n}) - f(j2^{-n})}{f(j2^{-n}) - f((j-1)2^{-n})},$$

$n > 0, 0 < j < 2^n$, are bounded away from 0 and ∞. Moreover, if

$$q = \sup_{n>0,\, 0<j<2^n} k_{nj}^{\pm 1}, \tag{3}$$

then f is k-quasisymmetric for some $k = k(q)$ such that $k \to 1$ as $q \to 1$, as follows from [T1, Proposition 4] (on p. 135 of [T1] one has the explicit bound $q(1 + q + q^2 + q^3 + q^4)$ for k). Actually, in [T1] one considers maps of R but one sees that if $f\colon I \to I$ is extended to a homeomorphism of R by (2), then (3) is not increased if the supremum is taken over all $n, j \in \mathsf{Z}$.

D. SALEM'S FUNCTION. The starting point of our construction is the singular function considered by Salem [S]. We describe it here. This function is not quasisymmetric but a slight change in the construction will give us a function with this property. I am indebted to Matti Lehtinen for bringing Salem's function to my attention.

Let $t \in (-\frac{1}{2}, \frac{1}{2})$ and define inductively intervals I^t_{nj}, $n \geqq 0$ and $0 \leqq j < 2^n$, by the rule that $I^t_{00} = I$ and that the invervals $I^t_{(n+1)j}$ (j varies) are obtained from I^t_{nj} by dividing each interval I^t_{nj} into two intervals $I^t_{n+1,2j}$ and $I^t_{n+1,2j+1}$; the indexation is such that I^t_{nj} are consequtively on I as j varies from 0 to $2^n - 1$ (so that I^t_{n0} is the "lowest" and $I^t_{n,2^n-1}$ is the "highest" interval) and that

$$\text{(4)} \qquad \frac{|I^t_{n+1,2j}|}{|I^t_{nj}|} = \tfrac{1}{2} + t \text{ and } \frac{|I^t_{n+1,2j+1}|}{|I^t_{nj}|} = \tfrac{1}{2} - t,$$

where $|J|$ is the length of an interval J. If $t = 0$, we denote I^0_{nj} also by I_{nj} and

$$I_{nj} = I^0_{nj} = [j2^{-n}, (j+1)2^{-n}].$$

Salem's singular function h is the homeomorphisms h of I such that $h(I_{nj}) = I^t_{nj}$. One easily sees that

$$\frac{h(\frac{1}{2} + \frac{1}{2}^{-n}) - h(\frac{1}{2})}{h(\frac{1}{2}) - h(\frac{1}{2} - \frac{1}{2}^{-n})} = \frac{(\frac{1}{2} - t)(\frac{1}{2} + t)^{n-1}}{(\frac{1}{2} + t)(\frac{1}{2} - t)^{n-1}}.$$

If $t \neq 0$, this tends to 0 or to ∞ as $n \to \infty$ and hence h is not quasisymmetric.

We can obtain the quasisymmetry condition if we change the construction as follows. Otherwise the construction is valid as above but (4) is valid only for even j; if j is odd, then we interchange $\frac{1}{2} + t$ and $\frac{1}{2} - t$ in (4). After this small change we define a map f^n_t by the formula

$$\text{(5)} \qquad f^n_t(I_{nj}) = I^t_{nj} \quad (n \text{ fixed}, 0 \leqq j < 2^n)$$

and by the requirement that each $f^n_t \mid I_{nj}$ is linear and increasing. These maps have a limit f_t such that

$$\text{(6)} \qquad f_t(I_{nj}) = I^t_{nj}$$

for all n, j. This map f_t is quasisymmetric: it is easy to see that the numbers k_{nj} from C (for $f = f_t$) satisfy

$$k^{\pm 1}_{nj} = \frac{\frac{1}{2} + t}{\frac{1}{2} - t} \quad (n > 0, 0 < j < 2^n)$$

and hence (see C) f_t is quasisymmetric and its quasisymmetry constant tends to 1 as $t \to 0$. Note that f_0 is the identity map of I.

Thus f_t is the limit of maps f^n_t which are affine on intervals and the next function f^{n+1}_t in the sequence is obtained by dividing each interval into two pieces such that f_t^{n+1} changes the ratio of the pieces (but does not change the endpoints). This was the underlying idea of [T1], and Carleson [C] has also used a similar construction to find singular quasisymmetric maps.

E. We will show that the map $f = f_t f_{-t}^{-1}$ satisfies the conditions of our theorem for suitable $t > 0$. Except for the quasisymmetric part, the result is valid also if f_t and f_{-t} are defined as for Salem's function, or in fact if they are limits of maps as in (5) where the intervals I_{nj}^t satisfy (4) up to a permutation of the numbers $\frac{1}{2} + t$ and $\frac{1}{2} - t$ (that is, the order of $\frac{1}{2} - t$ and $\frac{1}{2} - t$ may vary in (4) depending on n and j but the same order is used for f_t and f_{-t}).

Even if we do not require that the intervals I_{nj}^t are consecutively on I, but require only that $I_{n+1,2j}^t, I_{n+1,2j+1}^t$ is a subdivision of I_{nj}^t into two intervals which satisfies (4) (now no permutation of $\frac{1}{2} + t$ and $\frac{1}{2} - t$ and the order of $I_{n+1,2j}^t$ and $I_{n+1,2j+1}^t$ on I may depend on t), (5) still defines a map f_t^n except for endpoint of the intervals I_{nj}. These maps have a limit f_t (possible non-homeomorphic) which is well-defined for points $x \neq j2^{-n}$. Again, apart from quasisymmetry, our results are valid for these maps.

However, we fix the situation and let f_t be as defined in Section D in which case the maps are quasisymmetric.

F. If $I_{n+m,i}^t \subset I_{nj}^t$, then there is a well-defined sequence

$$I_{nj}^t = I_{nj_0}^t \supset I_{n+1,j_1}^t \supset \dots \supset I_{n+m,j_m}^t = I_{n+m,i}^t$$

of intervals. The length of each interval is either $\frac{1}{2} + t$ or $\frac{1}{2} - t$ times the length of the preceding one and hence there is r such that

$$|I_{n+m,i}^t| = (\tfrac{1}{2} + t)^r(\tfrac{1}{2} - t)^{m-r}|I_{nj}^t|. \tag{7}$$

The number of such subintervals $I_{n+m,i}^t$ (i varies) of I_{nj}^t which satisfy (7) is given by the binomial coefficient $\binom{m}{r}$.

We denote by $A_t(n,j,m,r) = A_t(I_{nj}^t, m, r)$ the family of subintervals $I_{n+m,i}^t$ of I_{nj}^t for which (7) i true and define

$$\begin{aligned} A_t(n,j,m) &= A_t(I_{nj}^t, m) = \bigcup_{r \leq m/2} A_t(n,j,m,r), \quad \text{and} \\ B_t(n,j,m) &= B_t(I_{nj}^t, m) = \bigcup_{r > m/2} A_t(n,j,m,r). \end{aligned} \tag{8}$$

(If $t = 0$, then these families are no more well-defined by (7), but an equivalent definition is easily given combinatorially or by a limit process.) Thus if $K = I_{nj}^t$, then $A_t(K,m) \cup B_t(K,m)$ is the family of intervals $J = I_{n+m,k}^t$ such that $J \subset K$.

By (6), we have for every $K = I_{nj}^0$ and $m > 0$

$$f_t(A_0(K,m)) = A_t(f_t(K),m) \tag{9}$$

and similarly for the B_t-families.

Our theorem is based on some measure properties of these families. Define the number d_t by

$$\tfrac{1}{2} = \left(\sqrt{\tfrac{1}{4} - t^2}\right)^{d_t}; \tag{10}$$

then $0 < d_t = d_{-t} \leqq 1$ and $d_t = 1$ only if $t = 0$. We need also a number d_t' which is between d_t and 1. It is not very important how it is defined but it is convenient to fix it by

$$d_t' = \sqrt{d_t}. \tag{11}$$

The first lemma we need is

LEMMA 1. *Let $K = I_{nj}^t$. Then for all $m > 0$*

$$\sum_{J \in A_t(K,m)} |J|^{d_t} < |K|^{d_t} \tag{12}$$

it $t \geqq 0$, and if $t \leqq 0$,

$$\sum_{J \in B_t(K,m)} |J|^{d_t} < |K|^{d_t}. \tag{13}$$

PROOF. We prove (12). Exactly the same calculation proves (13). We have

$$\sum_{J \in A_t(K,m)} |J|^{d_t} = \sum_{r \leqq m/2} \binom{m}{r} [(\tfrac{1}{2} + t)^r (\tfrac{1}{2} - t)^{m-r} |K|]^{d_t}.$$

Since

$$(\tfrac{1}{2} + t)^r (\tfrac{1}{2} - t)^{m-r} = (\tfrac{1}{4} - t^2)^r (\tfrac{1}{2} - t)^{m-2r} \leqq \left(\sqrt{\tfrac{1}{4} - t^2}\right)^m$$

and $\left(\sqrt{\tfrac{1}{4} - t^2}\right)^{d_t} = \tfrac{1}{2}$, we obtain

$$\sum_{J \in A_t(K,m)} |J|^{d_t} \leqq \sum_{r \leqq m/2} \binom{m}{r} 2^{-m} |K|^{d_t} < |K|^{d_t}.$$

We do not need so much (13) than the following consequence of it. If $t < 0$, and $K = I_{nj}^t$, then for every $c' > 0$ there is $m = m(c', t, K)$ such that

$$\sum_{J \in B_t(K,m)} |J|^{d_t'} < c'. \tag{14}$$

This follows from (13) since $d_t < d_t'$ and, if $J \in B_t(K,m)$, $|J| \leqq (\tfrac{1}{2} + |t|)^m \to 0$ as $m \to \infty$.

REMARK. Actually, in (13) we could replace $|K|^{d_t}$ by $\frac{1}{2}|K|^{d_t}$ and similarly in (12) we could have the majorant $c_m|K|^{d_t}$ where $c_m \leqq \frac{3}{4}$ and $c_m \to \frac{1}{2}$ when $m \to \infty$. But this will suffice to us.

G. We can now construct a Cantor set X such that $\dim_H f_t(X) < 1$ ($t > 0$) although $f_{-t}(X)$ is fairly big, for instance of positive linear measure.

LEMMA 2. *Let $t > 0$ and $c > 0$. Let K be an (open, half-open or closed) subinterval of I. Then there is a Cantor set $X \subset K$ such that* $\dim_H f_t(X) \leqq d_t$ *and that if J_k, $k > 0$, are the intervals of $K \setminus X$, then*

$$\sum_{k>0} |f_{-t}(J_k)|^{d'_t} \leqq c. \tag{15}$$

PROOF. We can assume that K is open. We can represent K as a locally finite union of intervals K_j such that each K_j is of the form I_{nk} and that two intervals K_p and K_q have at most an endpoint in common. It is clear that it suffices to prove the lemma for each $K'_j = \operatorname{int} K_j$ with individually chosen $c = c_j$ such that $\Sigma_j c_j < c$. When we have found for each j the Cantor set $X_j \subset K'_j$, we set $X = \cup_j X'_j$ where either $X'_j = X_j$ or $X'_j = \varnothing$.

Thus we can assume that $K = \operatorname{int} I_{nk}$ for some n, k. We define inductively families F_j of intervals $J \subset K$ of the form I_{pq} with the following properties. For every $J \in F_j$ there is a number m_J such that

$$F_{j+1} = \bigcup_{J \in F_j} A_0(J, m_J) \tag{16}$$

where A_0 is as in (8). Furthermore, if E_j is the family components of $K \setminus (\cup F_j)$, then

$$\sum_{J \in E_j} |f_{-t}(J)|^{d'_t} \leqq c(2^{-1} + 2^{-2} + \ldots + 2^{-j}) < c. \tag{17}$$

We define $F_1 = \{K\}$ (it clearly satisfies (17)) and having defined $F_1, \ldots, F_j$, define F_{j+1} as follows. By (14) (and (9)), there is for every $J \in F_j$ a number m_J such that

$$\sum_{J' \in B_0(J, m_J)} |f_{-t}(J')|^{d'_t} \leqq c^{2-j-1} |f_{-t}(J)|. \tag{18}$$

We fix for every $J \in F_J$ such a number m_J and can now define F_{j+1} by (16). Every interval $J' \in E_{j+1}$ is a union of intervals J'' such that either $J'' \in E_j$ or $J'' \in B_0(J, m_J)$ for some $J \in F_j$. Since $d'_t < 1$, (17) and (18) imply that

$$\sum_{J' \in E_{j+1}} |f_{-t}(J')|^{d'_t} \leqq c(2^{-1} + \ldots + 2^{-j}) + c^{2-j-1} \sum_{J \in F_j} |f_{-t}(J)|$$
$$\leqq c(2^{-1} + \ldots + 2^{-j-1}).$$

(For the last inequality, we have used the fact that $\sum_{J \in F_j} |f_{-t}(J)| < 1$.)

We can now define the Cantor set X. Let

$$X_j = \cup F_j \text{ and}$$
$$X = \cap_j X_j$$

which is clearly a Cantor set. Let J be a component of $K \setminus X$. Then, beginning from some j, there is a unique $J_j \in E_j$ such that $J \supset J_j$. Then $|f_{-t}(J_j)|$ is an increasing sequence such that $|f_{-t}(J_j)| \to |f_{-t}(J)|$ as $j \to \infty$. It follows by (17) that (15) is true.

To estimate the Hausdorff dimension of $f_t(X)$, we observe that $\{f_t(J)\colon J \in F_J\}$ is a cover of $f_t(X)$ by intervals. Applying (12) (and (9)) to the the definition (16) of F_{j+1}, we obtain

$$\sum_{J' \in F_{j+1}} |f_t(J')|^{d_t} = \sum_{J \in F_j} \sum_{J' \in A_0(J, m_J)} |f_t(J')|^{d_t}$$

$$< \sum_{J \in F_j} |f_t(J)|^{d_t} < \ldots < |K|^{d_t} \leqq 1.$$

Since the maximum length of the intervals of F_j tends to 0 as $j \to \infty$, it follows that $\dim_H f_t(X) \leqq d_t < 1$.

H. Conclusion of the proof. We have that $t > 0$ and hence $d'_t < 1$. Consequently the linear measure $|f_{-t}(X)|$ of the Cantor set X of Lemma 2 is $\geqq 1 - c$. Here $c > 0$ is an arbitrary constant and it would be an easy matter to construct a set Y of full linear measure such that $\dim_H f_t f_{-t}^{-1}(Y) < 1$. However, in order to obtain that $\dim_H I \setminus Y < 1$, one more step is necessary.

We construct inductively Cantor sets $Y_1 \subset Y_2 \subset \ldots$ such that

$$\dim_H f_t(Y_j) \leqq d_t \tag{19}$$

for all j and such that if J_{jk}, $k > 0$, are the components of $I \setminus f_{-t}(Y_j)$, then

$$\sum_{k>0} |J_{jk}|^{d'_t} \leqq 2^{-j}. \tag{20}$$

For Y_1 we can take the Cantor set X given by Lemma 2 with $K = I$ and $c = \frac{1}{2}$. Suppose then that $Y_1, \ldots, Y_j$ have been constructed and satisfy (19 and (20). Let J_k, $k > 0$, be the components of $I \setminus Y_j$. For each J_k fix a Cantor set $X = Z_k \subset J_k$ as in Lemma 2 with $K = J_k$ and $c = 2^{-j-1} |f_{-t}(J_k)|$. Let $Y_{j+1} = Y_j \cup (\cup_{k>0} Z_k)$. It is clearly a Cantor set which satisfies (19). It satisfies also (20) since $\Sigma_k |f_{-t}(J_k)| \leqq 1$.

Let

$$Y = \cup_j Y_j.$$

Since $\dim_H f_t(Y_j) \leqq d_t$ for all j, also $\dim_H f_t(Y) \leqq d_t$.

We claim that $\dim_H I \setminus f_{-t}(Y) \leqq d_t'$. As above, let J_{jk}, $k > 0$, be the components of $I \setminus f_{-t}(Y_j)$. Then

$$I \setminus f_{-t}(Y) = \cap_j (\cup_k J_{jk}).$$

Thus for each j, $\{J_{jk}\colon\ k > 0\}$, is a cover of $I \setminus f_{-t}(Y)$ by intervals. We can now use (20) to conclude that $\dim_H I \setminus f_{-t}(Y) \leqq d_t'$.

Let

$$f = f_t f_{-t}^{-1}.$$

We have shown that f maps the set $f_{-t}(Y)$ onto the set $f_t(Y)$ such that $\dim_H f_t(Y) \leqq d_t < 1$ and $\dim_H I \setminus f_{-t}(Y) \leqq d_t' < 1$. By (10) and (11), d_t and d_t' can be chosen to be arbitrarily small. Thus to conclude the proof of the theorem we must only show that f is quasisymmetric and, moreover, that the quasisymmetry constant of f tends to zero as $t \to 0$.

It seems that the proof is non-trivial if one uses only the definition of quasisymmetry, but we can note that every f_r can be extended by (2) (see C) to a k_r-quasisymmetric map of R where $k_r \to 1$ as $r \to 0$. Hence f_t and f_{-t}, and consequently also f, have a Beurling-Ahlfors extensions to the upper half-plane whose dilatation tends to 0 as $t \to 0$. By the estimate in [BA, p. 131], f is q_t-quasisymmetric where $q_t \to 0$ as $t \to 0$. The proof is complete.

I. FUCHSIAN GROUPS. There is an area where singular quasisymmetric maps frequently turn up. Let G be a Fuchsian group with finite-volume quotient such that $\bar{\mathsf{R}} = \mathsf{R} \cup \{\infty\}$ is the invariant circle. Let $\varphi\colon\ G \to H$ be an isomorphism onto another Fuchsian group and suppose that there is homeomorphism f of $\bar{\mathsf{R}}$ such that $fgf^{-1} = \varphi(g)$ on $\bar{\mathsf{R}}$. If f fixes ∞ and is increasing then $f \mid \mathsf{R}$ is quasisymmetric but f is completely singular in the measure theoretic sense unless f is a Möbius transformation. (It seems Kuusalo [K] was the first to observe this).

It is natural to conjecture that in this situation, unless f is a Möbius transformation, f is as singular as the map of our theorem. One can mention some facts to support this conjecture. If G is torsionless and the quotient is non-compact, then there is a G-invariant triangulation of the upper half-plane U with ideal hyperbolic triangles whose vertices are on $\bar{\mathsf{R}}$. We can imagine that f is obtained by gliding two triangles with a common side along the common side (and doing this simultaneously for all sides). It would seem that topologicaly the situation resembles very much the construction of our singular map f. (Cf. [T2, Sections 2 and 3 and Remark 2 p. 370]).

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