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ON THE UPPER SEMICONTINUITY INTERSECTION DEFECT

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Abstract.

Let X be a locally noetherian scheme. We prove that the complete intersection defect function is upper semi-continuous on X in the following two cases: i) X is locally immersible in a locally complete intersection scheme, ii) X is excellent.

Let (A, m, K) be a local noetherian ring. The complete intersection defect of A is defined to be the integer

$$d(A) = \dim(A) - \varepsilon_0(A) + \varepsilon_1(A)$$

where dim denotes Krull dimension, and $\varepsilon_i(A)$ is the i-th deviation of A [2].

The function $d(\ )$ has the following properties [2]:

i) $d(A) \geqq 0$, and equality holds if and only if A is a complete intersection (i.e the m-adic completion $\hat{A}$ of A is isomorphic to a quotient of a regular local ring by an ideal generated by a regular sequence),

ii) $d(\hat{A}) = d(A)$,

iii) $d(A_p) \leqq d(A)$ for every $p \in \operatorname{Spec}(A)$

iv) If $A = R/I$, where R is a regular local ring, then

$$d(A) = \mu(I) - (\dim(R) - \dim(A)),$$

where $\mu(I)$ = minimum number of generators of I.

Using the equality of iv) and previous results of A. Grothendieck, L. L. Avramov has proved in [2, Proposition 3.4] the following result:

If X is a localy noeterian scheme, locally immersible in a regular scheme, then the function $x \mapsto d(\mathcal{O}_{X,x})$ is upper semi-continuous.

It is also noted in [2, Remark 3.5] that not every X has this property.

In the present paper we obtain a generalization of Avramov's result, namely:

THEOREM. *Let X be a locally noetherian scheme. Then, the function $x \mapsto d(\mathcal{O}_{X,x})$ is upper semi-continuous in the following two cases:*

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a) *X is locally immersible in a locally complete intersection scheme*
b) *X is excellent.*

We shall use the André-Quillen homology functors $H_n(A, B,\text{-})$ [1] and some results on the openness of the complete intersection loci due to S. Greco and M. G. Marinari [3]. The ideas used here have been applied by A. Ragusa [5] to smilar problems, namely the study of the semicontinuity of André deviations $\delta_n(A) = \dim_K H_n(A, K, K)$.

LEMMA 1. *Let A be a ring, B an A-algebra and n an integer number. Assume that* $H_i(A, B, B)$ *is a flat B-module for* $0 \leqq i \leqq n$. *Then* $H_u(A, B, W) \simeq H_n(A, B, B) \otimes_B W$ *for every B-module W.*

PROOF. This follows from the spectral sequence

$$E^2_{p,q} = \mathrm{Tor}^B_p(H_q(A, B, B), W) \Rightarrow H_{p+q}(A, B, W).$$

LEMMA 2. *Let* (A, m, K) *be a local noetherian ring and let* $p \in \mathrm{Spec}(A)$ *be such that* $H_i(A, A/p, A/p)$ *is A/p-free,* $i = 1, 2$. *Then*

$$-\varepsilon_0(A) + \varepsilon_1(A) \leqq -\varepsilon_0(A_p) + \varepsilon_1(A_p) - \varepsilon_0(A/p) + \varepsilon_1(A/p).$$

PROOF. Recall that $\varepsilon_{i-1}(A) = \dim_k H_i(A, K, K)$, $i = 1, 2$ [2, Remark 1.4].
Let $k(p) = A_p/pA_p$ be the residue field of A_p. Since $H_i(A, A/p, A/p$ is A/p-free, $i = 1, 2$, we obtain from lemma 1 and [1, lemme 3.20 and corollaire 5.27]

$$\dim_K H_i(A, A/p, K) = \dim_K H_i(A, A/p, A/p) \otimes_{A/p} K =$$

$$\dim_{k(p)} H_i(A, A/p, A/p) \otimes_{A/p} k(p) = \dim_{k(p)} H_i(A_p, k(p), k(p)) = \varepsilon_{i-1}(A_p), i = 1, 2.$$

Consider the Jacobi-Zariski exact sequence [1, theoreme 5.1]associated to the homomorphisms $A \longrightarrow A/p \longrightarrow K$

$$\begin{array}{l} H_2(A, A/p, K) \longrightarrow H_2(A, K, K) \xrightarrow{\phi} H_2(A/p, K, K) \longrightarrow \\ H_1(A, A/p, K) \longrightarrow H_1(A, K, K) \longrightarrow H_1(A/p, K, K) \longrightarrow 0. \end{array}$$

Let $N = \mathrm{Ker}\phi$. We obtain

$$\dim_k N - \varepsilon_1(A) + \varepsilon_1(A/p) - \dim_K H_1(A, A/p, K) + \varepsilon_0(A) - \varepsilon_0(A/p) = 0.$$

Moreover $\dim_K N \leqq \dim_K H_2(A, A/p, K) = \varepsilon_1(A_p)$. Therefore

$$\varepsilon_1(A_p) - \varepsilon_1(A) + \varepsilon_1(A/p) - \varepsilon_0(A_p) + \varepsilon_0(A) - \varepsilon_0(A/p) \geqq 0.$$

COROLLARY 3. *Let* (A, m, K) *be a local noetherian ring and let* $p \in \mathrm{Spec}(A)$ *be such that* $\dim(A) = \dim(A_p) + \dim(A/p)$ *(this is true for all* $p \in \mathrm{Spec}(A)$ *if A is catenary). Assume that* $H_i(A, A/p, A/p)$ *is A/p-free for* $i = 1, 2$. *Then*

$$d(A) \leqq d(A_p) + d(A/p).$$

PROPOSITION 4. *Let A be a noetherian catenary ring and $p \in \operatorname{Spec}(A)$. Then there exists an open neighbourhood U of p in $\operatorname{Spec}(A)$ such that, for every $q \in U \cap V(p)$, we have*

$$d(A_q) \leqq d(A_p) + d(A_q/pA_q).$$

PROOF. Since $H_i(A, A/p, A/p)$ is A/p-finite for each i [1, proposition 4.55], there exist $f_i \notin p$ such that $H_i(A_{f_i}, A_{f_i}/p_{f_i}, A_{f_i}/p_{f_i}) \simeq H_i(A, A/p, A/p)_{f_i}$ is A_{f_i}/p_{f_i}-free [4, 22.A, lemma 1].

Now we can shrink to the open neighbourhood of p, $A_{f_1 f_2}$, which from now on is denoted by A, and then we have $H_i(A, A/p, A/p)$ is A/p-free for $i = 1, 2$; furthermore, by localization in every $q \supseteq p$ of such an open set, we can assume A to be a local noetherian ring and we have to prove

$$d(A) \leqq d(A_p) + d(A/p).$$

This follows from corollary 3.

PROOF OF THE THEOREM. We only shall prove part a) since part b) is analogous.

The assertion being local on X, one can assume that X is a closed subscheme $\operatorname{Spec}(B/I)$ of the scheme $\operatorname{Spec}(B)$, where B is a locally complete intersection ring. Let $A = B/I$. B is catenary and therefore, so is A. Let n be an integer. We have to show that the set $U_n(A) = \{p \in \operatorname{Spec}(A) \mid d(A_p) \leqq n\}$ is open in $\operatorname{Spec}(A)$. By [2, Proposition 3.8] $U_n(A)$ is stable under generalization. Hence [4, 22.B, lemma 2] we only have to show that for any $p \in U_n(A)$, $U_n(A) \cap V(p)$ contains a non-empty open subset of $V(p)$.

Take $p \in \operatorname{Spec}(A)$; by proposition 4 we can find a neighbourhood of p, U', such that for $q' \in U' \cap V(p)$, we have

$$d(A_{q'}) \leqq d(A_p) + d(A_{q'}/pA_{q'}).$$

A/p is a quotient ring of locally complete intersection ring and therefore $U_{\mathrm{CI}}(A/p) = U_0(A/p)$ is open [3, corollary 3.4]. Then there exist another neighbourhood of p, U'', such that for $q'' \in U'' \cap V(p)$, $d(A_{q''}/pA_{q''}) = 0$.

Now for any $q \in U' \cap U'' \cap V(p)$ we have

$$d(A_q) \leqq d(A_p) \leqq n.$$

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