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ADJUNCTION PROPERTIES OF POLARIZED SURFACES VIA REIDER'S METHOD (*)

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Introduction.

In [S] Sommese investigated the spannedness and the very ampleness properties of the adjoint bundle $K_X \otimes \mathscr{L}$ to a very ample line bundle $\mathscr{L}$ on a smooth complex algebraic surface. In [LP_1] the ampleness of $K_X \otimes \mathscr{L}$ was studied for surfaces polarized by an ample line bundle $\mathscr{L}$.

Recently Reider's method [R], [Be] has changed the perspective in the study of adjoint bundles allowing one to consider more general line bundles $\mathscr{L}$. By this method Sommese and Van de Ven [SV] completed the study of the very ampleness of $K_X \otimes \mathscr{L}$ when $\mathscr{L}$ is very ample, started in [S].

The purpose of this paper is to use Reider's method to obtain information on the spannedness and the very ampleness of the adjoint bundles $K_X \otimes \mathscr{L}^t$ for surfaces polarized by an ample and spanned line bundle. More generally let $\mathscr{L}_1, \ldots, \mathscr{L}_t$ be ample line bundles on a smooth surface X. We prove the following facts.

(1.1) *If $t \geqq 3$, then $K_X \otimes \mathscr{L}_1 \otimes \ldots \otimes \mathscr{L}_t$ is very ample unless $t = 3$ and either*
a) $(X, \mathscr{L}_i) = (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1))$ *for each i, or*
b) *no $\mathscr{L}_i$ is spanned and X contains an effective divisor E numerically equivalent to each $\mathscr{L}_i$ such that $E^2 = 1$ and $h^0(E) = 1, 2$.*

Many examples as in b) are discussed.

(1.6) *If $t = 2$ and $c_1(\mathscr{L}_1)^2 \geqq 2, c_1(\mathscr{L}_2)^2 \geqq 3$, then $K_X \otimes \mathscr{L}_1 \otimes \mathscr{L}_2$ is very ample unless X contains an irreducible curve E satisfying $c_1(\mathscr{L}_{i|E}) = 1$, $i = 1, 2$ and $E^2 = 0$; if $(X, \mathscr{L}_i)$ is not a scroll for one i at least, then $\mathscr{L}_1$ and $\mathscr{L}_2$ are not spanned.*

As to the spannedness we prove

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(1.4) *If* $t \geqq 2$, *then* $K_X \otimes \mathscr{L}_1 \otimes \ldots \otimes \mathscr{L}_t$ *is spanned unless* $t = 2$, $\mathscr{L}_1 \underset{n}{\sim} \mathscr{L}_2$ *and* $c_1(\mathscr{L}_1)^2 = 1$; *if* $(X, \mathscr{L}_i) \neq (\mathsf{P}^2, \mathcal{O}_{\mathsf{P}^2}(1))$ *for at least one* i, *then no* $\mathscr{L}_i$ *is spanned.*

In [F_2] Fujita showed how results on the base locus of adjoint systems can follow from a slight variation of Reider's method; (1.4) provides some more detail.

The previous results have many applications.

In section 2 we provide some results on the spannedness and the very ampleness of powers of an ample line bundle on a minimal surface of Kodaira dimension 0. They unify and generalize results in [SD], [Co], [Ra].

In section 3 by combining (1.6) with the main result of [SV] we obtain the classification of surfaces polarized by an ample and spanned line bundle of genus 2 (see also [BLP] for related results). The corresponding classification in higher dimension is obtained via Bădescu's results on ample divisors [B].

In section 4 we get some specification of our previous results on polarized surfaces [LP_1] under the further assumption that the polarizing line bundle is spanned.

As a last thing, in section 5 we deduce the ampleness of the jacobian of an ample net on a smooth surface and obtain results on the ramification divisor of branched coverings of P^2 and of $\mathsf{P}^1 \times \mathsf{P}^1$.

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0. Notation and background.

Let X be a complex projective n-fold ($n \geqq 2$) and $\mathscr{L} = \mathcal{O}_X(L)$ a line bundle on X. We shall always confuse a line bundle with the associated invertible sheaf. Let

$$\begin{aligned}
|\mathscr{L}| &= \text{the complete linear system defined by } \mathscr{L};\\
\mathscr{L}^i &= \text{the } i\text{-th tensor power of } \mathscr{L};\\
h^i(\mathscr{L}) &= \dim_{\mathbf{C}} H^i(X, L);\\
\Delta(X, \mathscr{L}) &= n + c_1(\mathscr{L})^n - h^0(\mathscr{L});\\
g(\mathscr{L}) &= 1 + 1/2\,(c_1(K_X \otimes \mathscr{L}) \cdot c_1(\mathscr{L})^{n-1}),
\end{aligned}$$

where K_X is the canonical bundle of X.

A polarized pair is a pair $(X, \mathscr{L})$ consisting of a projective n-fold X and an ample line bundle $\mathscr{L}$ on it. $(Q^n, \mathcal{O}_{Q^n}(1))$ will stand for the smooth hyperquadric $Q^n \subset \mathsf{P}^{n+1}$ polarized by its hyperplane bundle. We recall the standard names of some classes of polarized pairs which will frequently occur in what follows. $(X, \mathscr{L})$ is a *scroll* if X is a P^{n-1}-bundle over a smooth curve an $\mathscr{L}_{|f} = \mathcal{O}_{\mathsf{P}^{n-1}}(1)$ for every fibre f of X. Note that $(Q^2, \mathcal{O}_{Q^2}(1))$ is a scroll in two different ways. $(X, \mathscr{L})$ is a *quadric bundle* (*conic bundle* when $n = 2$) if there is a morphism $p\colon X \to C$ over a smooth curve C, whose general fibre F satisfies $(F, \mathscr{L}_{|F}) = (Q^{n-1}, \mathcal{O}_{Q^{n-1}}(1))$. $(X, \mathscr{L})$ is a *Del Pezzo pair* if $K_X \otimes \mathscr{L}^{n-1}$ is trivial.

We say that a line bundle $\mathscr{L}$ is *spanned* to mean that it is spanned by its global sections.

Let X be a surface, i.e. $n = 2$, and C, $D \in \mathrm{Div}(X)$; CD will stand for the intersection index of C and D and C^2 for CC.

Assume that the surface X is a P^1-bundle over a smooth curve. The *invariant* e of X is the opposite of the minimum of the self-intersection indexes of the sections of X. A fundamental section of X is a section whose self-intersection index is $-e$. We shall always denote by ξ and f a fundamental section and a fibre of X respectively. The rational P^1-bundle of invariant e, $\mathsf{P}(\mathcal{O}_{\mathsf{P}^1} \oplus \mathcal{O}_{\mathsf{P}^1}(-e))$, will be denoted by F_e.

(0.1) Let X be a surface and $\mathscr{L}$ an ample line bundle on X. A smooth rational curve $E \subset X$ is said a (-1)-line (relative to $\mathscr{L}$) if $E^2 = -1$ and $EL = 1$. We recall that the number of the (-1)-lines is finite and that they are disjoint unless $(X, \mathscr{L})$ is a conic bundle. Apart from this case there exists a birational morphism r: $X \to X'$ onto a surface X' contracting all the (-1)-lines of X to a finite set $F \subset X'$.

Let $\mathscr{L}' = r_*\mathscr{L} = \mathcal{O}_{X'}(L)$; $\mathscr{L}'$ is ample and the pair $(X', \mathscr{L}')$ is referred to as the *reduction* of $(X, \mathscr{L})$. We recall that

$$(0.1.1) \qquad L'^2 \geqq L^2.$$

The main tool we use in this paper is Reider's method, which we recall in the following form.

(0.2) THEOREM. ([R], see also [SV]). *Let D be a numerically effective divisor on a surface X.*

(0.2.1) *If $D^2 \geqq 5$, then $K_X \otimes \mathcal{O}_X(D)$ is spanned unless X contains an effective divisor E satisfying either*

$$DE = 0,\ E^2 = -1 \text{ or } DE = 1,\ E^2 = 0.$$

(0.2.2) *If $D^2 \geqq 9$, then $K_X \otimes \mathcal{O}_X(D)$ is very ample unless X contains an effective divisor E satisfying either*

(i) $DE = 0$, $E^2 = -1$ *or* -2,
(ii) $DE = 1$, $E^2 = -1$ *or* 0
(iii) $DE = 2$, $E^2 = 0$, *or*
(iv) $D \underset{n}{\sim} 3E$ (numerically equivalent) *and* $E^2 = 1$

(0.3) LEMMA. *Let X be a surface polarized by an ample and spanned line bundle $\mathscr{L} = \mathcal{O}_X(L)$. If X contains an effective divisor E satisfying $EL = 1$, $E^2 = 0$, then $(X, \mathscr{L})$ is a scroll and E is a fibre.*

PROOF. Let φ: $X \to \mathsf{P}^N$ be the morphism associated with $|\mathscr{L}|$. From the equality

$$1 = LE = \deg \varphi_{|E} \deg \varphi(E)$$

we see that $E \simeq \mathsf{P}^1$ and then X is ruled by the Noether-Enriques theorem. The condition $LE = 1$ also says that $(X, \mathscr{L})$ is a scroll.

1. Very ampleness of certain adjoint bundles.

In this section X will be a surface and $\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_t$ ample line bundles on X. As usual we shall put $\mathscr{L}_i = \mathcal{O}_X(L_i)$. We investigate the very ampleness and the spannedness of the line bundle $K_X \otimes \mathscr{L}_1 \otimes \ldots \otimes \mathscr{L}_t$.

(1.1) PROPOSITION. *Let $t \geqq 3$. If $K_X \otimes \mathscr{L}_1 \otimes \ldots \otimes \mathscr{L}_t$ is not very ample, then $t = 3$ and either*

(1.1.1) $(X, \mathscr{L}_i) = (\mathsf{P}^2, \mathcal{O}_{\mathsf{P}^2}(1))$, $i = 1, 2, 3$, *or*

(1.1.2) *X contains an effective ample divisor E with $E^2 = 1$, $\Delta(X, [E]) = 1, 2$ and $L_i \underset{n}{\sim} [E]$, $i = 1, 2, 3$.*

If furthermore $\mathscr{L}_1$ is spanned, then $K_X \otimes \mathscr{L}_1 \otimes \ldots \otimes \mathscr{L}_t$ is very ample unless $t = 3$ and (1.1.1) *holds.*

PROOF. As $c_1(\mathscr{L}_1 \otimes \ldots \otimes \mathscr{L}_t)^2 \geqq 9$, if $K_X \otimes \mathscr{L}_1 \otimes \ldots \otimes \mathscr{L}_t$ is not very ample, (0.2.2) with $[D] = \mathscr{L}_1 \otimes \ldots \otimes \mathscr{L}_t$ shows that X contains an effective divisor E such that $3 \geqq DE \geqq t$. Therefore, $t = 3$ and E has to be as in (0.2.2) (iv). Moreover, $L_i E = 1$ and so $(L_i - E)E = 0$. Since $(L_i - E)^2 \geqq 0$, the Hodge index theorem implies that $L_i \underset{n}{\sim} [E]$. In particular $[E]$ is ample. As $0 \leqq \Delta(X, [E]) \leqq 2$, if we are not in case (1.1.2), then $\Delta(X, [E]) = 0$, which, combined with $E^2 = 1$, gives $(X, [E]) = (\mathsf{P}^2, \mathcal{O}_{\mathsf{P}^2}(1))$ [F$_1$]. Since $L_i \underset{n}{\sim} [E]$, this shows that we are in case (1.1.1). The last assertion is immediate once we consider that $h^0(\mathscr{L}_1) \geqq 3$ by the spannedness and so $\Delta(X, \mathscr{L}_1) = 0$.

Here is a list of pairs $(X, \mathscr{L}_i)$ as in (1.1.2); for all of them $K_X \otimes \mathscr{L}_1 \otimes \mathscr{L}_2 \otimes \mathscr{L}_3$ is not very ample.

(1.2) EXAMPLES. 1. Let X be a Del Pezzo surface of degree 1, i.e. $X = B_{p_1 \ldots p_8}(\mathsf{P}^2)$, the blow-up of P^2 at 8 points in general position and let $\mathscr{L}_i = K_X^{-1}$, $i = 1, 2, 3$. If $E \in |K_X^{-1}|$, then $\Delta(X, [E]) = 1$. $K_X \otimes \mathscr{L}_1 \otimes \mathscr{L}_2 \otimes \mathscr{L}_3 = K_X^{-2}$ is not very ample: the associated map expresses X as a double cover of the quadric cone Q branched at the vertex and along the transverse intersection of Q with a cubic surface.

2. Let X be the P^1-bundle of invariant $e = -1$ over a smooth elliptic curve and let $\mathscr{L}_i \underset{n}{\sim} [\xi]$, $i = 1, 2, 3$. Take $E = \xi$; then $\Delta(X, [E]) = 2$. $K_X \otimes \mathscr{L}_1 \otimes \mathscr{L}_2 \otimes \mathscr{L}_3 \underset{n}{\sim} [\xi + f]$ is not very ample: actually, its restriction to the elliptic curve ξ has degree 2.

3. Let $X = C^{(2)}$ be the symmetric product of a smooth curve C of genus 2, let π: $C \times C \to X$ be the obvious projection and let $D = \pi(C)$; D is a smooth curve of

genus two and $\pi^*D = (C \times \{a\}) + (\{a\} \times C)$. Hence $\mathscr{L}_i = [D], i = 1, 2, 3$ is ample. Letting $E = D$, we get $\Delta(X, [E]) = 2$. Moreover, $K_X \otimes \mathscr{L}_1 \otimes \mathscr{L}_2 \otimes \mathscr{L}_3$ is not very ample; actually, its restriction to D has degree $4D^2 = 4$ and so it cannot be very ample.

4. Let X be a Kynev surface, i.e. a minimal surface of general type with $c_1^2(X) = p_g(X) = 1$, $q(X) = 0$. The canonical system of X consists of a single smooth curve K of genus 2. If, in addition, X contains no (-2)-curves, then K_X is ample. Take $\mathscr{L}_i = K_X$, $i = 1, 2, 3$ and $E = K$. Then, $\Delta(X, [E]) = 2$ and $K_X \otimes \mathscr{L}_1 \otimes \mathscr{L}_2 \otimes \mathscr{L}_3 = K_X^4$ is not very ample: actually, its restriction to K cannot be very ample, as its degree is

$$\deg K^4_{X|K} = 4\,c_1^2(X) = 4.$$

5. Let X be a minimal surface of general type with $c_1^2(X) = 1$, $p_g(X) = 2$, $q(X) = 0$. If X does not contain any (-2)-curves, then K_X is ample. Let $\mathscr{L}_i = K_X$ and $E \in |K_X|$. Then $\Delta(X, [E]) = 1$ and $K_X \otimes \mathscr{L}_1 \otimes \mathscr{L}_2 \otimes \mathscr{L}_3 = K_X^4$ is not very ample. Actually, its restriction to a general element of $|K_X|$, which is a smooth curve of genus two, cannot be very ample, having degree $4c_1^2(X) = 4$.

(1.3) Many results are known on polarized surfaces $(X, \mathscr{L})$ with $\Delta(X, \mathscr{L}) = 1$ or 2, but a complete classification in case $c_1(\mathscr{L})^2 = 1$ is not yet available. This prevents us from getting a better statement in (1.1) without any further assumption on the $\mathscr{L}_i$'s. However, we note the following fact.

(1.3.1) PROPOSITION. *If $\mathscr{L}_1 \otimes \mathscr{L}_2 \otimes \mathscr{L}_3$ is very ample, then $K_X \otimes \mathscr{L}_1 \otimes \mathscr{L}_2 \otimes \mathscr{L}_3$ is very ample unless $(X, \mathscr{L}_i)$ is either as in* (1.1.1) *or as in the examples* (1.2.1), (1.2.2).

PROOF. Actually, for every curve $C \subset X$

$$\deg(\mathscr{L}_1 \otimes \mathscr{L}_2 \otimes \mathscr{L}_3)_{|C} \geqq 3.$$

Therefore the pair $(X, \mathscr{L}_1 \otimes \mathscr{L}_2 \otimes \mathscr{L}_3)$ can be neither $(\mathbf{P}^2, \mathscr{O}_{\mathbf{P}^2}(e))$, $e = 1, 2$, nor a scroll, nor a conic bundle and it is a Del Pezzo pair only in case (1.1.1). The assertion follows from the main result of [SV] since $(X, \mathscr{L}_1 \otimes \mathscr{L}_2 \otimes \mathscr{L}_3)$ cannot admit nontrivial reductions.

In the examples (1.2.3) (1.2.4) (1.2.5) $\mathscr{L}_1 \otimes \mathscr{L}_2 \otimes \mathscr{L}_3$ is not very ample, but it is spanned. As to (1.2.4) (1.2.5) this is due to the properties of the pluricanonical bundles (see [C, Th. 1] and [Ho, pp. 128–129] respectively), while it follows from a computation in case (1.2.3). We ask the following

(1.3.2) QUESTION. *Assume that $\mathscr{L}_1 \otimes \mathscr{L}_2 \otimes \mathscr{L}_3$ is spanned. How many exception to the very ampleness of $K_X \otimes \mathscr{L}_1 \otimes \mathscr{L}_2 \otimes \mathscr{L}_3$ can be found in addition to* (1.1.1) *and* (1.2)?

(1.4) PROPOSITION. *Let* $t \geqq 2$. *If* $K_X \otimes \mathscr{L}_1 \otimes \ldots \otimes \mathscr{L}_t$ *is not spanned, then* $t = 2$ *and* $\mathscr{L}_1 \underset{\bar{n}}{} \mathscr{L}_2$, $c_1(\mathscr{L}_1)^2 = 1$. *If furthermore* $\mathscr{L}_1$ *is spanned, then* $K_X \otimes \mathscr{L}_1 \otimes \ldots \otimes \mathscr{L}_t$ *is spanned unless* $t = 2$ *and* $(X, \mathscr{L}_i) = (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1))$, $i = 1, 2$.

PROOF. If $t = 2$ and $L_1^2 = L_2^2 = L_1 L_2 = 1$, the Hodge index theorem immediately shows that $\mathscr{L}_1 \underset{\bar{n}}{} \mathscr{L}_2$. In all the remaining cases, by the obvious inequality

$$c_1(\mathscr{L}_1 \otimes \ldots \otimes \mathscr{L}_t)^2 = \sum_{i,j} L_i L_j \geqq 5,$$

(0.2.1) applies with $[D] = \mathscr{L}_1 \otimes \ldots \otimes \mathscr{L}_t$. If $K_X \otimes \mathscr{L}_1 \otimes \ldots \otimes \mathscr{L}_t$ is not spanned, there exists an effective divisor E which has to satisfy

$$1 \geqq DE \geqq t.$$

This gives a contradiction. The last assertion is immediate since the spannedness of $\mathscr{L}_1$ implies that $h^0(\mathscr{L}_1) \geqq 3$ and so $\Delta(X, \mathscr{L}_1) = 0$.

(1.5) REMARK. If in (1.4) we simply assume $\mathscr{L}_1 \otimes \mathscr{L}_2$, instead of $\mathscr{L}_1$, to be spanned, then we get more exceptions to the spannedness of $K_X \otimes \mathscr{L}_1 \otimes \mathscr{L}_2$, e.g. (1.2.1) and (1.2.2). On the contrary in (1.2.4), (1.2.5) $K_X \otimes \mathscr{L}_1 \otimes \mathscr{L}_2 = K_X^3$ is spanned and gives a birational morphism [C], [Ho].

(1.6) PROPOSITION. *Let* $t = 2$ *and assume* $c_1(\mathscr{L}_1)^2 \geqq 2$, $c_1(\mathscr{L}_2)^2 \geqq 3$. *If* $K_X \otimes \mathscr{L}_1 \otimes \mathscr{L}_2$ *is not very ample, then* X *contains an irreducible curve* E *satisfying*

(1.6.1) $$EL_1 = EL_2 = 1, \ E^2 = 0.$$

If furthermore $\mathscr{L}_1$ *is spanned, then* $K_X \otimes \mathscr{L}_1 \otimes \mathscr{L}_2$ *is very ample unless* $(X, \mathscr{L}_i)$ *is a scroll for* $i = 1, 2$.

PROOF. Due to the assumption, by the Hodge index theorem we get

(1.6.2) $$c_1(\mathscr{L}_1 \otimes \mathscr{L}_2)^2 \geqq 2 + 2\sqrt{6} + 3 > 9.$$

So, if $K_X \otimes \mathscr{L}_1 \otimes \mathscr{L}_2$ is not very ample, (0.2.2) with $[D] = \mathscr{L}_1 \otimes \mathscr{L}_2$ shows that X contains an effective divisor E such that $DE < 3$ (strict inequality due to (1.6.2)). Then, in view of the ampleness of $\mathscr{L}_i$, E can only be as in (0.2.2, iii) and so (1.6.1) holds. In particular E is an irreducible curve. As to the last assertion, if $K_X \otimes \mathscr{L}_1 \otimes \mathscr{L}_2$ is not very ample, it follows from (1.6.1) and (0.3) that $(X, \mathscr{L}_1)$ is a scroll; but then $(X, \mathscr{L}_2)$ is a scroll as well.

(1.7) REMARK. Note that, if $c_1(\mathscr{L}_1)^2 = 2$ and $\mathscr{L}_1$ is spanned, then $\pi: X \to \mathbf{P}^2$ is a double cover, $\mathscr{L}_1 = \pi^* \mathcal{O}_{\mathbf{P}^2}(1)$. In this case $K_X \otimes \mathscr{L}_1 \otimes \mathscr{L}_2$ is very ample by (1.6) unless $c_1(\mathscr{L}_2)^2 \leqq 2$.

From (1.6) and (1.7) we immediately get the following

(1.8) COROLLARY. *Let X be a surface polarized by an ample and spanned line bundle $\mathscr{L}$. Then $K_X \otimes \mathscr{L}^2$ is very ample unless either*

(1.8.1) $(X, \mathscr{L}) = (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1))$,
(1.8.2) $\pi\colon X \to \mathbf{P}^2$ *is a double cover and* $\mathscr{L} = \pi^* \mathcal{O}_{\mathbf{P}^2}(1)$, *or*
(1.8.3) $(X, \mathscr{L})$ *is a scroll.*

2. Surfaces with numerically trivial canonical bundle.

In this section we give an application of the results of section 1 to minimal surfaces of Kodaira dimension zero.

(2.1) PROPOSITION. *Let X be a surface with numerically trivial canonical bundle and let $\mathscr{L}_1, \ldots, \mathscr{L}_t$ be ample line bundles on X. Then $\mathscr{L}_1 \otimes \ldots \otimes \mathscr{L}_t$ is*

(2.1.1) *very ample for $t \geqq 3$;*
(2.1.2) *spanned for $t = 2$, and very ample if in addition $\mathscr{L}_1$ is spanned and $c_1(\mathscr{L}_2)^2 > 2$.*

PROOF. To prove (2.1.1) we can assume $t = 3$. We have $\mathscr{L}_1 \otimes \mathscr{L}_2 \otimes \mathscr{L}_3 = K_X \otimes \mathscr{M}$, where $\mathscr{M} = \mathscr{L}_1 \otimes \mathscr{L}_2 \otimes (\mathscr{L}_3 \otimes K_X^{-1})$. Then $\mathscr{L}_1 \otimes \mathscr{L}_2 \otimes \mathscr{L}_3$ is very ample by (1.1) unless X contains an effective divisor E with $E^2 = 1$. However, this cannot occur since, being $K_X \underset{n}{\sim} 0$, the genus formula shows that E^2 is even.

To prove (2.1.2) we can assume $t = 2$. We have $\mathscr{L}_1 \otimes \mathscr{L}_2 = K_X \otimes \mathscr{N}$ with $\mathscr{N} = \mathscr{L}_1 \otimes (\mathscr{L}_2 \otimes K_X^{-1})$. Then $\mathscr{L}_1 \otimes \mathscr{L}_2$ is spanned by (1.4) since, being $K_X \underset{n}{\sim} 0$, it cannot happen that $c_1(\mathscr{L}_i)^2 = 1$. The last assertion in (2.1.2) follows from (1.6).

By taking $\mathscr{L}_i = \mathscr{L}$ for $i = 1, \ldots, t$, the above proposition provides a meaningful generalization of results proved by St. Donat [SD, Th. 8.3] for K3 surfaces and by Cossec [Co, Cor. 8.3.2] for Enriques surfaces. See also [Ra] for the case of abelian surfaces.

The very ampleness result in (2.1.2) can be further specified.

(2.2) PROPOSITION. *Let X be a surface with numerically trivial canonical bundle and let $\mathscr{L}_1$ be an ample and spanned line bundle and $\mathscr{L}_2$ an ample line bundle on X. Then $\mathscr{L}_1 \otimes \mathscr{L}_2$ is very ample unless $\pi\colon X \to \mathbf{P}^2$ is the* K3 *double cover branched along a smooth sextic and $\mathscr{L}_1 = \mathscr{L}_2 = \pi^* \mathcal{O}_{\mathbf{P}^2}(1)$.*

PROOF. We have $\mathscr{L}_1 \otimes \mathscr{L}_2 = K_X \otimes \mathscr{L}_1 \otimes \mathscr{N}_2$ where $\mathscr{N}_2 = \mathscr{L}_2 \otimes K_X^{-1}$. Since $K_X \underset{n}{\sim} 0$ we know that

$$c_1(\mathscr{L}_1)^2 \geqq 2, \quad c_1(\mathscr{N}_2) = c_1(\mathscr{L}_2)^2 \geqq 2$$

and by the Hodge index theorem

$$(*) \qquad L_1 N_2 \geqq \sqrt{L_1^2}\sqrt{N_2^2} \geqq 2.$$

In view of (2.1.2) we can assume $c_1(\mathcal{N}_2)^2 = 2$. Thus, since

$$c_1(\mathcal{L}_1 \otimes \mathcal{N}_2)^2 \geqq c_1(\mathcal{L}_1)^2 + 2 + 2L_1N_2,$$

by using (*) we see that

$$c_1(\mathcal{L}_1 \otimes \mathcal{N}_2)^2 > 9$$

if either $c_1(\mathcal{L}_1)^2 \geqq 3$ or $L_1N_2 \geqq 3$. In both cases (0.2) shows that if $\mathcal{L}_1 \otimes \mathcal{L}_2$ is not very ample, then X contains an effective divisor E such that

$$L_1E = 1, \quad E^2 = 0.$$

But this would mean that $(X, \mathcal{L}_1)$ is a scroll by (0.3), a contradiction. Therefore it only remains to consider the following case:

$$c_1(\mathcal{L}_1)^2 = c_1(\mathcal{N}_2)^2 = L_1N_2 = 2.$$

In this case if $\mathcal{L}_1 \otimes \mathcal{L}_2$ is not very ample, then $\pi\colon X \to \mathbf{P}^2$ is a double cover and $\mathcal{L}_1 = \pi^* \mathcal{O}_{\mathbf{P}^2}(1)$ by (1.7). Since $K_X \underset{n}{\sim} 0$, it follows that π is branched along a smooth sextic and X is a K3 surface. Finally as $(L_1 - L_2)L_1 = 0$ and $c_1(\mathcal{L}_1 \otimes \mathcal{L}_2^{-1})^2 = 0$, the Hodge index theorem implies that $\mathcal{L}_1 \underset{n}{\sim} \mathcal{L}_2$ and then we conclude that $\mathcal{L}_1 = \mathcal{L}_2$ since X is regular.

3. Sectional genus 2.

Reider's method combined with some recent results by Sommese and Van de Ven [SV] can also be used to classify projective manifolds polarized by an ample and spanned line bundle of genus 2. As to surfaces this supplies a generalization of a classical result by Castelnuovo (e.g. [I, Prop. 3.1]) and at the same time provides some specification to a more general result in [BLP].

(3.1) THEOREM. *Let X be a surface polarized by an ample and spanned line bundle $\mathcal{L}$ satisfying $g(\mathcal{L}) = 2$. Then either*

(3.1.1) *$(X, \mathcal{L})$ is a scroll over a smooth curve of genus 2,*

(3.1.2) *X is a $\mathbf{P}^1$-bundle over an elliptic curve, with invariant $e = -1$ and $L \underset{n}{\sim} 2\xi$, where ξ is a fundamental section,*

(3.1.3) *X is an $\mathbf{F}_e$ $(e \leqq 2)$ blown-up at $s \leqq 9$ points $p_1, \ldots, p_s$ on distinct fibres and $L = \sigma^* L_0 - E_1 - \ldots - E_s$, where $\sigma\colon X \to \mathbf{F}_e$ is the blowing-up, $E_i = \sigma^{-1}(p_i)$ and $L_0 \underset{n}{\sim} 2\xi + (e+3)f$,*

(3.1.4) *$\pi\colon X \to \mathbf{P}^2$ is the* K3 *double cover branched along a smooth sextic and $\mathcal{L} = \pi^* \mathcal{O}_{\mathbf{P}^2}(1)$, or*

(3.1.5) *$\pi\colon X \to Q \subset \mathbf{P}^3$ is a double cover of a quadric cone Q branched at the vertex and along the transverse intersection of Q with a cubic surface and $\mathcal{L} = \pi^* \mathcal{O}_Q(1)$.*

PROOF. Let $C \in |\mathscr{L}|$ be a general element. Since C has genus 2 the 2-canonical map of C is not an embedding, hence by adjunction, $K_X^2 \otimes \mathscr{L}^2$ cannot be very ample not even birationally. Note that $K_X^2 \otimes \mathscr{L}^2 = K_X \otimes \mathscr{M}$, where $\mathscr{M} = K_X \otimes \mathscr{L}^2$ and $\mathscr{M}$ is very ample unless $(X, \mathscr{L})$ is as in the exceptions of (1.8). But $g(\mathscr{L}) = 2$ can only happen in cases (1.8.2), (1.8.3), which respectively lead to (3.1.4) and (3.1.1). So apart from these cases we can assume that $\mathscr{M}$ is very ample. Now assume that $(X, \mathscr{M})$ is none of the following pairs

(3.1.6) $(\mathbf{P}^2, \mathscr{O}_{\mathbf{P}^2}(e))$, $(e = 1, 2)$, a scroll, a Del Pezzo pair, a conic bundle.

Then, by [SV] $K_X \otimes \mathscr{M}$ is very ample (out of the (-1)-lines of $(X, \mathscr{M})$) unless $(X, \mathscr{M})$ is one of the following pairs:

a) X is $\mathbf{P}^2$ blown up at seven points in general position and $\mathscr{M} = K_X^{-2}$;
b) X is the blowing-up at one point p of a surface $\hat{X}$ as in a) and $\mathscr{M} = \sigma^* K_{\hat{X}}^{-2} \otimes \mathscr{O}_X(-\sigma^{-1}(p))$, where $\sigma \colon X \to \hat{X}$ is the blow-up;
c) X is $\mathbf{P}^2$ blown-up at eight points in general position and $\mathscr{M} = K_X^{-3}$;
d) X is the $\mathbf{P}^1$-bundle of invariant $e = -1$ over an elliptic curve and $\mathscr{M}$ is numerically equivalent to $\mathscr{O}_X(3\xi)$.

Cases a), b) and d) lead to numerical contradictions. Actually, in case a) we get $\mathscr{L}^2 = \mathscr{M} \otimes K_X^{-1} = K_X^{-3}$, hence

$$4c_1(\mathscr{L})^2 = 9c_1^2(X) = 18,$$

absurd. In case b) let $E = \sigma^{-1}(p)$; then $\mathscr{L}^2 = \mathscr{M} \otimes K_X^{-1} = \sigma^* K_{\hat{X}}^{-3} \otimes [E]^{-2}$, hence

$$4c_1(\mathscr{L})^2 = 9c_1^2(\hat{X}) - 4 = 14,$$

absurd. In case d) we have $\mathscr{L}^2 = \mathscr{M} \otimes K_X^{-1} \,\tilde{_n}\, \mathscr{O}_X(5\xi - f)$ and the degree of $\mathscr{L}^2_{|f}$ would be odd, absurd. Note that all these numerical contradictions are independent of the assumption $g(\mathscr{L}) = 2$. In case c) we get $\mathscr{L} = K_X^{-2}$and this gives (3.1.5). So it only remains to consider what happens when $(X, \mathscr{M})$ is as in (3.1.6). Since $g(\mathscr{L}) = 2$ it cannot be $X = \mathbf{P}^2$. Were $(X, \mathscr{M})$ a Del Pezzo pair, then $(X, \mathscr{L})$ would be a Del Pezzo pair too and then $g(\mathscr{L}) = 1$, contradiction. Were $(X, \mathscr{M})$ a scroll, by restricting $\mathscr{M}$ to a fibre we would get a numerical contradiction. Finally, if $(X, \mathscr{M})$ is a conic bundle, by restricting to a fibre, we see that $(X, \mathscr{L})$ is a conic bundle too. By restricting the ruling projection to C, the Riemann-Hurwitz theorem shows that $h^{1,0}(X) \leqq 1$; so either X is rational or ruled over an elliptic curve. Since $X \neq \mathbf{P}^2$ there exists a birational morphism η: $X \to X_0$ onto a $\mathbf{P}^1$-bundle X_0. Let s be the number of the blowing-ups η factors through. Then $c_1^2(X) = 8(1 - h^{1,0}(X)) - s$. Since $(X, \mathscr{L})$ is a conic bundle we get

$$0 = c_1(K_X \otimes \mathscr{L})^2 = 8(1 - h^{1,0}(X)) - s + 4 - L^2,$$

and then $L^2 + s = 12$ or 4 according to whether X is rational or not. Note that $c_1(\mathscr{L})^2 \geqq 3$; actually, since it is a conic bundle with $g(\mathscr{L}) = 2$, $(X, \mathscr{L})$ can be neither $(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1))$, $(Q^2, \mathcal{O}_{Q^2}(1))$, nor a double cover of $\mathbf{P}^2$. Therefore $s \leqq 9$ or $s \leqq 1$ according to whether X is rational or not. Since $(X_0, \mathcal{L}_0 = \eta^* \mathcal{L})$ is again a conic bundle, then $\mathcal{L}_0 \underset{n}{\approx} \mathcal{O}_{X_0}(2\xi + bf)$, where the integer b satisfies the ampleness conditions for $\mathcal{L}_0$ [H, p. 382]. In the rational case the proof now continues as in [I, Prop. 3.1] and we get case (3.1.3). In the irrational case, let e be the invariant of the elliptic $\mathbf{P}^1$-bundle X_0. The genus formula allows one to compute b and then the ampleness conditions for $\mathcal{L}_0$ show that either

$$e = 0 \quad \text{and} \quad \mathcal{L}_0 \underset{n}{\approx} \mathcal{O}_{X_0}(2\xi + f), \text{ or}$$
$$e = -1 \quad \text{and} \quad \mathcal{L}_0 \underset{n}{\approx} \mathcal{O}_{X_0}(2\xi).$$

Since $\mathcal{L}_0 = K_{X_0} \otimes \mathcal{N}$, with $\mathcal{N}$ ample, we have $h^1(\mathcal{L}_0) = h^2(\mathcal{L}_0) = 0$; so in both cases the Riemann-Roch theorem gives $h^0(\mathcal{L}_0) = 3$ and therefore if $s > 0$, the spannedness of $\mathcal{L}$ gives

$$3 \leqq h^0(\mathcal{L}) < h^0(\mathcal{L}_0) = 3,$$

contradiction. This shows that $(X, \mathcal{L}) = (X_0, \mathcal{L}_0)$. In case $e = 0$, $\mathcal{L} = \mathcal{L}_0$ cannot be spanned since $\mathcal{L}_{0|\xi}$ is not spanned, having degree 1. Therefore $(X, \mathcal{L})$ is as in (3.1.2).

Known results on ample divisors [B, I] allow us to extend the above theorem to higher dimensions.

(3.2) COROLLARY. *Let X be a projective n-fold, $n \geqq 3$, polarized by an ample and spanned line bundle $\mathcal{L}$ with $g(\mathcal{L}) = 2$. Then either*

(3.2.1) *$(X, \mathcal{L})$ is a scroll over a smooth curve of genus* 2,

(3.2.2) *$(X, \mathcal{L})$ is a quadric bundle over $\mathbf{P}^1$, or*

(3.2.3) *$\pi: X \to \mathbf{P}^n$ is a double cover branched along a smooth sextic hypersurface and $\mathcal{L} = \pi^* \mathcal{O}_{\mathbf{P}^n}(1)$.*

PROOF. A general element $X_{n-1} \in |\mathcal{L}|$ is a smooth $(n-1)$-fold and $\mathcal{L}_{n-1} = \mathcal{L}_{|X_{n-1}}$ is ample and spanned. Iterate this procedure until you obtain a smooth 3-fold X_3 and an ample and spanned line bundle $\mathcal{L}_3 = \mathcal{L}_{|X_3}$. Then the pair $(X_2, \mathcal{L}_2)$ produced by a further step is as in Theorem (3.1). Note that $(X_2, \mathcal{L}_2)$ cannot be as in (3.1.2) [B]. Moreover if $(X_2, \mathcal{L}_2)$ is as in (3.1.1), then $(X, \mathcal{L})$ is as in (3.2.1) [B]. Similarly, cases (3.1.3), (3.1.4) ascend to cases (3.2.2), (3.2.3) respectively (see [I, Prop. 1.11]). On the other hand case (3.1.5) does not ascend. In fact, if $\mathcal{N}$ is an ample and spanned line bundle on a 3-fold there is no smooth $S \in |\mathcal{N}|$ with $(S, \mathcal{N}_{|S})$ as in (3.1.5). To see this adapt the argument in [SV, Th. 1.7, first paragraph of the proof].

4. Further results about adjunction.

Reider's theorem and the argument used in section 2 show that if X is a surface, $\mathscr{L}$ an ample and spanned line bundle and $c_1(\mathscr{L})^2 \geqq 5$, then $K_X \otimes \mathscr{L}$ is spanned unless $(X, \mathscr{L})$ is a scroll.

As is known, if $\mathscr{L}$ is ample, to get the ampleness of the adjoint bundle a reduction is needed [LP, Thm. 2.5]. We can provide a specification of the quoted result in the case of spanned ample line bundles.

(4.1) PROPOSITION. *Let X be a surface and let $\mathscr{L}$ be an ample and spanned line bundle such that $c_1(\mathscr{L})^2 \geqq 5$. If $(X, \mathscr{L})$ is neither a scroll, a conic bundle, nor a Del Pezzo pair, then $(X, \mathscr{L})$ admits a reduction $(X', \mathscr{L}')$ such that $K_{X'} \otimes \mathscr{L}'$ is ample and spanned.*

PROOF. Let $(X', \mathscr{L}')$ be the reduction of $(X, \mathscr{L})$ where $K_{X'} \otimes \mathscr{L}'$ is ample [LP, Th. 2.5] and let $r\colon X \to X'$ be the reduction morphism contracting all the (-1)-lines of X to points $p_1, \ldots, p_s$ of X'. By contradiction, assume that $K_{X'} \otimes \mathscr{L}'$ is not spanned. In view of (0.1.1) we have $L'^2 \geqq 5$ and so we can apply Theorem (0.2) with $\mathcal{O}_X(D) = \mathscr{L}'$ and conclude that X' contains an effective divisor E, which, due to the ampleness of $\mathscr{L}'$, has to satisfy

(4.1.1) $$L'E = 1, \quad E^2 = 0.$$

Let $\tilde{E} = r^{-1}(E)$ be the proper transform of E on X and let $m_i \geqq 0$ be the multiplicity of E at p_i, $i = 1, \ldots, s$. We have

$$0 < L\tilde{E} = L'E - \sum m_i = 1 - \sum m_i.$$

Hence $m_i = 0$ for $i = 1, \ldots, s$. Then we get from (4.1.1)

$$L\tilde{E} = 1, \quad \tilde{E}^2 = 0,$$

which gives a contradiction in view of (0.3).

The following Corollary is another immediate consequence of [LP, Th. 2.5] and (1.4).

(4.2) COROLLARY. *Let X be a surface polarized by an ample and spanned line bundle $\mathscr{L}$. Then $K_X \otimes \mathscr{L}^2$ is ample and spanned unless $(X, \mathscr{L})$ is either $(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1))$ or a scroll.*

The very ampleness of $K_X \otimes \mathscr{L}$ up to a reduction can be studied under the assumption that $c_1(\mathscr{L})^2 \geqq 9$ following the outline of [SV], but, since $\mathscr{L}$ is not very ample, this leads to a large number of exceptions [P]. However, the method we used to prove Theorem (3.1) provides information on the exceptions to the very ampleness of $(K_X \otimes \mathscr{L})^2$ up to a reduction even with no assumption on $c_1(\mathscr{L})^2$.

(4.3) THEOREM. *Let X be a surface polarized by an ample and spanned line bundle $\mathscr{L}$.* Assume that *$(X, \mathscr{L})$ is neither* $(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(e))$, $e = 1, 2$, *a scroll, a conic bundle, a Del Pezzo pair, nor as in* (3.1.4), (3.1.5). *Then $(X, \mathscr{L})$ admits a reduction $(X', \mathscr{L}')$ on which $(K_{X'} \otimes \mathscr{L}')^2$ is very ample.*

PROOF. The proof goes along the same lines as that of Theorem (3.1). Actually, if $(X, \mathscr{L})$ is not as in the exceptions of Corollary (1.8), then $\mathscr{M} = K_X \otimes \mathscr{L}^2$ is very ample. Note however that if $(X, \mathscr{L})$ is as in (1.8.2) and the branch locus of π: $X \to \mathbf{P}^2$ has degree $\delta > 6$, then K_X is ample, so that $\mathscr{M}$ is the tensor power of three ample line bundles. Then, by using (0.2.1) again, one can see that $(K_X \otimes \mathscr{L})^2$ is very ample. On the other hand, if $\delta = 6$, then $(X, \mathscr{L})$ is as in (3.1.4), while if $\delta \leqq 4$, $(X, \mathscr{L})$ fits into the remaining exceptions.

So we can assume that $\mathscr{M}$ is very ample. Assume that $(X, \mathscr{M})$ is not as in (3.1.6). Then, by [SV] $(X, \mathscr{M})$ admits a reduction $(X', \mathscr{M}')$ where $K_{X'} \otimes \mathscr{M}'$ is very ample. Let π: $X \to X'$ be the reduction morphism and put $\mathscr{L}' = \pi^* \mathscr{L}$. Thus $(K_{X'} \otimes \mathscr{L}')^2 = K_{X'} \otimes \mathscr{M}'$ is very ample. On the other hand, if $E \subset X$ is a (-1)-curve, we have

$$EM = 2EL - 1$$

and this shows that $(X' \mathscr{L}')$ is exactly the reduction of $(X, \mathscr{L})$. The remaining exceptions come from the pairs listed in (3.1.6).

5. A final application.

Consider a net $\mathscr{N}$ of ample divisors on a surface X. Classically the Jacobian $J_{\mathscr{N}}$ of $\mathscr{N}$ is defined as the locus of the singularities of the elements of $\mathscr{N}$ (e.g. see [SR, p. 427]). Let $\mathscr{L}$ be the line bundle whose complete linear system contains $\mathscr{N}$; as is known the line bundle $[J_{\mathscr{N}}]$ depends on $\mathscr{L}$ since

$$[J_{\mathscr{N}}] = K_X \otimes \mathscr{L}^3.$$

Then Theorem (1.1) implies the following

(5.1) COROLLARY. *Let X be a surface polarized by an ample line bundle $\mathscr{L}$ such that $|\mathscr{L}|$ contains a net $\mathscr{N}$. Then $[J_{\mathscr{N}}]$ is very ample unless $(X, L) = (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1))$.*

PROOF. If $(X, \mathscr{L}) \neq (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1))$ and $[J_{\mathscr{N}}]$ is not very ample, then $(X, \mathscr{L})$ has to be as in (1.1.2), which gives $c_1(\mathscr{L})^2 = 1$. Then, as $h^0(\mathscr{L}) \geqq \dim \mathscr{N} + 1 = 3$, we get $\Delta(X, \mathscr{L}) = 0$; contradiction.

In particular this implies the following fact.

(5.2) COROLLARY. *Let X be a surface and let f: $X \to \mathbf{P}^2$ be a finite morphism of degree $\geqq 2$. Then the ramification divisor R of f is very ample.*

PROOF. Actually, R is the jacobian of the net $f^*|\mathcal{O}_{\mathbf{P}^2}(1)|$.

Similarly, from proposition (1.6) one deduces the following corollary (see also [LP_2, Th. 3.2]).

(5.3) COROLLARY. *Let X be a surface and let $f: X \to Q^2$ be a finite morphism of degree $\geqq 2$. Then the ramification divisor of f is very ample unless $(X, f^*\mathcal{O}_{Q^2}(1))$ is a scroll and f gives rise to a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{p} & B \\ {\scriptstyle f}\downarrow & & \downarrow \\ Q^2 & \xrightarrow{q} & \mathbf{P}^1 \end{array}$$

where p and q are scroll projections.

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